3 Euler Circuits and Hamilton Cycles

An **Euler circuit** in a graph is a circuit which includes each edge exactly once. An **Euler trail** is a walk which contains each edge exactly once, i.e., a trail which includes every edge. A **Hamilton cycle** is a cycle in a graph which contains each vertex exactly once. A connected graph is called **Hamiltonian** is it contains a Hamilton cycle. There is a simple condition on the degree numbers of a connected graph which allows one to decide whether the graph can be expressed as an Euler circuit. In general, the problem of determining whether a graph is Hamiltonian is more difficult.

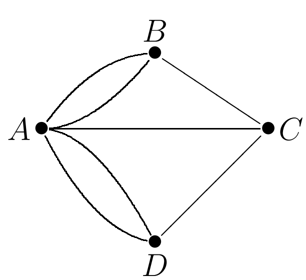
Definition 3.1 *A* ***multigraph*** *is a pair $G = (V, E)$, where V is the vertex set and E is the edge set which allows more than one edge between each pair $x \neq y \in V$. A* ***loop*** *is a an edge connecting a single vertex with itself. A* ***pseudograph*** *is a multigraph with loops. A* ***directed graph*** *is a graph in which the vertices are ordered pairs (x, y) with $x \neq y \in V$. A directed graph can have the two distinct edges (x, y) and (y, x). A* ***directed multigraph*** *allows multiple copies of directed edges.*

These terms are not employed uniformly in the literature. Some authors allow multigraphs and directed multigraphs to have loops. The degree of a vertex and the degree sequence apply to multigraphs. If the edge $\{x, y\}$, $x \neq y$, appears m times, then these make the contribution of m to the the degree of vertex x and m to that of y. If the pseudograph has k loops at x, these contribute $2k$ to the degree of x. In the case of directed (multi-)graphs, each vertex x has an **out-degree** $d^+(x)$ and an **in-degree** $d^-(x)$. The edge (x, y) contributes 1 to the out-degree of x and 1 to the in-degree of y. Each loop (x, x) contributes 1 to each of the in- and out-degrees of x.

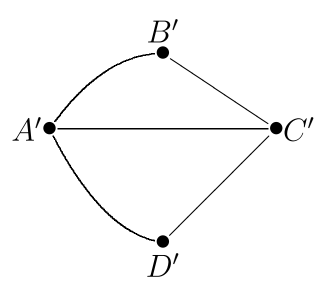
Theorem 3.1 *A connected pseudograph has a Euler circuit if, and only if, the degree of each vertex is even. It has an Euler trail, if, and only if, the degree sequence has exactly 2 odd entries.*

The graph corresponding to Euler's Königsberg is given by G. The town is now called Kaliningrad. The original bridges were destroyed in war. The rebuilt bridge structure is given in G' below. The degree of G is $3, 3, 3, 5$; that of G', $2, 2, 3, 3$. By the theorem G' has an Euler trail; G has neither Euler circuit nor Euler trail.

$G =$



$G' =$



Proof: That the degree of each vertex must be even in the case of an Euler circuit follows from the fact that along the circuit as a vertex is passed the incoming edge contributes 1 to the degree and the outgoing edge another 1 to add 2 to the degree for each time the circuit passes this vertex.

For the converse assume that the graph is connected and that the degree of each vertex is even. Note that loops may be added to a circuit, so we consider a multigraph without loops. We proceed by induction on the number of vertices. If there are only 2 vertices, they must be connected by an even multiple of the edge between then from which an Euler circuit is easily made. Now we assume the result to be valid for graphs of order k for all $2 \leq k \leq n$ (at this stage we only know that this works for $n = 2$). Now if there are $n+1$ vertices, select a fixed vertex which we call x_0 of G. Now $d(x_0) \geq 2$ because the graph is connected and the degree numbers are all even. Choose one edge from x_0, say $\{x_0, x_1\}$, and let $G_1 := G \setminus \{x_0, x_1\}$. Continue in this fashion until a vertex is reached, say x_m so that $G_m := G \setminus \{\{x_0, x_1\}, \ldots, \{x_{m-1}, x_m\}\}$ has no edges containing x_m. We claim $x_0 = x_m$, because if $x_m \neq x_0$, there are an odd number of edges in $x_0 x_1 \cdots x_m$ which contain x_m: strictly between x_0 and x_m pairs of edges share a common vertex. If there are no more edges in G_m adjacent to x_m and $x_m \neq x_0$, then x_m has odd degree in G. The assumption of $d(x)$ even thus implies that $x_0 x_1 \cdots x_m$ is a circuit. Unless G_m has no edges, this is not an Euler circuit for G. The vertices of G_m all have even degree and G_m has at least one isolated vertex x_0. Each of the components of G_m which is not an isolated vertex is a connected multigraph of order n or less whose degree sequence has only even entries. We apply the induction hypothesis to get an Euler circuit for each. It is not difficult to splice these circuits into $x_0 x_1 \cdots x_m$ to get an Euler circuit for G.

