First Arts Modular Degree
Mathematical Studies 2004–2005

Combinatorics and Number Theory Problem Sheet 5

1. Prove that $2^{340} \equiv 1 \bmod 341$ (note that 341 is **not** a prime).

2. Assuming the equality

$$m\binom{n}{m} = n\binom{n-1}{m-1}$$

previously proved when $0 \leq m \leq n$, prove that if r is a positive integer, then r divides $\binom{rm}{m}$. Hint: replace n by rm in the given formula.

3 Evaluate $\phi(2310)$ and $\phi((15)^3)$, where ϕ is the phi function.

4. Use Euler's Theorem to find the smallest positive integer b with $3^{404} \equiv b \bmod 1000$. Hence find the last three digits in the decimal expansion of 3^{404}.

5. Find the smallest positive integer x with $3^{25} \equiv x \bmod 25$.

6. Find the smallest positive integer that leaves a remainder of 14 on division by 15 and a remainder of 16 on division by 17.

7. Find a positive integer less than 60 which is divisible by 7 and leaves a remainder of 1 when divided by 17.

8. Find an integer solution x to the system of congruences

$$x \equiv 2 \bmod 5, \quad x \equiv 3 \bmod 7, \quad x \equiv 2 \bmod 12.$$

9. Find an integer x so that

$$x \equiv 3 \bmod 11, \quad x \equiv 6 \bmod 8, \quad x \equiv 14 \bmod 15.$$