Lecture 5: Put - Call Parity

Reminder: basic assumptions

1. There are no arbitrage opportunities, i.e. no party can get a riskless profit.
2. Borrowing and lending are possible at the risk-free interest rate $r > 0$ which is constant over time and is compounded continuously.
3. There are no transaction costs, and no dividends are payed on stocks.

Bounds for prices of European options:

$$S_0 - Ke^{-rT} \leq c \leq S_0$$
$$Ke^{-rT} - S_0 \leq p \leq Ke^{-rT}$$
We consider a relationship between the prices of European call and put options.

**Claim**
Let $p$ be the price of a European put option and $c$ be the price of a European call option with strike price $K$ and maturity $T$. Then

$$c + Ke^{-rT} = p + S_0.$$
Proof

We consider two portfolios:

Portfolio A: one European call and cash = $Ke^{-rT}$.

Portfolio B: one European put and one share.

At time $T$, both are worth $\max(S_T, K)$.

Hence they should also have identical values today, that is

$$c + Ke^{-rT} = p + S_0.$$  

$\square$
American options case

The put-call parity for European options says that

\[ c - p = S_0 - K e^{-rT}. \]

For American options there is no such simple relation but the following holds:

**Claim**

*Let \( P \) be the price of an American put option and \( C \) be the price of an American call option with strike price \( K \) and maturity \( T \). Then*

\[ S_0 - K \leq C - P \leq S_0 - K e^{-rT}. \]
**Proof**

Compare

**Portfolio A:** one American call and $K$ EUR in cash.

**Portfolio B:** one American put and one share.

The worth of both A and B depends on whether the options are exercised, and at which moment of time.

Suppose the put option is exercised at some moment $0 \leq t \leq T$. Then the worth of B at time $T$ is $\max(S_t, K)e^{r(T-t)}$.

But if one exercises the call option at the same moment $t$, the worth of A at time $T$ will be $\left(\max(S_t, K) + K(e^{rt} - 1)\right)e^{r(T-t)}$, which is bigger or equal than the worth of B:

$$\left(\max(S_t, K) + K(e^{rt} - 1)\right)e^{r(T-t)} \geq \max(S_t, K)e^{r(T-t)}.$$

Hence at $t = 0$ the worth of A should be also bigger or equal than the worth of B, that is

$$C + K \geq P + S_0.$$
To prove the upper bound for $C - P$, we compare

**Portfolio C:** one American call and $Ke^{-rT}$ EUR in cash.

**Portfolio B:** one American put and one share.

In this case the call option can’t be exercised until the cash grows up to $K$, which happens only at $t = T$. Therefore the value of $C$ at time $T$ is $\max(S_T, K)$.

As we have seen above, if the put option is exercised at time $0 \leq t \leq T$ then the worth of $B$ at time $T$ is

$$\max(S_t, K)e^{r(T-t)}$$

which can be made equal to the worth of $C$ taking $t = T$, and possibly can be made bigger.

Therefore the initial worth of $B$ should be bigger or equal to the initial worth of $C$, that is

$$P + S_0 \geq C + Ke^{-rT}.$$
Arbitrage opportunities: an example

If put-call parity doesn’t hold, there will be arbitrage opportunities.

Example
Suppose $S_0 = 31$ EUR, $K = 30$ EUR, $T = 3$ months, $r = 10\%$ p.a., $c = 3$ and $p = 2.25$ EUR.

Then portfolio A: “one call and cash $Ke^{-rT}$ is worth

\[ c + Ke^{-rT} = 32.26, \]

while portfolio B: “one put and one share” is worth

\[ p + S_0 = 33.25. \]

We have $A < B$, and we shall show that one can make an arbitrage profit.
**Arbitrage strategy: Sell the portfolio $B$ and buy $A$.**

More precisely, assume that the trader has already the stock at time $t = 0$ which is in portfolio $B$. At time $t = 0$ he sells a put and a stock (so portfolio $B$), so he obtains $p + S_0$ EUR.

Further, he buys a call and put the rest of the money to a deposit with $r\%$ interest rate.

The cashflow is

$$p + S_0 - c = 2.25 + 31 - 3 = 30.25.$$  

Invested for 3 months this gives $30.25e^{0.1\times\frac{3}{12}} = 31.02$ EUR.

If $S_T > K = 30$, he exercises the call, so he buys one share for the price $K$. Then he has one share and a profit of 1.02 EUR.

If $S_T < K = 30$, then the put will be exercised from the other party. Then he must buy one share from the other party for the price $K$, so he has one share and a profit of 1.02 EUR.

Riskless profit is 1.02 EUR.
Factors affecting option prices

In the contract of an option we specify

1. Strike price $K$;
2. Time to expiration $T$;

In case of an American option it is clear that the price of the option increases if we increase the maturity time. For a call option we have to the right to buy one stock at maturity time for the strike price $K$. If we increase the strike price in the option we expect that the price of the option decreases. For a put option we have to the right to sell one stock at maturity time for the strike price $K$. If we increase the strike price in the option we expect that the price of the option increases.
Factors affecting option prices

Six factors are affecting the price of a stock option are

1. Strike price $K$;
2. Time to expiration $T$;
3. Current stock price $S_0$;
4. Risk-free interest rate $r$;
5. Volatility of the stock price $\sigma$.

We have already seen that the interest rate $r$ is important for giving bounds for the price of an option. Usually there is a relationship between interest rates and stock prices (low interest rates usually boost stock prices as deposits do not create enough wealth).
Volatility

The stock price $S_0$ is fundamental for the price of an option. Until now we have no idea how to determine a price for an option. Clearly the price should depend on the future development of the stock price $S_T$. In order to determine a price we have to make assumptions about the development of the stock price $S_T$ which are based on probability theory.

Volatility of a stock price is (roughly speaking) a measure of how uncertain we are about the future stock price movements.

As volatility increases, the chances of a stock doing really well or really poorly increase. Generally the prices of calls and puts increase as volatility increases as the risk increases.