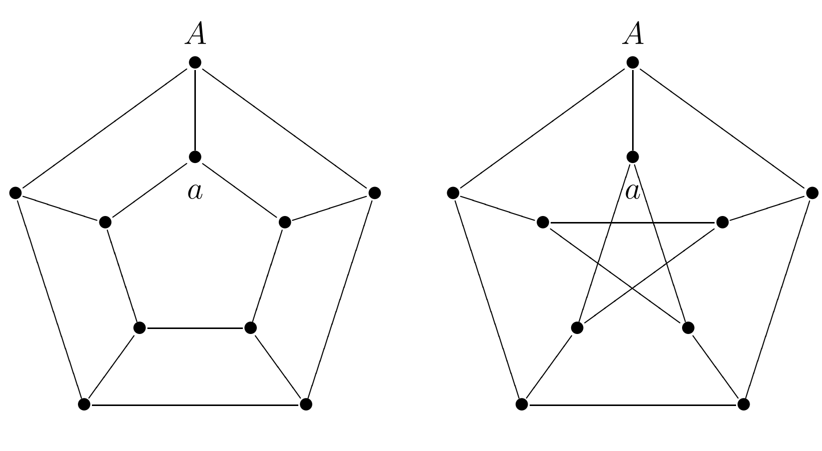
For the case of exactly two odd vertices, add an edge between these vertices to get an even degree sequence. The Euler circuit for this graph with the new edge removed is an Euler trail for the original graph. □

The corresponding result for directed multigraphs is

Theorem 3.2 *A connected directed multigraph has a Euler circuit if, and only if,* $d^+(x) = d^-(x)$. *It has an Euler trail if, and only if, there are exactly two vertices with* $d^+(x) \neq d^-(x)$, *one with* $d^+(x) = d^-(x) + 1$ *and one with* $d^+(x) = d^-(x) - 1$.

3.1 Hamilton Cycles

The above criteria completely characterize graphs with Euler circuits or trails. Now we consider the harder problem of deciding whether a graph has a Hamilton cycle: a connected subgraph which includes all the vertices and has degree 2 at each vertex. Clearly a Hamiltonian graph has no leaves. The simplest Hamiltonian graph is C_n, the cycle of order n, $n > 2$. Also K_n, $n > 2$, has a Hamilton cycle because it contains C_n. We consider two other graphs, which show that it is not always easy to see whether or not a graph is Hamiltonian.



The graph to the left above we call the double pentagon. The one on the right is called the Petersen graph. Both share the same degree sequence. However, the double pentagon is Hamiltonian, while Petersen's graph is not. A Hamilton path in either graph starting on the outside pentagon must finish on the outside pentagon, which means that it includes an even number of the edges joining the inner figure to the outer. Assume that four of the inner-to-outer edges are used, leaving the Aa edge omitted. Then the two outer edges at A and the two inner edges at a must be in the cycle. In the double pentagon this yields a cycle already, so a Hamilton cycle of the double pentagon cannot include four inner-to-outer edges. In the Petersen graph the two outer edges at A, plus the two inner edges at a plus the four inner-to-outer edges (excluding Aa) yield two disjoint paths, each with 5 vertices. There are no edges left in the graph which can join these two paths to make a cycle. Thus neither graph has a Hamilton cycle with 4 inner-to-outer edges. A Hamilton path of the double pentagon is achieved by traversing 4 of the 5 outer edges, taking a edge to the inner pentagon, which is traversed in the reverse direction, the path then returning to the outside start. Now we attempt a Hamilton cycle for Petersen's graph using exactly two inner-to-outer edges. We start with aA, one of the two inner-to-outer edges. Then we must include all of the outer edges before returning to the inner figure. We can get back to a via 2 edges or 3 edges, but we cannot include all the inner vertices in a cycle. This shows that double pentagon is Hamiltonian, while Petersen's graph is not.

We now consider partial results concerning Hamiltonian graphs. Recall that for $G = (V, E)$ and $S \subset V$, $G \setminus S$ means the graph with vertex set $V \setminus S$ and edges set consisting of all edges in E both of whose endpoints are in $V \setminus S$.

