First Arts Lecture Notes 2004-2005

Chapter 1. Mathematical induction, counting techniques and binomial theorem

We begin these lectures by describing the technique of mathematical induction. This is a powerful method of proving certain types of mathematical statement. We will not start by giving a rigorous definition of all that is involved in mathematical induction but will illustrate how it is used by examining some examples. It has to be admitted that students are often unconvinced about the validity of mathematical induction, although it is perfectly rigorous, and their use of the technique is correspondingly weak and unsure.

Let's begin by examining the sum of a certain arithmetic progression. We look at the sums of the odd whole numbers, as follows.

$$1 = 1 = 1^2$$

$$1 + 3 = 4 = 2^2$$

$$1 + 3 + 5 = 9 = 3^2$$

$$1 + 3 + 5 + 7 = 16 = 4^2$$

Notice that the sum appears to be the square of the total number of terms involved in the sum. So, suppose we have r terms in our sum, these terms being the first r positive odd numbers. The rth term is in fact $2r - 1$ and the sum is

$$1 + 3 + 5 + \cdots + 2r - 1$$

According to the pattern we have just seen, we suspect that this sum should equal r^2, so that

$$1 + 3 + 5 + \cdots + 2r - 1 = r^2$$

should hold. However, we have not *proved* this yet. We know that it is true for $r = 1$, 2, 3 and 4. In the method of mathematical induction, in order to *prove* a statement like that above, we assume that it is true for some *particular* value of r. This is called our *induction hypothesis.* We then try to prove the statement for the next value of r, namely $r + 1$, using the induction hypothesis. This process amounts to completing what is called the *induction step.* In our example, what we are required to prove is that

$$1 + 3 + 5 + \cdots + 2r - 1 + 2r + 1 = (r + 1)^2.$$

What we have done is to add the $(r+1)$th odd number, which is $2r+1$, to the previous sum to form the sum to $r+1$ terms. But now, by the induction hypothesis,

$$1 + 3 + 5 + \cdots + 2r - 1 = r^2$$

and so

$$1 + 3 + 5 + \cdots + 2r - 1 + 2r + 1 = r^2 + 2r + 1.$$

But $r^2 + 2r + 1 = (r+1)^2$, by a well known formula of algebra, and so we have completed the induction step. This establishes the formula for all values of r.

Now we will try to give a formal description of the method of mathematical induction. Suppose we have a statement $P(n)$ involving the positive integer n. For example, this might be the statement

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2.$$

Suppose that $P(1)$ is true-this will be easy to check. Suppose also that for any particular number k, the truth of $P(k)$ automatically implies the truth of $P(k+1)$. Then $P(n)$ is true for all positive integers n.

In practice, people experience problems in various ways. Firstly, they cannot formulate mathematically what the statement $P(k+1)$ is-this is very common. Secondly, and this is more likely, they cannot do the necessary mathematical manipulation that deduces the truth of $P(k+1)$ from the truth of $P(k)$, and quite frequently they make the calculations far too complicated. Finally, their proofs often have inconsistencies or logical flaws.

The checking of the truth of $P(1)$ is easy. However, you can give false proofs by induction if you forget to check it. Here is one such. Let $P(n)$ be the statement:

$$2 + 2^2 + \cdots + 2^n = 2^{n+1}.$$

We will show that the truth of $P(k)$ implies the truth of $P(k+1)$. Now $P(k)$ being true means

$$2 + 2^2 + \cdots + 2^k = 2^{k+1}.$$

What is the statement $P(k+1)$? It is what you get when you replace the n in statement $P(n)$ by $k+1$. So, it must be

$$2 + 2^2 + \cdots + 2^{k+1} = 2^{k+1+1} = 2^{k+2}.$$

Can we prove that this is true? Starting from

$$2 + 2^2 + \cdots + 2^k = 2^{k+1},$$

a result we assume to be true, we get on adding 2^{k+1} to each side

$$2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1}.$$

But $2^{k+1} + 2^{k+1} = 2 \times 2^{k+1} = 2^{k+2}$, by the law of indices. This show that $P(k+1)$ is true if $P(k)$ is true. Is then $P(n)$ true for all n. The answer is no, it is never true. For in fact, $P(1)$ is the statement $2 = 2^{1+1} = 4$, which is not true and it is actually easy to prove the correct statement

$$2 + 2^2 + \cdots 2^n = 2^{n+1} - 2.$$

Induction is not applied just to find the sum of certain series but is used for a variety of purposes. We give another example here.

