CHARACTERS OF π-SEPARABLE GROUPS INDUCED BY CHARACTERS OF LARGE SCHUR INDEX

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Let G be a finite group and let χ be an irreducible complex character of G. Let D be a complex representation of G affording χ and let $m(\chi)$ denote the Schur index of χ over the field $\mathbb{Q}$ of rational numbers. It is well known that $m(\chi)$ is a divisor of the degree $\chi(1)$ of χ. When the extreme case $m(\chi) = \chi(1)$ occurs, $D(G)$ is isomorphic to a finite subgroup of the multiplicative group of a division algebra. See, for example, [H, pp.548-549]. This imposes severe restrictions on the structure of $D(G)$. For example, if p is an odd prime divisor of $|D(G)|$, a Sylow p-subgroup of $D(G)$ is cyclic, and a Sylow 2-subgroup of $D(G)$ is either cyclic or generalized quaternion.

Suppose now that G is π-separable, where π is a set of primes, and let H be a Hall π-subgroup of G. Suppose also that H has an irreducible complex character θ with the property that $m(\theta) = \theta(1)$. We show in this paper that, modulo certain exceptions related to the quaternion group of order 8, G has an irreducible character χ with the property that $m(\chi) = \theta(1)$ and $\chi(1) = s\theta(1)$, where s is a π'-number. Furthermore, the field of values $\mathbb{Q}(\chi)$ of χ is contained in the field $\mathbb{Q}(\theta)$. The hypothesis that $m(\theta) = \theta(1)$ is somewhat restrictive and it is certainly true that it is not easy to construct non-linear characters θ with the desired property (see, for example, [I, Theorem 10.16]) but the possibility that θ is linear is included, and our main result in this case reduces to a theorem obtained by us in a previous paper, [G, Theorem 1]. Another interesting possibility is that H has an image isomorphic to a non-cyclic subgroup of the multiplicative group of the Hamiltonian quaternions

$\mathbb{H}$. In this case, H has a real-valued complex character θ with $\theta(1) = 2 = m(\theta)$ and so, modulo the exceptions which we will describe later, the hypotheses of our theorem apply. We also feel that, without any specific information about how H is embedded in G, it is unlikely that we can make any far reaching conclusions about properties of the characters of G on the basis only of properties of characters of H, unless we impose some fairly tight conditions onto these characters of H.

We now proceed to derive our main results. We require the following general property of the Schur index. A proof of the first part of the result we quote is given in [I, Lemma 10.4], and a proof of the second part can be modelled on the argument given there.

LEMMA 1. *Let H be a subgroup of the group G and let ψ and θ be irreducible characters of H and G, respectively. Then*

$$m(\theta) \text{ divides } m(\psi)|\mathbb{Q}(\psi,\theta):\mathbb{Q}(\theta)|[\psi^G,\theta]$$

and

$$m(\psi) \text{ divides } m(\theta)|\mathbb{Q}(\psi,\theta):\mathbb{Q}(\psi)|[\psi,\chi_H].$$

The next result we need is given as Problem 10.15 in [I, p.173]. We provide a proof on the basis of Lemma 1 above.

LEMMA 2. *Let θ be an irreducible character of G and suppose that $m(\theta) = \theta(1)$. Let H be a subgroup of G and let ψ be an irreducible constituent of θ_H. Then all the irreducible constituents of θ_H are Galois conjugate to ψ over $\mathbb{Q}(\theta)$. Moreover, $m(\psi) = \psi(1)$.*

Proof. Set $[\psi, \theta_H] = [\psi^G, \theta] = c$ and $|\mathbb{Q}(\psi,\theta) : \mathbb{Q}(\theta)| = r$. Let $\sigma_1 = 1, \ldots, \sigma_r$ be all the Galois automorphisms of $\mathbb{Q}(\psi,\theta)$ over $\mathbb{Q}(\theta)$. It follows then that $\psi = \psi^{\sigma_1}$, $\ldots$, ψ^{σ_r} are different irreducible constituents of θ_H and each has multiplicity c. Thus we have the inequality

$$\theta(1) \geq \psi(1)cr.$$

On the other hand,

$$m(\theta) \text{ divides } m(\psi)rc$$

by Lemma 1. Since $m(\psi)$ divides $\psi(1)$, we deduce that $m(\theta)$ divides $\psi(1)cr$. Our inequality above implies now that $m(\psi) = \psi(1)$ and

$$\theta_H = c(\psi^{\sigma_1} + \cdots + \psi^{\sigma_r}),$$

as required. ■

Our next result is a consequence of what has been mentioned in the introduction. There is more than one proof available, and we give one that occurs naturally in the theory of Frobenius complements.

