Lecture 16: Delta Hedging

We are now going to look at the construction of binomial trees as a first technique for pricing options in an approximative way.

These techniques were first proposed in:

Example

Suppose we want to value a European call option giving the right to buy a stock for the strike price $K = €21$ (the price in the contract) in 3 months (expiry date $T$). The current stock price $S_0$ is €20.
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in 3 months the stock price will be either €22 or €18. We do not know the probability for the occurrence of the stock prices €22 or €18.
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However, at expiry date, we know the value of the option: if the stock price goes up (to €22), the payoff $S_T - K$ is €1; if the stock price goes down (to €18), then the option is worthless, so the value is 0.
A portfolio with one stock and $\Delta$ shares

Look at a portfolio which is generated at time 0 by buying $\Delta$ many shares of a stock (long position) and selling one call option of the stock (short position).

If $f$ is the price of the option the portfolio at time 0 has value

$$20\Delta - f.$$ 

After three months the value of this portfolio can be computed: if the stock prices moves from €20 to €22 then the value of the shares is €22$\Delta$ and the value of the option is €1, so the total value of the portfolio is

$$22\Delta - 1$$

(in this case our contract partner has the right to buy from us one stock at the strike price €21, so we have to give him €1).

If the stock prices moves from €20 to €18 then the value of the shares is €18$\Delta$ and the value of the option is zero, so the total value of the portfolio 18$\Delta$. 

Strategy: delta hedging

The idea is now to choose \( \Delta \) such that the value of the portfolio in both cases (stock price up or down) is the same: so we require

\[
22\Delta - 1 = 18\Delta, \text{ so } \Delta = 0.25.
\]

Control: in 3 months
if \( S_T = \€ 22 \), then the portfolio is worth \( 22 \cdot 0.25 - 1 = \€ 4.50 \).
if \( S_T = \€ 18 \), then the portfolio is worth \( 18 \cdot 0.25 = \€ 4.50 \).
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Definition
The delta of an option is the number $\Delta$ of shares we should hold for one option (short position) in order to create a riskless hedge: so after maturity time $T$ the value of the portfolio containing $\Delta$ share and selling one call option is for both cases $S_T > K$ and $S_T \leq K$ the same.
The construction of a riskless hedge is called delta hedging.
So choosing $\Delta = 0.25$ leads to a portfolio where there is **no uncertainty** about the value of the portfolio in 3 months, namely €4.50. Since the portfolio has no risk we can compute its value at time 0 by discounting the price with the risk-free rate (no arbitrage opportunities). This means that we can find the value of the portfolio at time 0 by discounting the value in 3 months:

If we suppose that $r = 12\%$, then the portfolio at time 0 is worth $€4.367$, and so $f = 20 \times 0.25 - 4.367 = 0.633$. 

$\frac{5}{12}$
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If we suppose that \( r = 12\% \), then the portfolio at time 0 is worth

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4.50e^{-0.12 \cdot 0.25} = €4.367.
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If we suppose that $r = 12\%$, then the portfolio at time 0 is worth

$$4.50e^{-0.12\cdot0.25} = €4.367.$$

Let $f$ be the price of the call option today. Then the portfolio at time 0 has worth

$$20 \cdot 0.25 - f = 4.367,$$

and so

$$f = 20 \cdot 0.25 - 4.367 = 0.633.$$
The general case

We want to value the price \( f \) of an European option (put or call). The current stock price is \( S_0 \).
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We assume that after maturity time $T$ the stock price $S_T$ will be either

$$S_0 \cdot u \ (u > 1) \quad \text{or} \quad S_0 \cdot d \ (d < 1),$$

where $u$ stands for up, and $d$ for down.

After maturity time $T$ we compute the value (payoff) of the option depending on the stock price $S_T$ and the strike price $K$: there are only two possibilities and we denote the value of the option by $f_u$ if the stock has gone up and by $f_d$ if it has gone down.
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In our example:

if the stock price goes up (to €22), the value $f_u$ is €1;
if the stock price goes down (to €18), then the value $f_d = 0$. 
The delta of a stock

Look at a portfolio at time 0 generated by

buying $\Delta$ shares of a stock (long position) at stock price $S_0$
selling one option of the stock (short position).