<u>Example</u>

If $a_1, a_2, \ldots, a_n, \ldots,$ is a sequence of numbers, with $a_1 = 1$, and if the relation $a_{n+1} = 2a_n + 1$ holds for all positive whole numbers n, prove that

$$a_n = 2^n - 1.$$

Notice that

$$a_2 = 2a_1 + 1 = 3 = 2^2 - 1$$

$$a_3 = 2a_2 + 1 = 7 = 2^3 - 1$$

$$a_4 = 2a_3 + 1 = 15 = 2^4 - 1$$

and so on. Thus the result is certainly true when $n = 1$. Assume it is true when n has the particular value r. Thus

$$a_r = 2^r - 1.$$

Now try to prove that

$$a_{r+1} = 2^{r+1} - 1.$$

But $a_{r+1} = 2a_r + 1 = 2 \times (2^r - 1) + 1 = 2^{r+1} - 1$ by the law of indices. This proves what we want.

The main problem in trying to evaluate the sum of a series by induction is that of spotting a pattern. For instance, consider the sum

$$1^2 + 2^2 + 3^2 \cdots + n^2.$$

It is not at all obvious from experimentation what is a closed form for this sum. In fact, the result is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can prove this by induction. It is clear that the result is true when $n = 1$. We assume the result is true for $n = r$. So, we assume that

$$1^2 + 2^2 + \cdots + r^2 = \frac{r(r+1)(2r+1)}{6}$$

is true and try to prove the corresponding statement when r becomes $r + 1$. This is

$$\begin{aligned} 1^2 + 2^2 + \cdots + r^2 + (r+1)^2 &= \frac{(r+1)(r+1+1)(2(r+1)+1)}{6} \\ &= \frac{(r+1)(r+2)(2r+3)}{6} \end{aligned}$$

But, by the induction hypothesis, the sum to $r + 1$ terms above is

$$\begin{aligned} 1^2 + 2^2 + \cdots + r^2 + (r+1)^2 &= \frac{r(r+1)(2r+1)}{6} + (r+1)^2 \\ &= (r+1)(\frac{r(2r+1)}{6} + r + 1) \\ &= (r+1)\frac{(2r^2 + 7r + 6)}{6} \end{aligned}$$

But by easy factorization, $2r^2 + 7r + 6 = (2r + 3)(r + 2)$ and so the expression above is

$$\frac{(r+1)(r+2)(2r+3)}{6},$$

as required. Finding patterns for more complicated sums is not so easy and indeed there may not always be any obvious pattern to be found.

<u>Subsets of a set</u>

Suppose we are given a set S. A *subset* of S is a set T whose elements are contained in S. We allow the possibility that a subset actually contains no elements. We call such a subset the *empty subset* of S and denote it by $\emptyset$. We also consider S to be a subset of itself.

Now suppose that S contains exactly n elements. We raise the question: what is the total number of subsets of S, including S and $\emptyset$? The answer turns out to be quite straightforward. We will take S to consist of the numbers 1, 2, $\ldots$, n and write $S = \{1, 2 \ldots, n\}$. (Of course, it really does not matter precisely what the elements contained in S are.) Subsets will be written as $\{1, 2\}$ or $\{1, 2, 3\}$, and so on.

