Positive cones and gauges on algebras with involution

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(joint work with Vincent Astier)

The connections between quadratic forms, orderings and valuations on fields are well-known [8]. Building on our work on signatures of hermitian forms over algebras with an involution [1, 2], positivity, and our answer to a question of Procesi and Schacher analogous to Hilbert's 17th problem [3], we developed a theory of positive cones on algebras with involution [4].

The canonical "valuations" associated to positive cones turn out to be Tignol-Wadsworth gauges [9, 10, 11]. There is a natural notion of compatibility between positive cones and gauges, that can be described in several equivalent ways, reminiscent of the field case, and which also gives rise to a theorem in the style of Baer-Krull about lifting positive cones from the residue algebra [5].

We present some of our main results on these topics in this note. We refer to [4, 5] for the details. Let F be a field of characteristic not 2 and let A be an F-algebra, equipped with an F-linear involution σ.

1. POSITIVE CONES

Definition 1. A set $\mathscr{P} \subseteq \mathrm{Sym}(A, \sigma)$ is a prepositive cone on (A, σ) if

(P1) $\mathscr{P} \neq \emptyset$;
(P2) $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$;
(P3) $\sigma(a)\mathscr{P}a \subseteq \mathscr{P}$, for all $a \in A$;
(P4) $\mathscr{P}_F := \{u \in F \mid u\mathscr{P} \subseteq \mathscr{P}\}$ belongs to X_F, the space of orderings of F;
(P5) $\mathscr{P} \cap -\mathscr{P} = \{0\}$.

A maximal prepositive cone is called a positive cone. We denote the set of positive cones on (A, σ) by $X_{(A,\sigma)}$. For $\mathscr{P} \in X_{(A,\sigma)}$, $\leq_{\mathscr{P}}$ denotes the partial ordering on A induced by $\mathscr{P}$.

From now on we assume that (A, σ) is a central simple F-algebra with involution in the sense of the Book of Involutions [7].

Examples 2. (1) The only two positive cones on $(M_n(\mathbb{R}), t)$ are the set of positive semidefinite matrices and the set of negative semidefinite matrices.
(2) For $P \in X_F$, let $\mathscr{M}_P := \{a \in \mathrm{Sym}(A, \sigma) \cap A^\times$ of maximal signature at $P\} \cup \{0\}$. If A is a division algebra, then $\mathscr{M}_P \in X_{(A,\sigma)}$ if and only if $\mathscr{M}_P \neq \mathrm{Sym}(A, \sigma)$.

Theorem 3. *If A is division, then $X_{(A,\sigma)} = \{\mathscr{M}_P, -\mathscr{M}_P \mid \mathscr{M}_P \neq \mathrm{Sym}(A, \sigma),\ P \in X_F\}$. In general, $X_{(A,\sigma)} = \{\mathscr{C}_P(\mathscr{M}_P), -\mathscr{C}_P(\mathscr{M}_P) \mid \mathscr{M}_P \neq \mathrm{Sym}(A, \sigma),\ P \in X_F\}$, where $\mathscr{C}_P$ denotes the closure under* (P2)*,* (P3) *and* (P4) *with $\mathscr{P}_F = P$.*

Theorem 4 ("Artin-Schreier")**.** *The following are equivalent:*
(1) *(A, σ) is formally real, i.e., $X_{(A,\sigma)} \neq \emptyset$;*
(2) *There exists $a \in \mathrm{Sym}(A, \sigma) \cap A^\times$ such that $\langle a \rangle_\sigma$ is strongly anisotropic;*
(3) *The Witt group $W(A, \sigma)$ is not torsion.*

Theorem 5 ("Artin", simplified version)**.** *Assume that for every $\mathscr{P} \in X_{(A,\sigma)}$ we have $1 \in \mathscr{P} \cup -\mathscr{P}$. Then*

$$\bigcap\{\mathscr{P} \in X_{(A,\sigma)} \mid 1 \in \mathscr{P}\} = \{\sum_{i=1}^{s} \sigma(x_i)x_i \mid s \in \mathbb{N},\ x_i \in A\}.$$

We also use the techniques developed for the proofs of the above theorems to give a Sylvester decomposition of hermitian forms over (A, σ) with respect to a positive cone and obtain in this way another description of signatures of hermitian forms.

