Example
Suppose we have a 6 month European call option with $K = \€ 21$. Suppose $S_0 = \€ 20$ and in two time steps of 3 months the stock can go up or down by 10% ($u = 1.1$, $d = 0.9$). Let $r = 0.12$.

We want to calculate the value $f$ of the option at node $A$. We can do this by working back to $A$ in several steps.
We can calculate the value of the option at node $B$ by using the result we found for the one-step binomial tree.

Using the formulas

$$f = e^{-rT} \left[ pf_u + (1 - p)f_d \right],$$

$$p = \frac{e^{rT} - d}{u - d},$$

we find the price of the option at node $B$

$$p = (e^{-0.12 \cdot 0.25} - 0.9) / 0.2 = 0.6523,$$

$$f = e^{-0.12 \cdot 0.25} \left[ 0.6523 \cdot 3.2 + 0.3477 \cdot 0 \right] = 2.0257.$$
Similarly, we can compute the value of the option at node $C$ from the values at the nodes $E$ and $F$.

We find $f = 0,$ since the value at both the nodes $E$ and $F$ is 0.
Finally, we compute the value of the option at node A from the values at the nodes B and C.

\[
p = 0.6523, \\
f = e^{-0.12 \cdot 0.25} \left[ 0.6523 \cdot 2.0257 + 0.3477 \cdot 0 \right] = 1.2823. \\
\]

Hence the value of the option today is €1.2823.
The general case

We assume that the current stock price is $S_0$ and that it can go up by a factor of $u$ or down by a factor of $d$ during each time step $\delta t$ years. Assume that the risk-free rate is $r$. 

$$
\begin{align*}
&\text{uS}_0 &\quad \text{udS}_0 &\quad \text{S}_0 \\
&\quad \text{uuS}_0 &\quad \text{dS}_0 &\quad \text{ddS}_0
\end{align*}
$$

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Using the formulas for the one-step binomial tree, we find

\[ f_u = e^{-r\delta t} \left[ pf_{uu} + (1 - p)f_{ud} \right] \]
\[ f_d = e^{-r\delta t} \left[ pf_{ud} + (1 - p)f_{dd} \right] \]
\[ f = e^{-r\delta t} \left[ pf_u + (1 - p)f_d \right] \]

all with

\[ p = \frac{e^{r\delta t} - d}{u - d}. \]

If we substitute the first two results into the third, we find

\[ f = e^{-2r\delta t} \left[ p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd} \right]. \]
**Theorem**

Assume that the stock price \( S_0 \) goes either up or down by a factor \( u > 1 \) and \( d < 1 \) resp. in the time steps \( \delta t \). Let \( f_{uu} \) and \( f_{ud} \) and \( f_{dd} \) the payoffs of the option at maturity time \( T = 2\delta t \) in the different cases of stock movements. Let \( r \) be the riskless interest rate. Then the price \( f \) of the european option is

\[
f = e^{-2r\delta t} \left[ p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd} \right].
\]

where

\[
p = \frac{e^{r\delta t} - d}{u - d}.
\]
An example with puts

Suppose we have a 2 year European put with $K = € 52$ and $S_0 = € 50$.
Suppose we have 2 time steps each of 1 year, in which the stock goes up or down by 20%, so $u = 1.2$ and $d = 0.8$.
Suppose $r = 0.05$. 

![Diagram of a binomial lattice for pricing a European put option with inputs A, B, C, D, E, F and node values 50, 60, 40, 32, 48, 72. The lattice shows the possible stock prices at each time step with up and down factors p and 1-p, and the probabilities of each outcome.]
We get

\[ p = \frac{e^{r \delta t} - d}{u - d} = \frac{e^{0.05 \cdot 1} - 0.8}{1.2 - 0.8} = 0.6282, \]

and

\[ f_{uu} = 0, \]
\[ f_{ud} = 4, \]
\[ f_{dd} = 20, \]

and so

\[ f = e^{-2r \delta t} \left[ p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd} \right] = 4.1923. \]
An American put option

As an example we consider the same example as before, but now we assume we have an American put option.

Let now $F$ be the price of the American put option. Then the terminal payoffs are the same, i.e. that $F_{uu} = f_{uu}$ and $F_{ud} = f_{ud}$ and $F_{dd} = f_{dd}$. The strategy is as before: compute the payoffs at the previous nodes.
At node $B$ we compute the value $f_u$ using that $p = 0.6282$

\[
f_u = e^{-r\delta t} \left[ pf_{uu} + (1 - p)f_{ud} \right] \\
= e^{-0.05} \left[ 0.6282 \cdot 0 + 0.3718 \cdot 4 \right] = 1.4147.
\]

In contrast to the European option we have now the right to exercise the option. Thus we have to compute the payoff: in the case $S_T > K$ it is not interesting to exercise the option (we have the right to sell the option for the strike price $K$), so we conclude that the payoff of the American option $F_u$ is equal to $f_u$.

