1. As
\[ n \binom{n-1}{m-1} = m \binom{n}{m}, \]

\( n \) divides \( m \binom{n}{m} \). Now we are assuming that \( m \) and \( n \) are relatively prime and so it follows that \( n \) must divide \( \binom{n}{m} \). Note that what we have proved may not be true if \( \gcd(m, n) > 1 \).

2. Using the Euclidian algorithm for 31 and 41, we have \( 1 = 4 \times 31 - 3 \times 41 \). This means that
\[ 31 \times 4 \equiv 1 \pmod{41}. \]

Multiplying by 3,
\[ 31 \times 12 \equiv 3 \pmod{41} \]

and we therefore take \( x = 12 \).

3. Using the Euclidian algorithm for 317 and 409, we have \( 1 = 40 \times 317 - 31 \times 409 \). This means that
\[ 317 \times 40 \equiv 1 \pmod{409}. \]

Multiplying by 3,
\[ 317 \times 120 \equiv 3 \pmod{409} \]

and we therefore take \( x = 120 \).

4. Suppose that \( \gcd(b, c) = 1 \) and let \( d = \gcd(b^m, c^n) \). Suppose that \( d > 1 \). Then there is a prime \( p \) dividing \( d \) which divides both \( b^m \) and \( c^n \). However, as \( p \) is a prime, if \( p \) divides \( b^m \), \( p \) divides \( b \). Likewise, if \( p \) divides \( c^n \), \( p \) divides \( c \). But then \( p \) is a common divisor of \( b \) and \( c \), contradicting \( \gcd(b, c) = 1 \). Therefore, \( d = 1 \).

5. As
\[ \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}, \]

all three binomial coefficients cannot be odd, for the sum of two odd numbers is even.

6. There does not seem to be a quick way to do this question. We calculate as follows:
\[ 2^6 \equiv 64 \equiv 17 \pmod{47}, \quad 2^{12} \equiv 17^2 \equiv 289 \equiv 7 \pmod{47} \]
\[ 2^{18} \equiv 17 \times 7 \equiv 119 \equiv 25 \pmod{47}, \quad 2^{20} \equiv 100 \equiv 6 \pmod{47} \]

We therefore take \( x = 6 \).

7. The order is a divisor of 30. Note that
\[ 3^5 \equiv 243 \equiv -5 \pmod{31}, \quad 3^{10} \equiv 25 \pmod{32}, \quad 3^{15} \equiv -125 \equiv -1 \pmod{31}. \]
As the order of 3 modulo 31 is not 2 or 3, these calculations show that 3 must have order 30 modulo 31.

8. We have

$$a^{p-1} \equiv 1 \mod p$$

or equivalently,

$$a^{p-1} - 1 \equiv 0 \mod p.$$ 

Thus $p$ divides $a^{p-1} - 1$. But $a^{p-1} - 1$ factorizes as

$$a^{p-1} - 1 = (a - 1)(a^{p-2} + a^{p-3} + \cdots + a + 1)$$

and hence $p$ divides this product. But as $p$ is a prime, $p$ must divide one of the two factors above. However, $p$ cannot divide $a - 1$, as we have excluded the possibility $a \equiv 1 \mod p$. Hence the other possibility holds, meaning that

$$a^{p-2} + a^{p-3} + \cdots + a + 1 \equiv 0 \mod p.$$ 

9. We have

$$2^{p-1} \equiv 1 \mod p$$

and since $p - 1$ is even,

$$2^{(p-1)/2} = 4^{(p-1)/2} \equiv 1 \mod p.$$ 

This implies that the order of 4 modulo $p$ is a divisor of $(p-1)/2$.

10. As $n$ has order 2 modulo $p$, $n^2 \equiv 1 \mod p$. This means $p$ divides $n^2 - 1 = (n - 1)(n + 1)$. As $p$ is a prime, $p$ divides either $n - 1$ or $n + 1$. Now if $p$ divides $n - 1$, $n \equiv 1 \mod p$ and this means that $n$ has order 1 modulo $p$. Since $n$ has order 2, we must have the other case, namely, $p$ divides $n + 1$, or equivalently, $n \equiv -1 \mod p$. 