LEMMA 3. *Let D be a faithful irreducible complex representation of the group G with character θ. Suppose that $m(\theta) = \theta(1)$. Let p be a prime divisor of $|G|$. Then a Sylow p-subgroup of G is cyclic if p is odd, and is cyclic or generalized quaternion if $p = 2$. If q and r are different prime divisors of $|G|$, any subgroup of G of order qr is cyclic.*

Proof. Let H be a proper subgroup of G. It follows from Lemma 2 that θ_H does not contain the principal character 1_H. We deduce that $D(G)$ is a linear group isomorphic to G in which no non-identity element fixes a non-zero vector. The lemma now follows from [H, Lemma 8.12, p.502]. ■

Our next lemma must be well known, but we include a proof for the sake of completeness.

LEMMA 4. *Let S be either a dihedral, semi-dihedral or generalized quaternion 2-group. Suppose furthermore that S is not a quaternion group of order 8. Then $\mathrm{Aut}(S)$ is a 2-group.*

Proof. Our hypotheses imply that S has a unique cyclic subgroup of index 2, R, say. R is thus characteristic in S. Suppose, if possible, that σ is an automorphism of S of order r, where r is an odd prime. Let x be a generator of R. Then as $\mathrm{Aut}(R)$ is a 2-group, $\sigma(x) = x$. Now by a well known result in the theory of p-groups, σ induces an automorphism of order r of the quotient group S/Φ, where Φ is the Frattini subgroup of S. Since S/Φ has order 4, r can only be 3 and σ can have no non-trivial fixed points in its action on S/Φ. However, $x\Phi$ is certainly a non-trivial fixed point of σ, since $x \notin \Phi$. This is a contradiction and the lemma is proved. ∎

The following result plays an important role in the proof of our main theorem.

LEMMA 5. *Let N be a normal Hall subgroup of the group G. Suppose that N has a faithful irreducible complex character ψ satisfying $m(\psi) = \psi(1)$. Suppose also that N contains its centralizer in G. Then the inertia subgroup $I_G(\psi)$ of ψ in G is N, unless possibly $|I_G(\psi) : N| = 3$ and a Sylow 2-subgroup of N is isomorphic to a quaternion group of order 8.*

Proof. Set $I = I_G(\psi)$. By a theorem of Gallagher, [I, Lemma 13.3 and Corollary 13.4], we may extend ψ to a character ϕ of I satisfying $\mathbb{Q}(\phi) = \mathbb{Q}(\psi)$. We claim that

$$m(\phi) = m(\psi) = \psi(1) = \phi(1).$$

This follows since, from the first part of Lemma 1, $m(\phi)$ divides $m(\psi)$, while from the second part of Lemma 1, $m(\psi)$ divides $m(\phi)$. We note furthermore that, since ψ is assumed to be faithful and N contains its centralizer in G, ϕ is also faithful.