At node $C$ we have

\[
f_d = e^{-r\delta t} \left[ pf_{ud} + (1 - p)f_{dd} \right] \\
= e^{-0.05} \left[ 0.6282 \cdot 4 + 0.3718 \cdot 20 \right] = 9.436.
\]

Now we have the right to sell the option at the node $C$ for the strike price of 52 Euros, so the payoff is $52 - 40 = 12$ which is higher than the price of the option at this time of 9.436 Euro. So the value of the American option at node $C$, denoted by $F_d$, is now 12 and not 9.436.
At node A we have the formula

\[
f = e^{-r\delta t} \left[ pF_u + (1 - p)F_d \right]
\]

\[
e^{-0.05} \left[ 0.6282 \cdot 1.4147 + 0.3718 \cdot 12 \right]
\]

\[
= 5.0894.
\]

Pay off from early exercise at this node is 2 which is less than \( f = 5.0894 \). Thus \( F \) is equal to \( f = 5.0894 \) and the value of the option today is \( \varepsilon 5.0894 \).
American options

For European options we have a general formula for the two-step binomial tree.

For American options we can’t use it directly; but we can still use the binomial tree model.

The difference with European options is that we have to check at every node how big the payoff is: if it is better to exercise the option at the node we have to change the value of the option at the node.
Multiple-step binomial trees

So far we have seen one-step and two-step binomial trees.

The results for European option are

\[ f = e^{-r\delta t} \left[ pf_u + (1 - p)f_d \right] \]

for one-step, and

\[ f = e^{-2r\delta t} \left[ p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd} \right] \]

for two-step, both with

\[ p = \frac{e^{r\delta t} - d}{u - d}, \]

where \( \delta t \) is the length of one step.
If we continue adding more steps, we find

\[ f = e^{-3r\delta t} \left[ p^3 f_{uuu} + 3p^2(1-p)f_{uud} + 3p(1-p)^2f_{udd} + (1-p)^3f_{ddd} \right] \]

for three-step, and for \( n \) steps, all of length \( \delta t \)

\[ f = e^{-nr\delta t} \sum_{k=0}^{n} \binom{n}{k} p^{n-k}(1-p)^k f_{u^{n-k}d^k}, \]

again with

\[ p = \frac{e^{r\delta t} - d}{u - d}, \]

for European options.

Here \( f_{u^{n-k}d^k} \) means \( f \) with as index \((n-k)\) \( u \)'s and \( k \) \( d \)'s.
Some remarks

- The binomial model with just one or two steps is unrealistically simple. However, if we take a larger number of steps, say more than 30 steps, we can actually get a reasonable model.
- Traders use software for the calculations (strong advantage in comparison to students in MST30030), see Hull, p. 255.
- The most delicate point in the procedure is the assumption that we know only certain values of the stocks at time $T$ will occur. The price of the option will depend on the choice.
- Later we shall see that the so-called volatility $\sigma$ of a stock price per year is a basic ingredient of the price of an option. Cox, Ross and Rubenstein proposed in 1979

$$u = e^{\sigma \sqrt{\delta t}}, \text{ and } d = e^{-\sigma \sqrt{\delta t}}.$$  

where $\delta t$ is the step length of the binomial tree.
Delta hedging

Recall the example we started our discussion of binomial trees with: a European call option, with strike price $K = € 21$.

We considered a portfolio of long $\Delta$ shares and short one option. We then found

$$\Delta = \frac{1 - 0}{22 - 18} = \frac{“\text{difference” in price of option}}{“\text{difference” in price of stock”}}.$$

$\Delta$ is the number of shares we should have with one shorted option in order to maintain a riskless portfolio.

The construction of such a riskless hedge is referred to as delta hedging.
Recall another example we considered: again a European call option, with strike price \( K = \€ 21 \).

In the first time step

\[
\Delta = \frac{2.0257 - 0}{22 - 18} = 0.5064.
\]
In the second timestep, we find

\[ \Delta_u = \frac{3.2 - 0}{24.2 - 19.8} = 0.7273 \]

\[ \Delta_d = \frac{0 - 0}{19.8 - 16.2} = 0. \]

We see that in order to maintain a riskless hedge using an option and the stock, the holdings in the stock need to be adjusted at each time step.