# THE QUADRATIC TYPE OF CERTAIN IRREDUCIBLE MODULES FOR THE SYMMETRIC GROUP IN CHARACTERISTIC TWO

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Abstract

*Let $\lambda$ be a 2-regular partition of $n$ into two parts and let $D^\lambda$ denote the corresponding irreducible $\mathbb{F}_2\Sigma_n$-module. We say that $D^\lambda$ is of quadratic type if there is a non-degenerate $\Sigma_n$-invariant quadratic form defined on this module. In this paper, we show that $D^\lambda$ is not of quadratic type precisely when the smaller part of $\lambda$ is a power of 2, say $2^r$, where $r \geq 0$, and $n \equiv k \bmod 2^{r+2}$, where $k$ is one of the $2^r$ consecutive integers $2^{r+1}+2^r-1, \ldots, 2^{r+2}-2$.*

Let $\Sigma_n$ denote the symmetric group of degree $n$ and let $\mathbb{F}_2$ denote the field of order 2. We say that a partition $\lambda$ of $n$ is 2-regular if $\lambda$ has no repeated parts. Given a 2-regular partition of $n$, there is an absolutely irreducible $\mathbb{F}_2\Sigma_n$-module $D^\lambda$ that corresponds to $\lambda$. Moreover, every irreducible $\mathbb{F}_2\Sigma_n$-module is isomorphic to some $D^\mu$ for a suitable 2-regular partition $\mu$ of $n$. The module $D^{(n)}$ is the trivial module. Provided $\lambda$ is different from $(n)$, there is a non-degenerate $\Sigma_n$-invariant alternating bilinear form $f_\lambda$, say, defined on $D^\lambda \times D^\lambda$. We say that $D^\lambda$ is of *quadratic type* if there is a non-degenerate $\Sigma_n$-invariant quadratic form defined on $D^\lambda$, whose polarization is $f_\lambda$.

A description in terms of the parts of $\lambda$ of those $D^\lambda$ that are not of quadratic type does not appear to be known at present, and we may expect any complete solution of the quadratic type problem to involve some delicate combinatorial considerations. The purpose of this paper is to decide when $D^\lambda$ is not of quadratic type in the case that $\lambda$ is a two-part partition, that is, a partition of $n$ into exactly two non-zero parts. Our main result is that, when $\lambda$ is a two-part partition, $D^\lambda$ is not of quadratic type precisely when the smaller part of $\lambda$ is a power of 2, say $2^r$, where $r \geq 0$, and $n \equiv k \bmod 2^{r+2}$, where $k$ is one of the $2^r$ consecutive integers $2^{r+1}+2^r-1, \ldots, 2^{r+2}-2$.

An investigation of the problem of quadratic type involves some discussion of integral lattices. We first recall that, given a partition $\lambda$ of $n$, the Specht lattice $S^\lambda$ is a sublattice of the permutation lattice $M^\lambda$, defined by the permutation action of $\Sigma_n$ on the cosets of the Young subgroup that corresponds to $\lambda$. Restriction of the standard inner product on $M^\lambda$ defines a positive definite $\Sigma_n$-invariant integral symmetric bilinear form $F_\lambda$, say, on $S^\lambda \times S^\lambda$. Provided that $\lambda \neq (n)$, $F_\lambda$ is even, meaning that $F_\lambda(u,u)$ is an even integer for all $u$ in $S^\lambda$. We may thus define an integral quadratic form $Q_\lambda$ on $S^\lambda$ by setting

$$Q_\lambda(u) = 2^{-1}F_\lambda(u,u).$$

It is clear that $F_\lambda$ is the polarization of $Q_\lambda$.