We begin by considering some small values of n.

n	Subsets	number
1	S, $\emptyset$	2
2	S, $\emptyset$, $\{1\}$, $\{2\}$	4
3	S, $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$	8

The number of subsets appears to double at each stage and to have the form 2^n, where n is the number of elements in S. We shall now prove that this is true by induction on n

Theorem. *Let S be a set containing exactly n elements. Then S contains exactly 2^n subsets.*

Proof. We use induction on n and know the result is true for $n = 1$. Assume the result is true when n has the particular value r and then try to prove the corresponding result when n becomes $r + 1$. So, suppose that

$$S = \{1, 2, \ldots, r + 1\}.$$

We want to show that S contains 2^{r+1} subsets. Let T be the subset $\{1, 2, \ldots, r\}$ of S. By induction, T contains 2^r subsets.

Subsets of S are of two types:

(a) those that do not contain $r + 1$

(b) those that do contain $r + 1$

Subsets of type (a) are simply subsets of T and there are 2^r of these by induction. Subsets of type (b) are obtained by adjoining $r + 1$ to any of the 2^r subsets of T, and so there are also 2^r of these. The total number of subsets is therefore

$$2^r + 2^r = 2 \times 2^r = 2^{r+1},$$

by the law of indices, and this is precisely what we want.

Section 2. Counting techniques, permutations and subsets

In this section, we want to consider ways of selecting and arranging objects taken from a set. Most of our arguments are based on the following simple idea. Suppose we have two sets of objects, S and T, S consisting of i objects and T of j objects. Suppose now we have to choose one object from S and one from T. In how many ways can we do this? Let the objects in S be enumerated as $\{a_1, \ldots, a_i\}$ and those in T as $\{b_1, \ldots, b_j\}$. Then writing the object from S first, we can have any of

$$\begin{array}{ccc} (a_1, b_1), & \ldots, & (a_1, b_j) \\ (a_2, b_1), & \ldots, & (a_2, b_j) \\ \vdots & \vdots & \vdots \\ (a_i, b_1), & \ldots, & (a_i, b_j) \end{array}$$

We clearly have ij choices. If now we choose objects from r different sets, $S_1, \ldots, S_r$, where S_1 contains n_1 objects,..., S_r contains n_r, then we can argue that the number of ways we can make a sequence of r choices, consisting first of an object from S_1, followed by an object from S_2, and so on, is

$$n_1 \times n_2 \times \cdots \times n_r.$$

Technically, this can be proved by induction on r, but we will not go through this. The whole procedure can be reduced to the case when $r = 2$, treated above. An alternative enunciation of this counting principle is as follows.

Counting principle

If one event can occur in n_1 different ways, and following this, another event can occur in n_2 different ways, and then a third in n_3 ways, and so on, and if these events are unrelated to each other, the total number of ways that the sequence of events can occur is

$$n_1 \times n_2 \times n_3 \times \cdots \times n_r,$$

where r is the total number of events in the sequence.

Let us now recall the definition of a factorial. Let n be a positive whole number. We define $n!$ to be the number given by

$$n! = n \times n - 1 \times n - 2 \times \cdots \times 2 \times 1$$

and call this number n *factorial* or *factorial* n. We also define, for convenience, $0!$ to equal 1. We have the following table

$$n = 1\ 2\ 3\ 4\quad 5\quad 6\ \ldots$$

$$n! = 1\ 2\ 6\ 24\ 120\ 720\ \ldots$$

Notice that

$$n! = n \times (n-1)! = n(n-1) \times (n-2)!$$

and so on.

The number $n!$ factorial occurs in connection with arrangements of objects. Suppose we have n different objects which we wish to arrange in order. Suppose that the objects are numbered from 1 to n. Then we can arrange the objects in the order

$$1\ 2\ 3\ \ldots\ n,$$

which we call the standard arrangement or order. But we could also arrange as

$$2\ 1\ 3\ \ldots\ n$$

and so on. These various arrangements or orderings of the objects are called *permutations*.