We will now show that if $|I| = p|N|$, where p is a prime, then $p = 3$ and a Sylow 2-subgroup of N is quaternion of order 8. We will also show that we cannot have $|I| = 9|N|$. Suppose then that $|I| = p|N|$, and let g be an element of order p in I. Let q be any prime divisor of $|N|$. By the Frattini argument, g normalizes a Sylow q-subgroup Q of N. Clearly, either g centralizes Q or it induces by conjugation an automorphism of order p of Q. Suppose first that $q = 2$. Then Q is either cyclic or generalized quaternion. Now Lemma 4 shows that Q does not admit an automorphism of order p unless $p = 3$ and Q is isomorphic to a quaternion group of order 8. Thus g centralizes Q in this case, with one possible exception. Suppose next that q is an odd prime. Then g normalizes the unique subgroup Q_1, say, of order q in Q. If g does not centralize Q, it does not centralize Q_1, since any q'-automorphism of a cyclic q-group acts non-trivially on the subgroup of order q. But if g does not centralize Q_1, it follows that the subgroup of I of order qp generated by Q_1 and g is not cyclic. This contradicts Lemma 3 applied in the context of I and the character ϕ. Thus g centralizes Q. We deduce that unless possibly we are in the exceptional case described above, g centralizes a Sylow q-subgroup for each prime divisor q of $|N|$ and thus centralizes N, contrary to hypothesis. In the exceptional case, we cannot have $|I| = 9|N|$, since the order of the automorphism group of a quaternion group of order 8 is 24. ■

Suppose that N is a normal subgroup of the group G and let ψ be an irreducible complex character of N. We define the semi-inertia subgroup $S_G(\psi)$ of ψ in G by

$$S_G(\psi) = \{g \in G : \text{ there is } \sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|} : \mathbb{Q}) \text{ with } \psi^g = \psi^\sigma\}.$$

Here, $\mathbb{Q}_{|G|}$ denotes the field obtained by adjoining a primitive root of unity of order $|G|$ to the field of rational numbers and $\mathrm{Gal}(\mathbb{Q}_{|G|} : \mathbb{Q})$ is the associated Galois group. Given that the action of Galois automorphisms on the irreducible characters of N commutes with the conjugation action by G on the irreducible characters, it is easy to see that $S_G(\psi)$ is a subgroup of G, and the inertia subgroup $I_G(\psi)$ is a normal subgroup of $S_G(\psi)$. Furthermore, the quotient group $S_G(\psi)/I_G(\psi)$ is isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{Q}_{|G|} : \mathbb{Q})$, and is hence abelian.

We are now in a position where we can enunciate our main theorem and proceed to its proof.

THEOREM 1. *Let G be a π-separable group and let H and M be a Hall π-subgroup and a Hall π'-subgroup of G, respectively. Suppose that H has an irreducible complex character θ satisfying $m(\theta) = \theta(1)$. Suppose also that either $3 \in \pi$ or $2 \notin \pi$. Then θ^G contains a unique irreducible character $\chi = \chi_\theta$ with $[\chi_M, 1_M] > 0$. We have $[\chi_M, 1_M] = \theta(1)$, $\mathbb{Q}(\chi) \leq \mathbb{Q}(\theta)$ and $m(\chi) = \theta(1)$. Moreover, there is a subgroup U of G that contains H and an irreducible character ϕ of U with $\phi_H = \theta$ and $\phi^G = \chi$. In addition, if θ' is any irreducible character of H with $[\chi_H, \theta'] > 0$ and $m(\theta') = \theta'(1) = \theta(1)$, then θ and θ' are conjugate in the normalizer of H in G.*

Proof. We use induction on $|G|$. In the case that $H = G$, we clearly take χ to be θ and then all the other conclusions are trivial. We may thus assume that H is a proper subgroup of G. Now since M complements H, we have

$$(\theta^G)_M = \theta(1)\rho_M,$$

where ρ_M is the regular character of M. We may thus choose an irreducible constituent χ of θ^G satisfying $[\chi_M, 1_M] > 0$, and we intend to show that χ is unique and satisfies all the conclusions of the theorem above.

Let K be the kernel of χ. Since θ is an irreducible constituent of χ_H, by the Frobenius reciprocity theorem, we may consider θ to be an irreducible character of HK/K, which is a Hall π-subgroup of G/K. Now as χ is an irreducible character of G/K, the result follows by induction if $|K| > 1$. We can therefore assume that $K = 1$ and hence that χ is faithful. Let now $L = O_{\pi'}(G)$. As $L \leq M$ and $[\chi_M, 1_M] > 0$, we deduce that $[\chi_L, 1_L] > 0$. Clifford's theorem now implies that L is in the kernel of χ and is thus trivial. Let $N = O_\pi(G)$. Since G is π-separable and $O_{\pi'}(G) = 1$, N contains its centralizer in G. By Clifford's theorem,