Let $\overline{S}^\lambda$ denote the $\mathbb{F}_2\Sigma_n$-module defined by

$$\overline{S}^\lambda = S^\lambda/2S^\lambda.$$

Given $u \in S^\lambda$, let $\overline{u}$ denote the image of $u$ in $\overline{S}^\lambda$. As $F_\lambda$ is even, we may define an alternating bilinear form

$$\overline{F}_\lambda : \overline{S}^\lambda \times \overline{S}^\lambda \to \mathbb{F}_2$$

by setting

$$\overline{F}_\lambda(\overline{u},\overline{v}) = F_\lambda(u,v) + 2\mathbb{Z}.$$

If $\lambda$ is 2-regular, it is well known that the radical $\operatorname{rad}\overline{F}_\lambda$ of $\overline{F}_\lambda$ is the unique maximal $\mathbb{F}_2\Sigma_n$-submodule of $\overline{S}^\lambda$ and

$$\overline{S}^\lambda/\operatorname{rad}\overline{F}_\lambda \cong D^\lambda.$$

In order to discuss the problem of whether $D^\lambda$ is of quadratic type or not, it will be convenient to take a more general point of view and develop a theory for all finite groups. Thus, let $G$ be a finite group and let $L$ be a $\mathbb{Z}G$-lattice. Let $F : L \times L \to \mathbb{Z}$ be a $G$-invariant non-degenerate even positive definite symmetric bilinear form. We may scale $F$ so that it is not a non-trivial integral multiple of any other integral symmetric bilinear form defined on $L \times L$. We then define a quadratic form $Q$ on $L$ by $Q(u) = 2^{-1}F(u,u)$. Let $\overline{L} = L/2L$ and let $\overline{F}$ be the corresponding alternating bilinear form defined on

$\overline{L} \times \overline{L}$. Let $R$ denote the radical of $\overline{F}$. We suppose from now on that $R$ is the unique maximal $\mathbb{F}_2G$-submodule of $\overline{L}$. Then $\overline{L}/R$ is an irreducible self-dual $\mathbb{F}_2G$-module and our problem is to decide whether or not it is of quadratic type. We first define a quadratic form $\overline{Q}$ on $\overline{L}$ by setting

$$\overline{Q}(\overline{u}) = Q(u) + 2\mathbb{Z}.$$

We define the singular radical $R_0$ of $\overline{Q}$ by

$$R_0 = \{\overline{u} \in R : \overline{Q}(\overline{u}) = 0\}.$$

We now state without proof the following elementary properties of $R_0$.

LEMMA 1. *$R_0$ is an $\mathbb{F}_2G$-submodule of $R$. We either have $R_0 = R$ or $R_0$ has codimension 1 in $R$ and $R/R_0$ is the trivial one-dimensional $\mathbb{F}_2G$-module.*

It is straightforward to see that we may define a quadratic form $\overline{Q}_0$ on $\overline{L}/R_0$ by

$$\overline{Q}_0(\overline{u} + R_0) = \overline{Q}(\overline{u}).$$

LEMMA 2. *Under the given assumptions, $\overline{L}/R$ is of quadratic type if and only if $R = R_0$. If $\overline{L}/R$ is not of quadratic type, $\overline{L}$ has a trivial $\mathbb{F}_2G$-composition factor.*

*Proof.* If $R = R_0$, the result is clear from our observation above. Conversely, suppose that $R \neq R_0$. Then certainly $\overline{L}$ has a trivial $\mathbb{F}_2G$-composition factor, and our assumption on the uniqueness of $R$ as a maximal submodule implies that $R/R_0$ is the socle of $\overline{L}/R_0$. It follows from Lemma 1.3 of [2] that $\overline{L}/R$ is not of quadratic type. ■

Let $n$ be the rank of the lattice $L$. By the theory of the Smith normal form for integral matrices, there exist integral bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of $L$ and positive integers $r_i$ for $1 \leq i \leq n$ such that

$$F(x_i, y_j) = r_i \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. Let $\nu$ be the standard 2-adic valuation on the integers. We may order our bases so that

$$\nu(r_1) \leq \ldots \leq \nu(r_n),$$

and our scaling of $F$ implies that $\nu(r_1) = 0$.