Example For $n = 3$, we have exactly 6 possible arrangements, as follows:

$$1\ 2\ 3, \quad 2\ 1\ 3, \quad 3\ 1\ 2$$

$$1\ 3\ 2, \quad 2\ 3\ 1, \quad 3\ 2\ 1$$

Now $6 = 3!$ and this example is just a special case of a more general result.

Theorem. *The total number of permutations of a set of n different objects is $n!$.*

Proof We use induction. Let $P(n)$ be the statement that the number of permutations of a set of n different objects is $n!$. $P(1)$ is true. Assume that $P(r)$ is true. We must show that $P(r+1)$ is true. Let S be a set of $r+1$ objects, which we wish to arrange into order. We have to select an object to go in the first place. We can choose any object and so we get $r+1$ choices. Having chosen, we have r objects left. By induction, these can be permuted among themselves in in $r!$ ways, and then placed after our first object to get a permutation of $r+1$ objects. By the counting principle, the number of ways we can perform the two choices (choosing a first object and then a permutation of the remaining r objects) is

$$r + 1 \times r!$$

and this number is $(r+1)!$. Hence $P(r+1)$ is true and the theorem true in general.

Example Given the digits 2, 4, 6 and 9, how many 4 figure numbers can be formed

(a) if repeated digits are allowed;

(b) if no repeated digits are allowed;

(a) In this case, we have a choice of 4 digits at each position of the number, and so there are

$$4 \times 4 \times 4 \times 4 = 256$$

possible numbers.

(b) In this case, each number formed is a permutation of the 4 digits and thus there are $4! = 24$ possible numbers.

Example Find the number of ways of arranging 8 different books on a shelf if

(a) they can be arranged in any order

(b) two particular books must be next to each other

(c) two particular books must not be next to each other.

(a) There are clearly $8! = 40320$ possible arrangements.

(b) Suppose that the two books that must be next to each other are A and B. If they must be next to each other, treat these two as one. Then we are essentially arranging 7 books and this can be done in 7! ways. However, to each of these arrangements, there are two associated rearrangements, depending on the order of A and B, as follows

$$\ldots AB \ldots$$

$$\ldots BA \ldots$$

This gives $2 \times 7! = 2 \times 5040 = 10080$ arrangements.

(c) In this case, we just subtract the number of arrangements with the two books together from the total number of rearrangements to get those in which the two given books are not together. Thus, there are $40320 - 10080 = 30240$ arrangements of this type.

Consider now a set containing n different objects and suppose we have m slots in a line to fill, where $1 \leq m \leq n$. We may ask: in how many different ways can we arrange the objects into the slots? This is a slightly more general question than the previous one of finding the number of permutations or orderings of the set of n objects. We can choose the object for the first slot in n ways, that for the second in $n-1$ ways, and so on, finally, that for the rth and last slot in $n-m+1$ ways. The total number of choices is then

$$n \times n-1 \times \cdots \times n-m+1.$$

Note that when $m = n$, we are just permuting the objects and we get our earlier figure of $n!$ arrangements.

Definition An ordered arrangement of m objects taken from a set of size n is called an *ordered m-subset* of the set.

When we arrange different objects into m slots in a line, we are forming an ordered r-subset and thus we see that we have proved the following.

Theorem. *The number of ordered m-subsets of a set of size n is*

$$n \times n-1 \times \cdots \times n-m+1.$$

Now, suppose that we just select m objects from our set of size n, but do not arrange them into order. We call such a subset an *unordered m-subset.* A basic question arises: how many unordered m-subsets are there in a set of size n?

This can be answered as follows. Take any unordered m-subset of the master set. It contains m objects, which we can arrange in $m!$ ways. Each arrangement of the objects creates a different *ordered* m-subset. Thus, to any unordered m-subset, there correspond $m!$ ordered m-subsets containing the same obects. A different choice of unordered m-subset will lead to different ordered subsets. Therefore, if there are exactly N unordered m-subsets that can be formed from the set of n objects, there must be exactly

$$N \times m!$$

ordered m-subsets that can be formed from this set. However, we already know that there are

$$n \times n-1 \times \cdots \times n-m+1$$

ordered m-subsets and thus we obtain

$$N = \frac{n(n-1)\dots(n-m+1)}{m!}$$

unordered m-subsets.