$$\chi_N = r(\psi_1 + \cdots + \psi_t),$$

where r and t are positive integers and the ψ_i are a complete set of G-conjugate irreducible characters of N. Since $N \leq H$ and θ is a constituent of χ_H, we may take $\psi = \psi_1$ to be a constituent of θ_N. Let I and S be the inertia and semi-inertia subgroups of ψ in G. Clifford's theorem implies that there is an irreducible character ϕ of I with $\phi_N = r\psi$ and $\phi^G = \chi$. Thus, since $I \leq S$, $\xi = \phi^S$ is an irreducible character of S that induces χ. Furthermore, it is well known and easy to prove that $\mathbb{Q}(\chi) = \mathbb{Q}(\xi)$ and $m(\chi) = m(\xi)$.

Now $m(\theta) = \theta(1)$ and ψ is an irreducible constituent of θ_N. Lemma 2 implies that all irreducible constituents of θ_N are Galois conjugate over $\mathbb{Q}(\theta)$. As $\mathbb{Q}(\psi, \theta)$ is a normal subfield of $\mathbb{Q}_{|G|}$, each automorphism of $\mathbb{Q}(\psi, \theta)$ is the restriction of an automorphism of $\mathbb{Q}_{|G|}$ and it follows that all H-conjugates of ψ are Galois conjugate. Thus H is a subgroup of S. Write

$$\chi_S = \xi + \sum_i a_i \xi_i,$$

where the ξ_i are irreducible characters of S different from ξ. Since χ_H contains θ, either ξ_H or some $(\xi_i)_H$ contains θ. Now ξ_I contains ϕ, by Frobenius reciprocity, and since $\phi_N = r\psi$, we see that $[\xi_N, \psi] \geq r$. However, as $[\chi_N, \psi] = r$, we deduce that $[\xi_N, \psi] = r$ and $[(\xi_i)_N, \psi] = 0$ for all i. Finally, since θ_N contains ψ, it must be the case that ξ_H contains θ. Moreover, since $H \leq S$, $G = SM$ and hence $T = S \cap M$ is a Hall π'-subgroup of S. By Mackey's subgroup theorem, we have

$$\chi_M = (\xi^G)_M = (\xi_T)^M$$

and since $[\chi_M, 1_M] > 0$, we must have $[\xi_T, 1_T] > 0$.

Suppose now that $S \neq G$. Then we may apply induction to the character ξ of S. In this case, we have $[\xi_T, 1_T] = \theta(1)$ and thus $[\chi_M, 1_M] = \theta(1)$ also. Induction also implies that $m(\xi) = \theta(1)$ and $\mathbb{Q}(\xi) \leq \mathbb{Q}(\theta)$. Since $\chi = \xi^G$, we must also have

$$\mathbb{Q}(\chi) \leq \mathbb{Q}(\xi) \leq \mathbb{Q}(\theta).$$

Lemma 1 implies that $m(\xi)$ divides $m(\chi)$ and thus $\theta(1)$ divides $m(\chi)$. However, by Frobenius reciprocity, χ occurs with multiplicity $\theta(1)$ in the induced character 1_M^G. Lemma 1 thus implies that $m(\chi)$ divides $\theta(1)$ and we deduce that we have the equality $m(\chi) = \theta(1)$, as required. Furthermore, by induction there is a subgroup U of S containing H and an irreducible character τ of U with $\tau_H = \theta$ and $\tau^S = \xi$. But then $\tau^G = \chi$ and this proves that χ is induced from a proper subgroup in the manner claimed in the theorem. To complete the proof in this case, let θ' be an irreducible character of H with $[\chi_H, \theta'] > 0$ and $m(\theta') = \theta'(1) = \theta(1)$. Clifford's theorem implies that there exists some g in G such that $(\theta')_N$ contains ψ^g, since $(\theta')_N$ must consist of certain G-conjugates of ψ. Since $m(\theta') = \theta'(1)$, our earlier arguments imply that H is contained in the semi-inertia subgroup of ψ^g. However, the semi-inertia group of ψ^g is clearly S^g. Thus H and H^g are Hall π-subgroups of S^g, and so are conjugate by an element of S^g, since S^g is π-separable. A straightforward argument then shows that $g = sn$, for some $s \in S$ and some n in the normalizer of H in G. At this point, we may finish the conjugacy proof for θ and θ' by induction using an identical argument to that given in Theorem 1 of [G].