Theorem 6. *$X_{(A,\sigma)}$ is a spectral space with respect to the "Harrison" topology with basis $H_\sigma(a_1, \dots, a_\ell) := \{\mathscr{P} \in X_{(A,\sigma)} \mid a_1, \dots, a_\ell \in \mathscr{P}\}$.*

2. Gauges from positive cones

Gauges were defined by Tignol and Wadsworth, cf. [9, 10, 11]:

Definition 7. Let $v : F \to \Gamma_v \cup \{\infty\}$ be a valuation of F and let Γ be a totally ordered abelian group. A map $w : A \to \Gamma \cup \{\infty\}$ is a v-gauge if

(1) w is a v-value function on A, i.e. for all $x, y \in A$ and $\lambda \in F$, we have

$$w(x) = \infty \Leftrightarrow x = 0;\ w(x+y) \geq \min\{w(x), w(y)\};\ w(\lambda x) = v(\lambda) + w(x);$$

(2) w is surmultiplicative, i.e., $w(1) = 0$ and $w(xy) \geq w(x)+w(y)$, for all $x, y \in A$.

(3) w is a v-norm, i.e., A has a "splitting basis" $\{e_1, \dots, e_m\}$ such that

$$w(\sum_{i=1}^{m} \lambda_i e_i) = \min_{1\leq i\leq m} (v(\lambda_i) + w(e_i)), \quad \forall \lambda_1, \dots, \lambda_m \in F.$$

(4) the graded algebra $\mathrm{gr}_w(A)$ (with grading determined by w) is a graded semisimple $\mathrm{gr}_v(F)$-algebra.

A gauge w is σ-special if $w(\sigma(x)x) = 2w(x)$ for all $x \in A$. If w is a gauge on A, we define $R_w := \{a \in A \mid w(a) \geq 0\}$ and $I_w := \{a \in A \mid w(a) > 0\}$.

Let $\mathscr{P} \in X_{(A,\sigma)}$ such that $1 \in \mathscr{P}$ (this is always possible after scaling), and let $P = \mathscr{P}_F$. Following the standard definition in the field case, and inspired by Holland [6], we define for a subfield k of F,

$$R_{k,\mathscr{P}} := \{x \in A \mid \exists m \in k \quad \sigma(x)x \leq_{\mathscr{P}} m\},$$
$$I_{k,\mathscr{P}} := \{x \in A \mid \forall \varepsilon \in k^\times \cap P \quad \sigma(x)x \leq_{\mathscr{P}} \varepsilon\}.$$

It is not difficult to see that $R_{k,\mathscr{P}}$ is a subring of A and that $I_{k,\mathscr{P}}$ is a two-sided ideal of $R_{k,\mathscr{P}}$. Both are stable under σ. Note that $R_{k,\mathscr{P}}$ is in general not a total valuation ring, nor a Dubrovin valuation ring.

Theorem 8. *Let $v_{k,P}$ be the valuation on F whose valuation ring is $\{x \in F \mid \exists m \in k \quad -m \leq_P x \leq_P m\}$. There exists a $v_{k,P}$-gauge $w_{k,\mathscr{P}}$ on A such that $R_{k,\mathscr{P}} = R_{w_{k,\mathscr{P}}}$ and $I_{k,\mathscr{P}} = I_{w_{k,\mathscr{P}}}$. Moreover, $w_{k,\mathscr{P}}$ is the unique σ-special $v_{k,P}$-gauge on A.*

3. Compatibility between gauges and positive cones

Let w be a σ-special v-gauge on A, let σ_0 be the induced involution on the residue algebra $A_0 := R_w/I_w$, and let $\pi_w : R_w \to A_0$ be the canonical projection.

Theorem 9. *Let $\mathscr{P} \in X_{(A,\sigma)}$ such that $1 \in \mathscr{P}$. The following are equivalent:*
(1) $0 \leq_{\mathscr{P}} a \leq_{\mathscr{P}} b \Rightarrow w(b) \leq w(a)$, *for all* $a, b \in A$;
(2) R_w *is* $\mathscr{P}$*-convex;*
(3) $1 + \mathrm{Sym}(I_w, \sigma) \subseteq \mathscr{P}$.
The above statements imply that $\pi_w(\mathscr{P} \cap R_w)$ is a positive cone on (A_0, σ_0).

Definition 10. We say that w and $\mathscr{P}$ (with $1 \in \mathscr{P}$) are compatible if one of the above equivalent statements holds.

Theorem 11 ("Baer-Krull")**.** *If $\mathscr{Q} \in X_{(A_0,\sigma_0)}$, then there exists $\mathscr{P} \in X_{(A,\sigma)}$ such that $\mathscr{P}$ is compatible with w, $\pi_w(\mathscr{P} \cap R_w) = \mathscr{Q}$ and $w = w_{\mathscr{P}}$. If $r := \dim \Gamma_v/2\Gamma_v$ is finite, then there are 2^r such liftings of $\mathscr{Q}$.*

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