If we take the number N above and multiply top and bottom by $(n-m)!$, we get

$$N = \frac{n(n-1)\dots(n-m+1)\times(n-m)!}{m!(n-m)!}.$$

But the top line is just $n!$ and thus we obtain

$$N = \frac{n!}{m!(n-m)!}.$$

This number is usually referred to as 'n choose m', as it tells us in how many ways we may choose m objects from a set of n different objects. We have thus proved:

Theorem. *The total number of unordered m-subsets in a set of n different objects is $\binom{n}{m}$.*

While the form of $\binom{n}{m}$ in terms of factorials is quite compact, it is often easier to use the equivalent form

$$\binom{n}{m} = \frac{n(n-1)\dots(n-m+1)}{m!}$$

when m is no bigger than $n/2$. Thus for example, we have the following simple formulae

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = \frac{n(n-1)}{2}$$

and so on. Another useful formula is:

$$\binom{n}{m} = \binom{n}{n-m}.$$

This can be proved directly from the formula for $\binom{n}{m}$ in terms of factorials. However, a counting argument is also available, as follows. Every time we choose an unordered m-subset from a set of size n, we choose at the same time an unordered subset of size $n-m$. This set of size $n-m$ is just what is left in the original set when m objects are removed. Each set completely determines the other. So, there are equal numbers of each type.

<u>Example</u> Given a number m between $n/2$ and n, it is easier to work out $\binom{n}{m}$ using

$$\binom{n}{m} = \binom{n}{n-m}.$$

Thus,

$$\binom{90}{87} = \binom{90}{3} = \frac{90 \times 89 \times 88}{3 \times 2 \times 1} = 117480.$$

The numbers $\binom{n}{m}$ have a basic property that we can prove quite easily by a counting argument.

Theorem.

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$$

Proof Consider the set S consisting of the numbers 1, 2, $\ldots$, $n+1$. The number of m-subsets in S is $\binom{n+1}{m}$. We can choose an m-subset in two ways. First of all, we choose it not to contain $n+1$. Then the m-subset is an m-subset of the numbers 1, 2, $\ldots$, n and there are $\binom{n}{m}$ of these. Alternatively, we choose it to contain $n+1$. Then the m-subset is made up of $n+1$ and an $m-1$-subset of the numbers 1, 2, $\ldots$, n. But this $m-1$-subset can be formed in $\binom{n}{m-1}$ ways. This gives $\binom{n}{m-1}$ m-subsets of the second kind. The total number of m-subsets is then

$$\binom{n}{m} + \binom{n}{m-1}$$

and this expression must equal $\binom{n+1}{m}$.

Note that this argument is rather similar to the one that we used to find the total number of subsets in a set of size n. In fact, we can prove an interesting fact concerning the numbers $\binom{n}{m}$ using our earlier result.

Theorem.

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Proof Let S be a set of size n. We proved in the induction section that the total number of subsets of S is 2^n. However, each subset is an m-subset for some integer m lying between 0 and n, and the total number of m-subsets is $\binom{n}{m}$. Summing from 0 to n, we get the required result.

<u>Example</u> Out of the 21 consonants and 5 vowels of the alphabet, how many arrangements of 5 different letters can be made

(a) if 4 consonants and 1 vowel must be used;

(b) if the letters used must include A and Z.

(a) There are are $\binom{21}{4} = 5985$ ways of choosing 4 consonants from the 21 available and 5 ways of choosing 1 vowel. This gives us $5985 \times 5 = 29925$ ways of selecting 5 letters of the correct type. For each choice of 5 letters, there are 5! ways of arranging them, as they are all different. Thus we get

$$29925 \times 120 = 3591000$$

possible arrangements.