For each integer $k \geq 0$, we set

$$L_k = \{x \in L : F(x, u) \in 2^k\mathbb{Z} \text{ for all } u \in L\}$$

and

$$\overline{L}_k = (L_k + 2L)/2L \cong L_k/2L_{k-1}.$$

Clearly, $L_0 = L$ and it is straightforward to see that $\overline{L}_1 = R$.

LEMMA 3. *We may define a symmetric bilinear form $\overline{F}_1$ on $R \times R$ by*

$$\overline{F}_1(\overline{u}, \overline{v}) = \overline{F}_1(u + 2L, v + 2L) = 2^{-1}F(u, v) + 2\mathbb{Z}$$

*for all $\overline{u}$ and $\overline{v}$ in $R$. The radical of $\overline{F}_1$ is $\overline{L}_2$.*

*Proof.* We first show that $\overline{F}_1$ is well defined. Thus, given $\overline{u}$, $\overline{v}$ in $R$, suppose that

$$\overline{u} = u + 2L = u_1 + 2L, \quad \overline{v} = v + 2L = v_1 + 2L,$$

where $u$, $u_1$, $v$, $v_1$ are all in $L_1$. Then we can write $u = u_1 + 2x$, $v = v_1 + 2y$ where $x$ and $y$ are in $L$. We therefore have

$$F(u, v) = F(u_1, v_1) + 2F(u_1, y) + 2F(x, v_1) + 4F(x, y).$$

Since $u_1$ and $v_1$ are in $L_1$, $F(u_1, y) \in 2\mathbb{Z}$ and $F(x, v_1) \in 2\mathbb{Z}$, and hence

$$2^{-1}F(u, v) + 2\mathbb{Z} = 2^{-1}F(u_1, v_1) + 2\mathbb{Z}.$$

This implies that $\overline{F}_1$ is well defined, and it is clearly bilinear and symmetric.

Suppose that $\overline{u} \in R$ is in the radical of $\overline{F}_1$, and let $\overline{u} = u + 2L$, where $u \in L_1$. Then, by definition,

$$F(u, v) \in 4\mathbb{Z}$$

for all $v \in L_1$. Recall now the integral bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of $L$. Suppose that there are exactly $c$ indices $i$ such that $\nu(r_i) = 0$ and exactly $d$ indices $j$ such that $\nu(r_j) = 1$. Then we easily verify that

$$\{2x_1, \ldots, 2x_c, x_{c+1}, \ldots, x_n\} \text{ and } \{2y_1, \ldots, 2y_c, y_{c+1}, \ldots, y_n\}$$

are integral bases for $L_1$. We may therefore write

$$u = 2a_1x_1 + \cdots + 2a_cx_c + a_{c+1}x_{c+1} + \cdots + a_nx_n$$

for unique integers $a_1, \ldots, a_n$. Then since $F(u, y_j) \in 4\mathbb{Z}$ for $c+1 \leq j \leq c+d$, we see that $a_j \in 2\mathbb{Z}$ for $c+1 \leq j \leq c+d$. It follows that $\overline{u} = u+2L$ is a linear combination of the elements $x_j + 2L$ where $j > c+d$. On the other hand, it is elementary to check that the elements $x_j + 2L$ where $j > c+d$ form a basis of $\overline{L}_2$. Moreover, any element of $\overline{L}_2$ is certainly in the radical of $\overline{F}_1$. Thus we have identified the radical of $\overline{F}_1$, as required. ■

Lemma 4. *We have $R = R_0$ if and only if $\overline{F}_1$ is alternating.*

*Proof.* Let $\overline{Q}$ be the quadratic form previously defined on $\overline{L}$. Then $\overline{Q}$ vanishes on $R$ precisely when

$$2^{-1}F(u,u) \in 2\mathbb{Z}$$

for all $u \in L_1$ and this is the condition for $\overline{F}_1$ to be alternating. ■

The following two lemmas must be well known, but we provide a proof in one case.

