Lecture 4: Properties of stock options

Reading: J.C.Hull, Chapter 9

An European call option is an agreement between two parties giving the holder the right to buy a certain asset (e.g. one stock unit) for a price $K$ (strike price) at time $T$ (maturity).

An European put option is an agreement between two parties giving the holder the right to sell a certain asset (e.g. one stock unit) for a price $K$ (strike price) at time $T$ (maturity).

An American option is an agreement where one can use the above right any time between 0 and time $T$.

The word “American” or “European” does NOT refer to a geographic place.

Most exchange-traded stock options have a life less than 1 year.
Claim

The price $C$ of an American option is larger than the price $c$ of an European option (with the same strike price $K$ and maturity $T$):

$$c \leq C.$$

Proof

Suppose that $C < c$. Instead of buying the European option we buy the American option. Then we can exercise the option at time $T$ and there is no need to buy the more expensive European option. This shows that $C$ should be larger than $c$. □
Claim

The price $C$ of an American call option (and therefore also the price $c$ of an European call option) is smaller or equal than the spot price $S_0$:

$$c \leq C \leq S_0.$$ 

Proof

Suppose that $C > S_0$. We issue (i.e. sell) an American call option (for one stock with strike price $K$ and maturity $T$) to the market and obtain $C$(EUR).

At the same time we buy one stock for the price $S_0$. So we have still $C - S_0 > 0$ (EUR).
The other party of our contract has the right to buy from us one stock at any time up to the maturity date $T$. If he exercises his right we give him the stock which we have already bought, otherwise the stock remains our property.

In any case we obtained $C - S_0$ (EUR) from this deal and we are happy to have earned money without any risk (this is a typical arbitrage profit). Since arbitrage is not possible in our model we conclude that $C \leq S_0$. □
Basic assumptions in the theory of options

Throughout our discussions in this course we will make the following assumptions:

1. There are no arbitrage opportunities, i.e. no party can get a riskless profit.
2. Borrowing and lending are possible at the risk-free interest rate $r > 0$ which is constant over time and is compounded continuously.
3. There are no transaction costs.

For simplicity, we shall consider only stocks which do not pay dividends.
The lower bound for prices of European call options

Claim
Let \( c \) be the price of an European call option with strike price \( K \) and maturity \( T \). Then the following inequality

\[
S_0 - Ke^{-rT} \leq c \leq S_0
\]

holds under the assumption that no dividend is payed on the stock.

Proof
The inequality \( c \leq S_0 \) was already proved. Suppose that \( c + Ke^{-rT} < S_0 \). We will show that somebody can make an arbitrage profit in this case.
First Proof

Suppose one trader has already the stock in his portfolio. Now he sees at the market the offer for a call option for the stock with strike price $K$ and maturity $T$ for a price $c$ such that

$$c + Ke^{-rT} < S_0.$$ 

He immediately sells one stock, obtains $S_0$ EUR, buys the call option and puts the cash $Ke^{-rT}$ on a deposit. After the maturity date $T$ he then has $K$ (EUR). At maturity date he exercises the call option and buys the stock with the cash $K$, and the stock is again in his portfolio. Thus he earned from this business $S_0 - (c + Ke^{-rT})$ EUR without any risk. $\square$
Second Proof

We consider two portfolios.

**Portfolio A**: 1 European call and cash = $Ke^{-rT}$.

**Portfolio B**: 1 share.

We shall compare the performance of these portfolios. After time $T$ we have

**Portfolio A**: The cash is worth $K$.
If $S_T > K$, the option is exercised and the portfolio is worth $S_T$.
If $S_T < K$, option is not exercised and the portfolio is worth $K$.
Thus the portfolio is worth $\max(S_T, K)$.

**Portfolio B**: the portfolio is worth $S_T$.

Here we use the assumption that no dividends are payed on the stock.
We see that portfolio A is worth at least as much as portfolio B at time $T$.

Thus the worth of portfolio A at time 0 must be also bigger or equal than the worth of portfolio B, since otherwise we could make an arbitrage profit. Therefore

$$c + Ke^{-rT} \geq S_0.$$

□
Bounds for the price of a European put option

Claim

Let $p$ be the price of an European put option with strike price $K$ and maturity $T$. Then the following inequality

$$Ke^{-rT} - S_0 \leq p \leq Ke^{-rT}$$

holds under the assumption that no dividend is payed on the stock.
Bounds for the price of a European put option

Claim

Let $p$ be the price of an European put option with strike price $K$ and maturity $T$. Then the following inequality

$$Ke^{-rT} - S_0 \leq p \leq Ke^{-rT}$$

holds under the assumption that no dividend is payed on the stock.

Proof

(upper bound) Suppose that $p > Ke^{-rT}$ is valid. One can sell (write) one put option for the stock obtaining $p$ EUR. We take $Ke^{-rT}$ EUR of this sum and make a deposit with interest rate $r$. After maturity time $T$ the deposit has $K$ EUR. The other party has the right to sell to us at maturity date $T$ one stock for the price $K$. If this right is exercised we give the other party $K$ EUR and we obtain the stock and still we have $p - Ke^{-rT} > 0$ EUR in our pocket. A good deal!
Remark
Similarly one proves that the price of an American put option is bounded by $K$. Here we can not deposit the money since the other party has the right to sell at any time between 0 and $T$.

(lower bound) We consider two portfolios $C$ and $D$ and compare their performance.
**Portfolio C:** 1 European put and 1 share.
**Portfolio D:** cash $= Ke^{-rT}$.

After time $T$ we have
**Portfolio C:** If $S_T < K$, the put option is exercised, so we sell the share using our option and we obtain $K$ EUR. If $S_T > K$, the option is not exercised and the portfolio is worth $S_T$.
Thus the portfolio is worth $\max(S_T, K)$.
**Portfolio D:** The cash is worth $K$. 
We see that portfolio C is worth at least as much as portfolio D at time $T$.
Thus the worth of portfolio C at time 0 must be bigger or equal to the worth of portfolio D (since otherwise we could make an arbitrage profit):

$$p + S_0 \geq Ke^{-rT}$$
$$p \geq Ke^{-rT} - S_0$$