(b) In this case, we are looking at arrangements such as

AZBCP or ZYPAQ.

We must use A and Z and so want 3 more letters drawn from the 24 in B-Y. We can choose these 3 letters in $\binom{24}{3}$ ways and with A and Z, we have this same number of ways of choosing the 5 different letters. As above, we can arrange the 5 letters in 5! ways, giving an answer of

$$\binom{24}{3} \times 120 = 242880.$$

<u>Example</u> In poker, 5 cards are dealt from the standard pack of 52 cards. Thus the total number of hands of 5 cards that you can receive (neglecting order of the cards) is

$$\binom{52}{5} = 2598960.$$

In bridge, 13 cards are dealt. So the number of possible bridge hands a person can receive is

$$\binom{52}{13} = 635,013,559,600.$$

How many of these hands contain exactly one card from each of the 13 denominations? Each denomination has 4 cards, so we can choose a card of a given denomination in 4 ways. This is done for each of the 13 denominations, leading to

$$4 \times 4 \times \cdots \times 4,$$

there being 13 4's above. We get

$$4^{13} = 67,108,864$$

possible hands of this kind.

<u>Example</u> Out of 5 men and 7 women, a committee of 2 men and 3 women is to be formed. In how many ways can this be done if

(a) there is no restriction;

(b) one particular man must be on the committee

(c) two particular men must not be on the committee and one particular woman must be on the committee.

(d) one man and one woman are married and cannot both be on the committeee.

(a) We have $\binom{5}{2}$ ways of choosing 2 men from 5 and $\binom{7}{3}$ ways of choosing 3 women from 7. As these choices are made independently, the total number of choices is

$$\binom{5}{2} \times \binom{7}{3} = 10 \times 35 = 350.$$

(b) If one particular man must be on the committee, we have to choose 1 more from the 4 remaining. This can be done in $\binom{4}{1} = 4$ ways and so this time we get

$$4 \times \binom{7}{3} = 4 \times 35 = 140.$$

(c) In this case, we have only 3 men to choose from and we can choose 2 of these in $\binom{3}{2} = 3$ ways. For the women, we need 2 more chosen from 6, and this can be done in $\binom{6}{2}$ ways. The total number of ways we can make the choices is

$$3 \times \binom{6}{2} = 3 \times 15 = 45.$$

(d) If the given man and woman are both on the committee, we can choose the remaining people in

$$\binom{4}{1} \times \binom{6}{2} = 60$$

ways. If we subtract this number from the total number of 350 possibilities, we get the number of possibilities with at most one of the pair a member. Answer is $350 - 60 = 290$.

We have considered ways of arranging different objects in our work so far. However, in some questions involving rearrangements, it has to be assumed that certain of the objects cannot be distinguished. For example, consider the word

SUCCESSFULLY.

This has 12 letters. If the letters were all different, we could rearrange them in 12! ways. However, we assume that the 3 S's are indistinguishable, as are the 2 U's, 2 L's and 2 C's. We first choose 3 numbers from 1 to 12 and place the 3 S's in positions corresponding to these 3 numbers. We can choose the 3 numbers in $\binom{12}{3}$ ways. There remain 9 letters to assign to 9 places between 1 and 12. Choose 2 numbers from these and put the 2 C's in these positions. Then choose 2 more numbers from the 7 remaining and put the 2 L's there and then 2 more from the 5 remaining and put the 2 U's there. Finally, we can assign the 3 different letters to 3 remaining positions in 3! ways. Total number of ways of making the choices is

$$\binom{12}{3} \times \binom{9}{2} \times \binom{7}{2} \times \binom{5}{2} \times 3! = \frac{12!}{3! \times 2! \times 2! \times 2!}$$

which works out to be 9979200.