Lemma 5. *Let $W$ be an $\mathbb{F}_2G$-module and let $f : W \times W \to \mathbb{F}_2$ be a non-degenerate $G$-invariant symmetric bilinear form. Suppose that $f$ is not alternating. Then*

$$W_0 = \{w \in W : \, f(w,w) = 0\}$$

*is an $\mathbb{F}_2G$-submodule of codimension 1 in $W$, and $W/W_0$ is the trivial $\mathbb{F}_2G$-module.*

Lemma 6. *Let $W$ be an $\mathbb{F}_2G$-module and let $f : W \times W \to \mathbb{F}_2$ be a non-degenerate $G$-invariant symmetric bilinear form. Suppose that the trivial one-dimensional $\mathbb{F}_2G$-module occurs with odd multiplicity as a composition factor of $W$. Then $\dim W$ is odd and hence $f$ is not alternating.*

*Proof.* We proceed by induction on $\dim W$, the result being trivial when $\dim W = 1$. Let $M$ be an irreducible $\mathbb{F}_2G$-submodule of $W$. As $M$ is irreducible, we have two possibilities: either $M \leq M^{\perp}$ or else $W = M \perp M^{\perp}$.

We first consider the case that $M \leq M^{\perp}$. Then there is an $\mathbb{F}_2G$-module isomorphism $W/M^{\perp} \cong M^*$, where $M^*$ is the (irreducible) dual space of $M$. Moreover, $f$ induces a non-degenerate $G$-invariant symmetric bilinear form on $(M^{\perp}/M) \times (M^{\perp}/M)$. Clearly, as $\dim M = \dim M^*$, we have

$$\dim W \equiv \dim (M^{\perp}/M) \bmod 2.$$

Since $M$ is the trivial module if and only if $M^*$ is the trivial module, it follows that $M^{\perp}/M$ contains the trivial $\mathbb{F}_2G$-module as a composition factor with odd multiplicity. Hence $\dim (M^{\perp}/M)$ is odd by induction and thus $\dim W$ is odd also.

In the second case, we have $W = M \perp M^{\perp}$. If $M^{\perp}$ contains an irreducible $\mathbb{F}_2G$-submodule $N$ with $N \leq N^{\perp}$, we return to the first case and are finished by induction. We may therefore assume that

$$W = N_1 \perp \ldots \perp N_r$$

is the orthogonal direct sum of irreducible $\mathbb{F}_2G$-submodules $N_i$. If $N_i$ is a non-trivial summand, $f$ must be alternating on $N_i$ by Lemma 5. Thus $\dim N_i$ is even. Consequently, $\dim W \equiv s \bmod 2$, where $s$ is the number of trivial composition factors, which implies that $\dim W$ is odd, as required. ∎

We can now describe our main tool for investigating the quadratic type problem.

Lemma 7. *Assume the notation and hypotheses previously introduced. Suppose that the trivial $\mathbb{F}_2G$-module occurs with multiplicity 1 as a composition factor of $\overline{L}$. Then $\overline{L}/R$ is not of quadratic type if and only if the trivial $\mathbb{F}_2G$-module occurs as a composition factor of $\overline{L}_1/\overline{L}_2$.*

*Proof.* We know that $\overline{L}/R$ is of quadratic type if and only if $R = R_0$, and Lemma 4 shows that $R = R_0$ if and only if $\overline{F}_1$ is alternating. Now if $\overline{F}_1$ is not alternating, Lemma 5 shows that the trivial $\mathbb{F}_2G$-module occurs as a composition factor of $\overline{L}_1/\overline{L}_2$. If $\overline{F}_1$ is alternating, Lemma 6 implies that the trivial $\mathbb{F}_2G$-module occurs as a composition factor of $\overline{L}_1/\overline{L}_2$ with even multiplicity, $2r$, say. As the trivial $\mathbb{F}_2G$-module occurs as a composition factor of $\overline{L}$ with multiplicity 1, it is clear that $r$ must be 0 and the lemma is proved. ■