Proposition 3.1 *If G is Hamiltonian and $x \in V(G)$, then $G \setminus \{x\}$ is connected.*

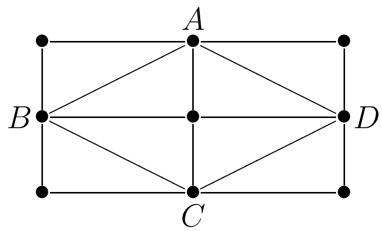
Proof: If one omits a vertex from a cycle, a path remains.

A more general version is the following.

Theorem 3.3 *If $G = (V, E)$ is Hamiltonian and $S \subset V$, then $G \setminus S$ has not more than $|S|$ components.*

Proof: If G is Hamiltonian, let $G' \subset G$ be a Hamilton cycle. It suffices to prove the result for G', since adding edges does not increase the number of components. The conclusion corresponds to the fact that if n items are chained together to form a loop, the removal of k items and the chains connecting them, yields at most k pieces. One can remove the items one at a time. The first removed yields an open connected chain, which is a path of the graph. If an item is removed from the end of the chain, the number of components remains the same. If an isolated item is removed, the number of components of the remaining graph is reduced by one. If an item is removed from the interior of 3 or more linked items, the number of components is increased by 1. In no case can the number of components be made greater than the number of items removed. $\square$

The above gives a necessary condition for a graph to be Hamiltonian, but it does not suffice. The Petersen graph has the component property of the theorem but is not Hamiltonian. On the other hand, if one can show that a graph does not have the component property of the theorem, the graph cannot be Hamiltonian.

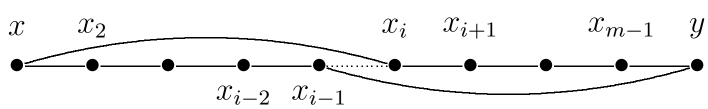


In the graph above the removal of the 4 vertices $ABCD$ yields a graph with 5 components, so the original graph has no Hamilton cycle.

We now present results which show that if a graph has *enough* vertices, then it must be Hamiltonian. Recall that $\Gamma_G(x)$ denotes $\{y \in V(G) : \{x, y\} \in E\}$.

Lemma 3.1 *Assume that for $G = (V, E)$ the vertices $x \neq y$ are not adjacent, i.e., $y \notin \Gamma_G(x)$, and there exists a path $x = x_1x_2 \cdots x_m = y$ and i, $2 < i < m$ so that $x_{i-1} \in \Gamma_G(y)$, while $x_i \in \Gamma_G(x)$. Then there exists an m-cycle which includes x and y.*

Proof: Since $x = x_1x_2 \cdots x_m = y$ is a path, it includes m distinct vertices. By hypothesis $\{y, x_{i-1}\} \in E$ and $\{x, x_i\} \in E$ so $xx_ix_{i+1} \cdots x_{m-1}yx_{i-1}x_{i-2} \cdots x_2x$ is the required m-cycle. □



For $G = (V, E)$ and $\{x, y\} \notin E$, the notation $G + \{x, y\}$ means the graph with vertex set V and edge set $E \cup \{x, y\}$.

Theorem 3.4 *(Bondy and Chvátal, 1976) Let $G = (V, E)$ be a graph of order n in which $x \neq y \in V$ and $\{x, y\} \notin E$ and $d(x) + d(y) \geq n$, Then G is Hamiltonian if, and only if, $G + \{x, y\}$ is.*

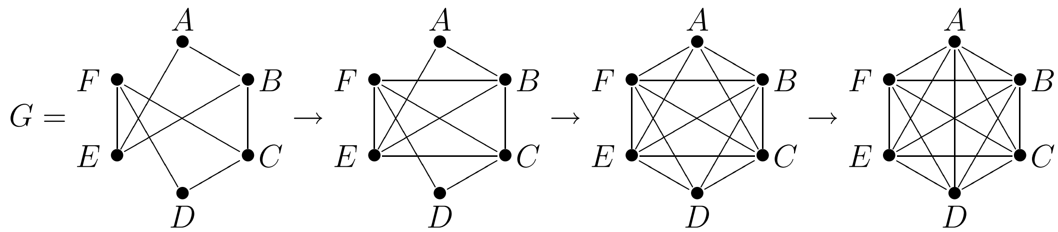
Proof: A Hamilton cycle in G is a Hamilton cycle in $G + \{x, y\}$, which proves the "if" part. Assume that $G + \{x, y\}$ has a Hamilton cycle. If this does not include $\{x, y\}$, then it is a Hamilton cycle in G. Assume the Hamilton cycle includes $\{x, y\}$, which we write as $x = x_1x_2 \cdots x_nx_1$, $y = x_n$. Now the $d(x)$ vertices adjacent to x appear in $x_2, \ldots, x_{n-1}$. If $x_i \in \Gamma(x)$ and $x_{i-1} \in \Gamma(y)$, we may invoke the lemma above to obtain the required Hamilton cycle which does not use $\{x, y\}$. The number of vertices not adjacent to or equal to y is $k = n - d(y) - 1$. Unless $k \geq d(x)$, there must be the pair $\{x_{i-1}, x_i\}$ as used in the lemma. For this to be the case $k = n - d(y) - 1 \geq d(x) \Rightarrow d(x) + d(y) \leq n - 1$. By hypothesis $d(x) + d(y) \geq n$ so there is a pair $\{x_{i-1}, x_i\}$ as required by the lemma, so there exists a Hamilton cycle in G. □