If $f$ is the price of the option at time 0 then the portfolio at time 0
has value

$$\Delta S_0 - f.$$

After expiry date $T$ we can compute the value of the portfolio:

if stock goes up the portfolio is worth $S_0 \cdot u \cdot \Delta - f_u$,
if stock goes down the portfolio is worth $S_0 \cdot d \cdot \Delta - f_d$,

We choose $\Delta$ such that the value of the portfolio at time $T$ in
both cases (stock price up or down) is the same:

$$S_0 \cdot u \cdot \Delta - f_u = S_0 \cdot d \cdot \Delta - f_d,$$
The delta of a stock

From \( S_0 \cdot u \cdot \Delta - f_u = S_0 \cdot d \cdot \Delta - f_d \), we find

\[
\Delta = \frac{f_u - f_d}{S_0(u - d)}.
\]

**Theorem**

Assume that the stock price \( S_0 \) after time \( T \) goes either up to \( S_0 \cdot u \) with \( u > 1 \) or down to \( S_0 \cdot d \) with \( d < 1 \). Let \( f_u \) and \( f_d \) be the payoffs at maturity time \( T \) in the case of up or down movement. Then the delta of one option is

\[
\Delta = \frac{f_u - f_d}{S_0(u - d)} = \frac{\text{difference of payoffs at time } T}{\text{difference of prices of stock at time } T}.
\]

The delta of a call option is positive, whereas the delta of a put option is negative.
Using the choice

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

the value of portfolio at time $T$ is in both cases (stock up or down) the same, namely:

$$S_0 \cdot u \cdot \Delta - f_u = S_0 \cdot u \cdot \frac{f_u - f_d}{S_0(u - d)} - f_u$$

which is equal to

$$u \cdot \frac{f_u - f_d}{u - d} - f_u = \frac{u \cdot (f_u - f_d) - (u - d) \cdot f_u}{u - d} = \frac{d \cdot f_u - u \cdot f_d}{u - d}.$$

Thus the value of the portfolio at time $T$ is

$$P_T := \frac{d \cdot f_u - u \cdot f_d}{u - d}.$$
The portfolio is riskless. Since no arbitrage opportunities exist the portfolio at time 0 is worth

\[ P_0 = e^{-rT} P_T = e^{-rT} \frac{df_u - uf_d}{u - d}. \]

The portfolio at time 0 is also worth

\[ P_0 = S_0 \Delta - f = \frac{f_u - f_d}{u - d} - f. \]
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Comparing these two values, we find

\[ f = \frac{f_u - f_d}{u - d} - e^{-rT} \frac{df_u - uf_d}{u - d} \]

\[ = \frac{e^{-rT}}{u - d} \left( e^{rT} (f_u - f_d) - (df_u - uf_d) \right) \]

\[ = \frac{e^{-rT}}{u - d} \left( f_u (e^{rT} - d) + f_d (u - e^{rT}) \right). \]
Put now

\[ p = \frac{e^{rT} - d}{u - d}. \]

Then

\[ f = e^{-rT} \left( pf_u + \frac{u - e^{rT}}{u - d} f_d \right). \]

Since

\[ 1 - p = \frac{u - d}{u - d} - \frac{e^{rT} - d}{u - d} = \frac{u - e^{rT}}{u - d} \]

we obtain the final formula for the price \( f \) of an option:

\[ f = e^{-rT} \left[ pf_u + (1 - p) f_d \right]. \]
Summary

Theorem
Assume that the stock price $S_0$ after time $T$ goes either up to $S_0 \cdot u$ with $u > 1$ or down to $S_0 \cdot d$ with $d < 1$. Let $f$ be the price of an option (either call or put) at time 0 with payoff $f_u$ and $f_d$ respectively at time $T$. If $r$ is the riskless interest rate then

$$f = e^{-rT} \left[ pf_u + (1 - p)f_d \right] \quad \text{where } p = \frac{e^{rT} - d}{u - d}.$$
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Assume that the stock price $S_0$ after time $T$ goes either up to $S_0 \cdot u$ with $u > 1$ or down to $S_0 \cdot d$ with $d < 1$. Let $f$ be the price of an option (either call or put) at time 0 with payoff $f_u$ and $f_d$ respectively at time $T$. If $r$ is the riskless interest rate then

$$f = e^{-rT} \left[ pf_u + (1 - p)f_d \right] \quad \text{where} \quad p = \frac{e^{rT} - d}{u - d}.$$ 

In our previous example we had $u = 1.1$, $d = 0.9$, $f_u = 1$, $f_d = 0$, $r = 0.12$ and $T = 0.25$, so

$$p = \frac{e^{0.12 \cdot 0.25} - 0.9}{1.1 - 0.9} = 0.6523,$$

$$f = e^{-0.12 \cdot 0.25} \left( 0.6523 \cdot 1 + 0.3477 \cdot 0 \right) = 0.633.$$