We turn to considering how the theory developed in the previous section relates to the modules $D^\lambda$ of $\Sigma_n$. We may apply Lemma 7 whenever we know that the trivial module $D^{(n)}$ occurs with multiplicity 1 as a composition factor of $\overline{S}^\lambda$. Now James, [3, Theorem 24.15], has proved that when $\lambda$ is a two-part partition, $D^{(n)}$ has multiplicity at most 1 as a composition factor of $\overline{S}^\lambda$. Moreover, the multiplicity is 1 if and only if $\dim \overline{S}^\lambda$ is odd. If we take $\lambda = (n-m, m)$, where $2m < n$, then

$$\dim \overline{S}^\lambda = \binom{n}{m} - \binom{n}{m-1} \equiv \binom{n+1}{m} \bmod 2.$$

Thus $D^{(n)}$ is a composition factor of $\overline{S}^{(n-m,m)}$ precisely when $\binom{n+1}{m}$ is odd.

A theorem of Schaper, [4] (see also [1], for example), provides the machinery we need to check whether $D^{(n)}$ occurs as a composition factor of $\overline{S}_1^{(n-m,m)}/\overline{S}_2^{(n-m,m)}$. We use the language of the Grothendieck group to state the theorem. Thus, given an $\mathbb{F}_2\Sigma_n$-module $M$, we let $[M]$ denote its image in the Grothendieck group of $\mathbb{F}_2\Sigma_n$-modules. The elements $[D^\lambda]$, where $\lambda$ runs over the 2-regular partitions of $n$, form a free basis of the group. We can now state Schaper's theorem in the form that is most useful to us.

Lemma 8. *In the Grothendieck group of $\mathbb{F}_2\Sigma_n$-modules, we have the equality*

$$\sum_{i\geq 1} i\big[\overline{S}_i^{(n-m,m)}/\overline{S}_{i+1}^{(n-m,m)}\big] = \sum_{i=0}^{m-1} \nu((n+1-m-i)/(m-i))\big[\overline{S}^{(n-i,i)}\big].$$

We refer to this equality as Schaper's formula. In view of our earlier work, the formula shows that $D^{(n-m,m)}$ is not of quadratic type precisely when the coefficient of $\big[D^{(n)}\big]$ on the right hand side above is 1. Therefore, it is a purely combinatorial and numerical matter to solve our problem, but we use some representation-theoretic ideas, such as branching theorems, to reduce our dependence on Schaper's theorem.

We begin by looking at those two-part partitions $\lambda$ whose second part is a power of 2, since these provide the $D^\lambda$ that are not of quadratic type.

LEMMA 9. *Let* $n = (2k+1)2^{r+1} + 2^r - 1$, *where* $k$ *and* $r$ *are non-negative integers, and let* $\lambda = (n - 2^r, 2^r)$. *Then* $D^\lambda$ *is not of quadratic type.*

*Proof.* It is straightforward to check that

$$\nu(n + 1 - 2^r - i) = \nu(2^r - i)$$

for $1 \leq i < 2^r$, while

$$\nu(n + 1 - 2^r) = r + 1 = \nu(2^r) + 1.$$

Thus $[D^{(n)}]$ occurs exactly once in Schaper's formula, which implies that $D^\lambda$ is not of quadratic type. ■

We remark that Lemma 9 can be proved by more elementary means. For, under the conditions of the lemma, $\overline{S}^\lambda$ has only the composition factors $D^\lambda$ and $D^{(n)}$, by James's results, [3, Theorem 24.15]. Thus $\dim R = 1$ and, as we can identify $R$ explicitly in this case, it is elementary to check that $R_0 = 0$.

Let $D^\lambda \downarrow_{\Sigma_{n-1}}$ denote the restriction of $D^\lambda$ to $\Sigma_{n-1}$.

The following lemma is a consequence of Lemma 1.2 of [2].