Definition 3.2 *The **closure** $C(G)$ of the graph $G = (V, E)$ is the graph with vertex set V obtained by successively adding edges $\{x, y\}$ not in the current graph which have $d(x) + d(y) \geq n$, where d's are calculated for the current graph.*

Lemma 3.2 *Regardless of the order in which edges satisfying the $d(x) + d(y) \geq n$ are appended, the resulting closure $C(G)$ is uniquely defined.*

Proof: Let G_1 and G_2 be graphs obtained by successively adding edges, which in the current graph satisfy $d(x) + d(y) \geq n$, until no more edges remain which satisfy the condition. If $G_1 \neq G_2$, assume G_1 contains an edge which is not in G_2. Let $e_1, e_2, \ldots, e_m$ be the edges in order which were added to G to produce G_1. Let $e_k = \{x_k, y_k\}$ be the first of these which is not in G_2. This implies that $e_1, e_2, \ldots, e_{k-1}$ are in G_2. Since adding additional vertices does not decrease $d(x)$, it follows that $d_{G_2}(x_k) + d_{G_2}(y_k) \geq n$. This means that e_k can be appended to G_2 to get a graph with more edges. This contradicts the construction of G_2, so $G_1 = G_2$. □

Note: the Petersen graph has order 10 and each vertex has degree 3. Since $3 + 3 < 10$, there are no vertices to be appended, so the Petersen graph is its own closure. Below we illustrate the closure process.



For G, $(d(A), d(B), d(C), d(D), d(E), d(F)) = (2, 3, 3, 2, 3, 3)$. First we add BF, EC to get $(2, 4, 4, 2, 4, 4)$. Then AC, AF, BD, DE to get $(4, 5, 5, 4, 5, 5)$ and finally AD to get $(5, 5, 5, 5, 5, 5) = K_6$.

Corollary 3.1 *If G is of order $n > 2$ and $C(G) = K_n$, then G is Hamiltonian.*

Theorem 3.5 *(Dirac, 1952) Let G be a graph of order $n > 2$ in which each vertex satisfies $d(x) \geq n/2$. Then G is Hamiltonian.*

Theorem 3.6 *(Ore, 1960) Let G be a graph of order $n > 2$ in which satisfies $d(x) + d(y) \geq n$ whenever the distinct vertices x, y are not adjacent. Then G is Hamiltonian.*

The above are special cases of the Theorem of Bondy and Chvátal. We consider a more general result due to Chvátal. First we need a lemma about degree sequences.

Lemma 3.3 *Let $|V| = n$ and let $d : V \to \mathbf{R}$ and $d' : V \to \mathbf{R}$ satisfy $d(x) \leq d'(x)$ for each $x \in V$. Let $d_1, \ldots, d_n$ and $d'_1, \ldots, d'_n$ be the values taken on by d and d' written in increasing order $d_{i-1} \leq d_i$, $d'_{i-1} \leq d'_i$, for $i = 2, \ldots, n$. Then $d_i \leq d'_i$ for $i = 1, \ldots, n$.*

Proof: Let $d'_k = d'(x)$, for fixed k, $1 \leq k \leq n$. Then there are at least k vertices y with $d'(y) \leq d'(x)$. For these vertices $d(y) \leq d'(y) \leq d'(x)$. Hence $d_k \leq d'_k = d'(x)$. □

Theorem 3.7 *Let $G = (V, E)$ be a graph of order $n > 2$ and let $d_1 \leq d_2 \leq \cdots d_n$ be the degree sequence. If for each i, $1 \leq i \leq n/2$, at least one of the following is true*

$$\text{Chvátal's condition: } \begin{cases} (a) & d_i > i; \\ (b) & d_{n-i} \geq n - i. \end{cases}$$

Then $C(G) = K_n$ and G is Hamiltonian.