We can generalize as follows. Suppose we have a set of n objects, made up of r subsets of size n_1, n_2, $\ldots$, n_r where

$$n = n_1 + n_2 + \cdots + n_r.$$

Suppose that the objects in each of the described subsets are alike and indistinguishable. We want to find the total number of rearrangements of these objects (which might be the letters of a word, as we observed above). First choose any n_1 numbers from 1 to n. We will place the n_1 identical objects of the first type in positions corresponding to these numbers. We can choose the n_1 numbers in $\binom{n}{n_1}$ ways. There remain $n - n_1$ numbers and we choose n_2 of these and place the n_2 identical objects of the second type in positions corresponding to these numbers. We can choose these n_2 numbers in $\binom{n-n_1}{n_2}$ ways. Proceeding in this way we get a total number of arrangements equal to

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1-n_2}{n_3} \times \ldots$$

and on cancelling terms we find that this number equals

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_r!}$$

Thus we have proved:

Theorem. *Suppose we have a set of n objects, made up of r subsets of size n_1, n_2, $\ldots$, n_r where*

$$n = n_1 + n_2 + \cdots + n_r.$$

Suppose that the objects in each of the described subsets are alike and indistinguishable. Then the total number of different arrangements of the objects is

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_r!}$$

Example Consider the number

$$127123296771$$

In how many ways can we rearrange the digits to form new 12 digit numbers. We have 3 1's, 3 2's, 1 3, 1 6, 3 7's and 1 9. The duplicated numbers are taken to be indistinguishable. According to the theorem, we get

$$\frac{12!}{3! \times 3! \times 1! \times 1! \times 3! \times 1!} = 2217600$$

different rearrangements.

Example In how many ways can the letters of the word SUCCESSFULLY be rearranged if the 2 C's must be next to each other.

In this case, we treat the 2 C's as if they were one letter. Then we effectively have only 11 letters, 3 S's, 2 U's, 2 L's, 1 E, 1 Y, 1 F and 1 C. If we apply the theorem we get

$$\frac{11!}{3! \times 2! \times 2!} = 1663200$$

different rearrangements.

The binomial coefficients can be conveniently arranged into a triangular form, starting with $n = 0$, and increasing the value of n by 1 in each row.

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

The resulting triangle is then

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

This arrangement of binomial coefficients into triangular form is named Pascal's triangle, after the French mathematician Blaise Pascal, who wrote about the subject in the 1650's. It becomes apparent that a pattern is present here, namely, the entry corresponding to the binomial coefficient $\binom{n}{m}$ is the sum of the entry immediately to its left and the entry immediately to its right in the row above its row. This makes the construction of the table of binomial coefficients very easy. The next two rows are

$$\begin{array}{ccccccccccccccc} & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

Note that the pattern of numbers in each row is the same reading from the left or the right. This follows from the equality

$$\binom{n}{m} = \binom{n}{n-m}$$

which we previously noticed.

Binomial coefficients play an important role in algebra, combinatorial theory, probability and statistics. Their most obvious occurrence in algebra is in the context of the *binomial theorem*, from which their name derives.

In applications of the binomial theorem, we wish to expand a power $(x + y)^n$, where n is a positive whole number and x and y are any numbers, in terms of powers of x and y. To try to see a pattern, we look at some examples for small n.

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

If we arrange the powers of x and y to start with the highest powers of x on the left and work towards decreasing powers of x and increasing powers of y, the pattern of coefficients revealed is clearly that displayed in Pascal's triangle. Thus, if we generalize, the coefficient of $x^{n-m}y^m$ in $(x + y)^n$ should be the binomial coefficient $\binom{n}{m}$. We first prove a special case of the binomial theorem, from which the general case follows easily.