LEMMA 10. *Suppose that* $D^\lambda$ *is of quadratic type. Let* $\lambda'$ *be a partition of* $n-1$ *such that* $D^{\lambda'}$ *occurs with odd multiplicity in* $D^\lambda \downarrow_{\Sigma_{n-1}}$. *Suppose moreover that* $D^{(n-1)}$ *is not a composition factor of* $D^\lambda \downarrow_{\Sigma_{n-1}}$. *Then* $D^{\lambda'}$ *is also of quadratic type.*

LEMMA 11. *Suppose that* $n$ *satisfies*

$$k2^{r+1} + 2^r \leq n \leq (k+1)2^{r+1} - 2$$

*for some non-negative integers* $k$ *and* $r$, *and let* $\lambda = (n - 2^r, 2^r)$. *Then the composition factors of* $D^\lambda \downarrow_{\Sigma_{n-1}}$ *include* $D^{\lambda'}$, *where* $\lambda' = (n - 1 - 2^r, 2^r)$, *with multiplicity 1, and do not include* $D^{(n-1)}$.

*Proof.* Theorem 3.1 of [5] shows that when $n$ is even, $D^\lambda \downarrow_{\Sigma_{n-1}} = D^{\lambda'}$, and when $n$ is odd, $D^\lambda \downarrow_{\Sigma_{n-1}}$ contains $D^{\lambda'}$ with multiplicity 1. In the latter case, $D^\lambda \downarrow_{\Sigma_{n-1}}$ contains $D^{(n-1)}$ if and only if $n \equiv -1 \bmod 2^{r+1}$. As no value of $n$ in our permitted range satisfies this congruence, the lemma follows. ■

Lemma 12. *Suppose that $n$ satisfies*

$$(2k+1)2^{r+1} + 2^r - 1 \le n \le (2k+2)2^{r+1} - 2$$

*for some non-negative integers $k$ and $r$, and let $\lambda = (n - 2^r, 2^r)$. Then $D^\lambda$ is not of quadratic type.*

*Proof.* $D^\lambda$ is not of quadratic type when $n = (2k+1)2^{r+1} + 2^r - 1$, by Lemma 9. Suppose, by way of contradiction, that there is some $n$ in the range specified above for which $D^\lambda$ is of quadratic type. We may then assume that $n$ is the smallest such integer in the permitted range. By Lemma 11, the composition factors of $D^\lambda \downarrow_{\Sigma_{n-1}}$ include $D^{\lambda'}$, where $\lambda' = (n-1-2^r, 2^r)$, with multiplicity 1, and do not include $D^{(n-1)}$. It follows from Lemma 10 that $D^{\lambda'}$ is also of quadratic type, contradicting the minimality of $n$. This proves the lemma. ∎

The following result makes critical use of Schaper's formula.

Lemma 13. *Suppose that $n = k2^{r+2} + 2^{r+1} - 2$ for some non-negative integers $k$ and $r$, and let $\lambda = (n - 2^r, 2^r)$. Then $D^\lambda$ is of quadratic type.*

*Proof.* Let $i$ be an integer satisfying $0 < i < 2^r - 1$. Then we easily see that

$$\nu(n + 1 - 2^r - i) - \nu(2^r - i) = \nu(i + 1) - \nu(i).$$

Furthermore, we also have

$$\nu(n - 2^r + 1) - \nu(2^r) = -r, \quad \nu(n - 2^{r+1} + 2) = \nu(k) + r + 2.$$

We note also that each binomial coefficient $\binom{n+1}{i}$ is odd for each $i$ with $0 \le i \le 2^r - 1$. Thus $\left[D^{(n)}\right]$ occurs in $\left[\overline{S}^{(n-i,i)}\right]$ with multiplicity 1 for each $i$ in this range. It follows that the coefficient of $[D^{(n)}]$ in the right hand side of Schaper's formula is

$$-r + \sum_{i=0}^{2^r-2} (\nu(i+1) - \nu(i)) + \nu(k) + r + 2,$$

which is at least as large as 2. Our earlier remarks and Lemma 7 now imply the result. ∎