Proof: Chvátal's condition can be expressed as follows: for each k with $k \leq n/2$ and $d_k \leq k$ we must have $d_{n-k} \geq n-k$.

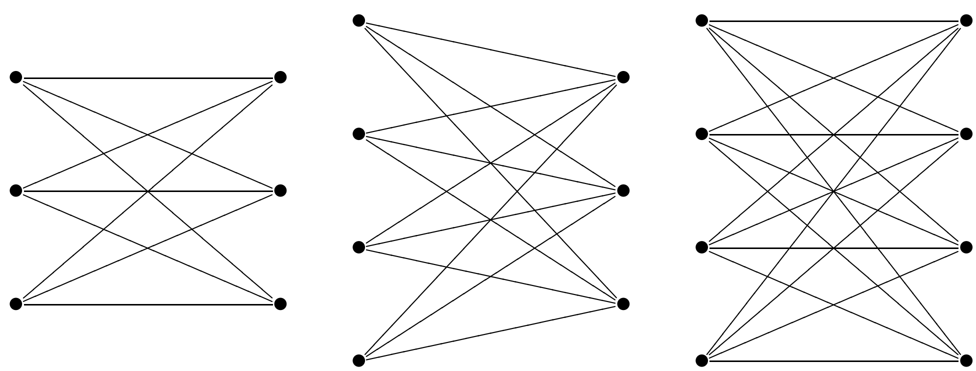
Let $\{d_i\}$ and $\{d'_i\}$ be the degree sequences respectively in G and $C(G)$. By the lemma $d_i \leq d'_i$, $i = 1, \ldots, n$. If $C(G) \neq K_n$, choose x, y so that $\{x, y\} \notin E(C(G))$ and $d'(x)+d'(y)$ is maximal. Since $C(G)$ is a closure, $d'(x) + d'(y) \leq n - 1$. Let $d'(x) \leq d'(y)$ so that $d'(x) < n/2$. Set $k := d'(x)$. The goal is to show that Chvátal's condition fails for k.

Let $M(x)$ and $M(y)$ denote the vertices in $C(G)$ which are not adjacent to x and not adjacent to y respectively. We have $|M(x)| = n - 1 - d'(x)$, $|M(y)| = n - 1 - d'(y)$. From $d'(x) + d'(y) \leq n - 1$ follow $|M(x)| \geq d'(y)$ and $|M(y)| \geq d'(x) = k$. The fact that $d'(x) + d'(y)$ is maximal for nonadjacent vertices implies that there are at least $|M(y)| \geq d'(x) = k$ vertices w with $d'(w) \leq d'(x)$, hence $d'_k \leq k = d'(x)$. Similarly $w \in M(x) \cup \{x\}$ implies $d'(w) \leq d'(y) \leq n - 1 - d'(x)$, so that there are at least $n - 1 - d'(x) + 1$ vertices w with $d'(w) \leq n - 1 - k$, so $d'_{n-k} \leq n - k - 1$. The choice of $d'(x) \leq d'(y)$ yields an extra w, i.e., $w = x$, in addition to $M(x)$ with $d'(w) \leq d'(y)$. This shows that Chvátal's condition fails for $k = d'(x)$.

□

Here we introduce a concept that provides useful examples.

Definition 3.3 *The graph $G = (V, E)$ is* ***bipartite*** *if $V = V_1 \cup V_2$ where $V_1 \cap V_2 = \emptyset$ and each edge has one endpoint in V_1 and the other endpoint in V_2. The complete bipartite graph $K_{m,n}$ consists of $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and $|V_1| = m$, $|V_2| = n$ and edge set $E = \{\{x, y\} : x \in V_1,\ y \in V_2\}$.*



The Bipartite Graphs $K_{3,3}$, $K_{4,3}$ and $K_{4,4}$

Note that $K_{4,4}$ is the only one of the above with an Euler circuit. Notice also that the closures of $K_{3,3}$ and $K_{4,4}$ are the corresponding complete graphs, so they are Hamiltonian. However $K_{4,3}$ is not Hamiltonian, as is the case for any $K_{m,n}$ with $m \neq n$. Any cycle in a bipartite graph must the same number of points from V_1 as from V_2. An alternative way to show that $K_{m,n}$ is not Hamiltonian is Theorem 3.3. For $m < n$, removing the m V_1 vertices leaves n isolated V_2 vertices. Since the number of remaining components n exceeds m, the theorem excludes a Hamilton cycle.