Finally suppose that $S = G$. It follows that I is normal in G, with abelian quotient group. Furthermore, χ_N consists entirely of Galois conjugates of ψ in this case. Since Galois conjugate characters have the same kernel, and χ is faithful, it follows that ψ is faithful. Under our hypothesis that either $3 \in \pi$ or $2 \notin \pi$, Lemma 5 now implies that $I = N$. Since this implies that G/N is abelian, it is clear that G has a normal Hall π-subgroup. Thus $N = H$ and $\chi = \theta^G$. Our theorem holds trivially in this case and the proof is complete. ■

We will now describe two examples of exceptional behaviour relating to Theorem 1 which show that some restrictive hypotheses are necessary to obtain the full conclusions of the theorem.

(a) We take $G = \mathrm{SL}_2(3)$ and H to be a Sylow 2-subgroup of G. Let θ be the faithful irreducible character of H. Then $m(\theta) = \theta(1) = 2$, but θ^G contains two irreducible characters χ_1 and χ_2 of degree 2 which satisfy

$$[(\chi_1)_M, 1_M] = [(\chi_2)_M, 1_M] = 1,$$

where M is a Sylow 3-subgroup of G.

(b) We take G to be the double cover of the symmetric group S_4 whose Sylow 2-subgroup H is a generalized quaternion group of order 16. Let θ be a faithful irreducible character of H. Then, as before, $m(\theta) = \theta(1) = 2$. It is also true that θ^G contains a unique irreducible constituent χ with $[\chi_M, 1_M] = 2$, where M is a Sylow 3-subgroup of G. However, $\chi(1) = 4$ and χ is not induced by an irreducible character of any proper subgroup of G containing H.

A key part of the proof of Theorem 1 is that the character θ of H is semi-primitive over $\mathbb{Q}$, meaning that there do not exist a proper subgroup L of H and an irreducible character ϕ of L satisfying $\phi^H = \theta$ and $\mathbb{Q}(\phi) = \mathbb{Q}(\theta)$. This has the consequence that if K is a normal subgroup of H and ϕ is an irreducible constituent of θ_K, then the semi-inertia subgroup $S_H(\phi)$ of ϕ is H. We will finish this paper by considering a case of similar behaviour which leads us to an analogous version of Theorem 1, although it is one where no Schur index hypothesis appears. We begin by presenting some of the details which we will need. We omit formal proofs of the next lemma, as it is surely well known.

Lemma 5. *Suppose that D is a dihedral, semi-dihedral, or generalized quaternion 2-group. Suppose also that D is not dihedral of order 8. Then any abelian normal subgroup of D is cyclic.*

The next result is a straightforward consequence of this lemma.

Lemma 6. *Suppose that D is a dihedral, semi-dihedral, or generalized quaternion 2-group. Suppose also that D is not dihedral of order 8. Let θ be a faithful irreducible complex character of D. Then θ is semi-primitive over $\mathbb{Q}$.*

Our next theorem can now be stated and proved.

Theorem 2. *Let G be a 2-solvable group and let H and M be a Sylow 2-subgroup and a Hall $2'$-subgroup of G, respectively. Suppose that H has an irreducible complex character θ such that $H/\ker\theta$ is either dihedral of order at least 16 or semi-dihedral or generalized quaternion of order at least 32. Then θ^G contains a unique irreducible character χ with $[\chi_M, 1_M] > 0$. We have $[\chi_M, 1_M] = 2$ and $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$. Moreover, there is a subgroup U of G that contains H and an irreducible character ϕ of U with $\phi_H = \theta$ and $\phi^G = \chi$. Thus χ is monomial and $\chi(1) = 2m$, where m is odd. In addition, if θ' is any irreducible character of H with $[\chi_H, \theta'] > 0$ and $H/\ker\theta \cong H/\ker\theta'$, then θ and θ' are conjugate in the normalizer of H in G.*