LEMMA 14. *Suppose that $n$ satisfies*

$$k2^{r+2} + 2^r - 1 \leq n \leq k2^{r+2} + 2^{r+1} - 2$$

*for some non-negative integers $k$ and $r$, and let $\lambda = (n - 2^r, 2^r)$. Then $D^\lambda$ is of quadratic type.*

*Proof.* By Lemma 13, $D^\lambda$ is of quadratic type when $n = k2^{r+2} + 2^{r+1} - 2$. It follows from repeated restrictions, using Lemma 11, that $D^\lambda$ is also of quadratic type when $k2^{r+2} + 2^r - 1 \leq n \leq k2^{r+2} + 2^{r+1} - 3$. ∎

We can now obtain our first major result on the quadratic type of those modules $D^\lambda$ when $\lambda$ is a two-part partition whose second part is a power of 2.

THEOREM 1. *Let $r$ be a non-negative integer and let $\lambda = (n - 2^r, 2^r)$, where $2^{r+1} < n$. Then $D^\lambda$ is not of quadratic type if and only if $n \equiv k \bmod 2^{r+2}$, where $k$ is one of the $2^r$ consecutive integers $2^{r+1} + 2^r - 1, \ldots, 2^{r+2} - 2$.*

*Proof.* Suppose first that $n \equiv k \bmod 2^{r+2}$ where $k$ is one of $0, \ldots, 2^r - 2$. Then it is straightforward to see that the binomial coefficient $\binom{n+1}{2^r}$ is even. It follows from our discussion after the proof of Lemma 7 that $D^\lambda$ is of quadratic type. Suppose next that $n \equiv k \bmod 2^{r+2}$ where $k$ is one of $2^r - 1, \ldots, 2^{r+1} - 2$. Then $D^\lambda$ is of quadratic type by Lemma 14. Now suppose that $n \equiv k \bmod 2^{r+2}$ where $k$ is one of $2^{r+1} - 1, \ldots, 2^{r+1} + 2^r - 2$. Then we again find that the binomial coefficient $\binom{n+1}{2^r}$ is even and it follows that $D^\lambda$ is of quadratic type in these cases. Lemma 12 shows that $D^\lambda$ is not of quadratic type when $n \equiv k \bmod 2^{r+2}$, and $k$ is any one of $2^{r+1} + 2^r - 1, \ldots, 2^{r+2} - 2$. Finally, when $n \equiv 2^{r+2} - 1 \bmod 2^{r+2}$, the binomial coefficient $\binom{n+1}{2^r}$ is even and it follows that $D^\lambda$ is of quadratic type in this case. This exhausts all possibilities and completes the proof. ∎

We now solve the problem of deciding the quadratic type of $D^\lambda$ when the smaller part of $\lambda$ is not a power of 2.

THEOREM 2. *Let $m \geq 3$ be an integer that is not a power of 2 and let $\lambda = (n - m, m)$ be a 2-regular partition of $n$. Then $D^\lambda$ is of quadratic type.*

*Proof.* We can find an integer $k \geq n$ such that the binomial coefficient $\binom{k+1}{m}$ is even. Then the module $D^{(k-m,m)}$ is of quadratic type for $\Sigma_k$. Now it follows from Theorem 3.1 of [5] that $D^{(k-m,m)} \downarrow_{\Sigma_{k-1}}$ contains $D^{(k-m-1,m)}$ but does not contain $D^{(k-1)}$ (here we use the fact that $m$ is not a power of 2). Thus $D^{(k-m-1,m)}$ is also of quadratic type by Lemma 10. It follows thus by $k-n$ repeated restrictions that $D^{\lambda}$ is of quadratic type. ■

*Acknowledgement*

The authors are grateful to Professor G. D. James for pointing out the relevance of Schaper's theorem to this problem.

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