Special Case of Binomial Theorem. *Let n be an integer ≥ 0. Then we have*

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{m}x^m + \cdots + \binom{n}{n}x^n.$$

Proof We proceed by induction on n. We already know that the result is true when $n = 1$. Suppose that it is true when $n = r$. Thus we assume that

$$(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \cdots + \binom{r}{m}x^m + \cdots + \binom{r}{r}x^r.$$

We say that x^m is the general term in this expansion, and our induction hypothesis is that its coefficient is $\binom{r}{m}$. Now we try to prove the corresponding formula when r becomes $r+1$. This means that the coefficient of x^m in $(1+x)^{r+1}$ should be $\binom{r+1}{m}$. But $(1+x)^{r+1} = (1+x)^r(1+x)$ and by induction, this expression is

$$\left(\binom{r}{0} + \binom{r}{1}x + \cdots + \binom{r}{m-1}x^{m-1} + \binom{r}{m}x^m + \cdots + \binom{r}{r}x^r\right)(1+x).$$

On multiplying out, we get

$$\begin{aligned}\binom{r}{0}+\binom{r}{1}x + \binom{r}{2}x^2 + \cdots + \binom{r}{m}x^m + \cdots + \binom{r}{r}x^r + \\ \binom{r}{0}x + \binom{r}{1}x^2 + \cdots + \binom{r}{m-1}x^m + \cdots + \binom{r}{r}x^{r+1}.\end{aligned}$$

The coefficient of x^m is thus

$$\binom{r}{m-1} + \binom{r}{m} \text{ which equals } \binom{r+1}{m},$$

as we showed in a previous theorem. This proves what we want.

For the general expansion $(x+y)^n$, we write

$$\begin{aligned}(x+y)^n &= \left(x(1+\frac{y}{x})\right)^n \\ &= x^n\left(1+\frac{y}{x}\right)^n \\ &= x^n\left(\binom{n}{0} + \binom{n}{1}\frac{y}{x} + \cdots + \binom{n}{m}\frac{y^m}{x^m} + \cdots\right) \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{m}x^{n-m}y^m + \cdots\end{aligned}$$

as required.

<u>Example</u> Find the coefficient of x^4 in the expansion of $(2+3x)^9$.

The expansion is

$$2^9 + \binom{9}{1} 2^8 (3x) + \binom{9}{2} 2^7 (3x)^2 + \binom{9}{3} 2^6 (3x)^3 + \binom{9}{4} 2^5 (3x)^4 + \cdots$$

Thus the required coefficient is

$$\binom{9}{4} \times 2^5 \times 3^4 = 126 \times 32 \times 81 = 326592.$$

<u>Example</u> Find the coefficient of x^2 and x^3 in the expansion of $(1 + 2x + 3x^2)^5$.

This is slightly more complicated as the expression inside the brackets is not a binomial but a trinomial. We regroup terms to obtain a binomial by putting

$$1 + 2x + 3x^2 = 1 + x(2 + 3x)$$

and then expand by the binomial theorem to get

$$(1 + x(2 + 3x))^5 = 1^5 + \binom{5}{1} 1^4 x(2 + 3x) + \binom{5}{2} 1^3 x^2 (2 + 3x)^2 + \binom{5}{3} 1^2 x^3 (2 + 3x)^3 + \cdots$$

We need not expand any further, as we are interested in x^2 and x^3, and the missing terms involve only higher powers of x. Looking at the first four terms, we have

$$1 + 5x(2 + 3x) + 10x^2(4 + 12x + 9x^2) + 10x^3(8 + \cdots).$$

Thus the coefficient of x^2 is $15 + 40 = 55$ and that of x^3 is $80 + 120 = 200$.

<u>Example</u> Find the coefficient of x^3 in the expansion of $(x + 2)^4(x - 1)^2$.

The binomial theorem is only partly applicable here. Expanding $(x + 2)^4$ and multiplying by $x^2 - 2x + 1$ we get

$$(x^4 + 8x^3 + 24x^2 + 32x + 16)(x^2 - 2x + 1)$$

and picking out the terms in the product that give us an x^3, we get the coefficient

$$32 - 2 \times 24 + 8 = -8.$$