Lecture 5: Put - Call Parity

Reading: J.C.Hull, Chapter 9

Reminder: basic assumptions

1. There are no arbitrage opportunities, i.e. no party can get a riskless profit.
2. Borrowing and lending are possible at the risk-free interest rate $r > 0$ which is constant over time and is compounded continuously.
3. There are no transaction costs, and no dividends are payed on stocks.

Bounds for prices of European options:

$$S_0 - Ke^{-rT} \leq c \leq S_0$$
$$Ke^{-rT} - S_0 \leq p \leq Ke^{-rT}$$

Put-call parity

We consider a relationship between the prices of European call and put options.

Claim

Let p be the price of a European put option and c be the price of a European call option with strike price K and maturity T. Then

$$c + Ke^{-rT} = p + S_0.$$

Proof

We consider two portfolios:

Portfolio A: one European call and cash $= Ke^{-rT}$.

Portfolio B: one European put and one share.

At time T, both are worth $\max(S_T, K)$.

Hence they should also have identical values today, that is

$$c + Ke^{-rT} = p + S_0.$$

$\square$

American options case

The put-call parity for European options says that

$$c - p = S_0 - Ke^{-rT}.$$

For American options there is no such simple relation but the following holds:

Claim

Let P be the price of an American put option and C be the price of an American call option with strike price K and maturity T. Then

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}.$$

Proof

Compare

Portfolio A: one American call and K EUR in cash.
Portfolio B: one American put and one share.

The worth of both A and B depends on whether the options are exercised, and at which moment of time.

Suppose the put option is exercised at some moment $0 \leq t \leq T$. Then the worth of B at time T is $\max(S_t, K)e^{r(T-t)}$.

But if one exercises the call option at the same moment t, the worth of A at time T will be $\Big(\max(S_t, K) + K(e^{rt} - 1)\Big)e^{r(T-t)}$, which is bigger or equal than the worth of B:

$$\Big(\max(S_t, K) + K(e^{rt} - 1)\Big)e^{r(T-t)} \geq \max(S_t, K)e^{r(T-t)}.$$

Hence at $t = 0$ the worth of A should be also bigger or equal than the worth of B, that is

$$C + K \geq P + S_0.$$

To prove the upper bound for $C - P$, we compare
Portfolio C: one American call and Ke^{-rT} EUR in cash.
Portfolio B: one American put and one share.

In this case the call option can't be exrcised until the cash grows up to K, which happens only at $t = T$. Therefore the value of C at time T is $\max(S_T, K)$.

As we have seen above, if the put option is exercised at time $0 \leq t \leq T$ then the worth of B at time T is

$$\max(S_t, K)e^{r(T-t)}$$

which can be made equal to the worth of C taking $t = T$, and possibly can be made bigger.

Therefore the initial worth of B should be bigger or equal to the inicial worth of C, that is

$$P + S_0 \geq C + Ke^{-rT}.$$

□

Arbitrage opportunities: an example

If put-call parity doesn't hold, there will be arbitrage opportunities.

Example

Suppose $S_0 = 31$ EUR, $K = 30$ EUR, $T = 3$ months, $r = 10\%$ p.a., $c = 3$ and $p = 2.25$ EUR.

Then portfolio A: "one call and cash Ke^{-rT} is worth

$$c + Ke^{-rT} = 32.26,$$

while portfolio B: "one put and one share" is worth

$$p + S_0 = 33.25.$$

We have $A < B$, and we shall show that one can make an arbitrage profit.

Arbitrage strategy: Sell the portfolio B and buy A. :
More precisely, assume that the trader has already the stock at time $t = 0$ which is in portfolio B.
At time $t = 0$ he sells a put and a stock (so portfolio B), so he obtains $p + S_0$ EUR.
Further, he buys a call and put the rest of the money to a deposit with $r\%$ interest rate.
The cashflow is

$$p + S_0 - c = 2.25 + 31 - 3 = 30.25.$$

Invested for 3 months this gives $30.25e^{0.1 \times \frac{3}{12}} = 31.02$ EUR.

If $S_T > K = 30$, he exercises the call, so he buys one share for the price K. Then he has one share and a profit of 1.02 EUR.
If $S_T < K = 30$, then the put will be exercised from the other party. Then he must buy one share from the other party for the price K, so he has one share and a profit of 1.02 EUR.

Riskless profit is 1.02 EUR.

Factors affecting option prices

In the contract of an option we specify

1. Strike price K;
2. Time to expiration T;

In case of an American option it is clear that the price of the option increases if we increase the maturity time.
For a call option we have to the right to buy one stock at maturity time for the strike price K. If we increase the strike price in the option we expect that the price of the option decreases.
For a put option we have to the right to sell one stock at maturity time for the strike price K. If we increase the strike price in the option we expect that the price of the option increases.

Factors affecting option prices

Six factors are affecting the price of a stock option are

1. Strike price K;
2. Time to expiration T;
3. Current stock price S_0;
4. Risk-free interest rate r;
5. Volatility of the stock price σ.

We have already seen that the interest rate r is important for giving bounds for the price of an option. Usually there is a relationship between interest rates and stock prices (low interest rates usually boost stock prices as deposits do not create enough wealth).

Volatility

The stock price S_0 is fundamental for the price of an option. Until now we have no idea how to determine a price for an option. Clearly the price should depend on the future development of the stock price S_T.
In order to determine a price we have to make assumptions about the development of the stock price S_T which are based on probability theory.

Volatility of a stock price is (roughly speaking) a measure of how uncertain we are about the future stock price movements.

As volatility increases, the chances of a stock doing really well or really poorly increase.
Generally the prices of calls and puts increase as volatility increases as the risk increases.