Proof. The proof is very similar to that of Theorem 1, so we will only provide the most important details. We use induction on $|G|$ and we will assume that $G \neq H$. We choose an irreducible constituent χ of θ^G satisfying $[\chi_M, 1_M] > 0$ and will show that χ is unique and satisfies all the conclusions listed above. As in the previous theorem, we may assume that χ is faithful and that $O_{2'}(G) = 1$. Let $N = O_2(G)$. Since G is 2-solvable and $O_{2'}(G) = 1$, N contains its centralizer in G. By Clifford's theorem,

$$\chi_N = r(\psi_1 + \cdots + \psi_t),$$

where r and t are positive integers and the ψ_i are a complete set of G-conjugate irreducible characters of N. Since $N \leq H$ and θ is a constituent of χ_H, we may take $\psi = \psi_1$ to be a constituent of θ_N. Let S be the semi-inertia subgroup of ψ in G.

Considering θ as a faithful irreducible character of $H/\ker\theta$, θ is semi-primitive over $\mathbb{Q}$ under the hypotheses we have adopted. It follows that H is contained in S. Provided that $S \neq G$, we may argue by induction that the conclusions of the theorem hold. We may therefore assume that $S = G$. As we have argued before, ψ must be faithful in this case and thus N is isomorphic to a normal subgroup of $H/\ker\theta$. Our hypotheses imply that N is not elementary abelian of order 4 and is hence either cyclic, dihedral, semi-dihedral or generalized quaternion. It follows from Lemma 4 that $\mathrm{Aut}(N)$ is a 2-group unless N is a quaternion group of order 8. Since N contains its centralizer in G, and G is not a 2-group by our earlier assumption, N must be quaternion of order 8. It follows immediately that $|G| = 24$ or 48, and we see in particular that H has order 8 or 16. Since these possibilities are ruled out by our hypotheses, we deduce that we cannot have $S = G$ and it follows that our theorem is true. ■

It is of interest to observe that there are exceptions to the conclusions of Theorem 2 if we relax the conditions on the order or structure of $H/\ker\theta$. The exceptions arise when $O_2(G)$ is either elementary abelian of order 4 or quaternion of order 8. Our earlier examples following the proof of Theorem 1 are relevant to this phenomenon and the following additional examples also illustrate the exceptional behaviour.

(c) We take $G = S_4$, and let H be a Sylow 2-subgroup of G. Then H is dihedral of order 8. Let θ be the faithful irreducible character of H of degree 2. We find that θ^G consists of two irreducible characters χ_1 and χ_2 each of degree 3. The restriction of each χ_i to a Sylow 3-subgroup M of G contains 1_M exactly once.

(d) We take G to be $\mathrm{GL}_2(3)$ and H to be a Sylow 2-subgroup of G. Then H is semi-dihedral of order 16. Let θ be a faithful irreducible character of H of degree 2. Then θ has degree 2 and is different from its complex conjugate $\overline{\theta}$. Let M be a Sylow 3-subgroup of G. Then θ^G contains a rational-valued irreducible character χ of degree 4 which satisfies $[\chi_M, 1_M] = 2$. Moreover, $\mathbb{Q}(\chi) \neq \mathbb{Q}(\theta)$ and $\chi_H = \theta + \overline{\theta}$, but θ is not conjugate to $\overline{\theta}$ in the normalizer of

H.

We conclude the paper by noting that Theorem 2 implies a connection between the global properties of a finite 2-solvable group and homomorphic images of its Sylow 2-subgroup.

Corollary 1. *Let G be a finite 2-solvable group and let H be a Sylow 2-subgroup of G.*

(a) Suppose that each character of G is real-valued. Then H has no homomorphic image that is semi-dihedral of order at least 32.

(b) Suppose that each irreducible character of G is 2-rational. Then H has no homomorphic image that is dihedral of order at least 16, or semi-dihedral or generalized quaternion of order at least 32.