First Arts Modular Degree
Mathematical Studies 2004–2005

Combinatorics and Number Theory Solution Sheet 3

1. The general term in the expansion of

$$\left(x^2 - \frac{3}{x}\right)^{10}$$

is

$$\binom{10}{r}(x^2)^{10-r}\left(\frac{-3}{x}\right)^r = (-3)^r\binom{10}{r}x^{20-3r}.$$

The power of x that appears here is $20-3r$. This equals 11 when $r = 3$. Thus the coefficient of x^{11} is

$$(-3)^3\binom{10}{3} = -27 \times 120 = -3240.$$

2. Note that

$$(x+2)^3(x-2)^5 = (x+2)^3(x-2)^3(x-2)^2 = (x^2-4)^3(x-2)^2.$$

We then have

$$(x^2-4)^3(x-2)^2 = (x^6 - 12x^4 + 48x^2 - 64)(x^2 - 4x + 4)$$

and this tells us that the coefficient of x^3 is $48 \times -4 = -192$.

3.

$$(1 + 2x(1+x))^5 = 1 + 5 \times 2x(1+x) + 10 \times 4x^2(1+x)^2 + \cdots$$

The expansion above tells us that we obtain x^2 in two ways in this expansion, with coefficient $10 + 40 = 50$.

4. The sum of the geometric series is given by

$$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

Setting $z = 1 + x$, our series sums to

$$\begin{aligned}\frac{(1+x)^{n+1} - 1}{x} &= \frac{1 + (n+1)x + \binom{n+1}{2}x^2 + \cdots}{x} \\ &= n + 1 + \binom{n+1}{2}x + \cdots\end{aligned}$$

The coefficient of x is thus

$$\binom{n+1}{2} = \frac{n(n+1)}{2}.$$

5. The gcd of 69 and 117 is 3. For working out s and t, we summarize the calculations as follows:

$$3 = 21 - 3 \times 6, \quad , \quad 6 = 48 - 2 \times 21, \quad 21 = 69 - 48, \quad 48 = 117 - 69.$$

Substituting back,

$$3 = 7 \times 21 - 3 \times 48, \quad 3 = 7 \times 69 - 10 \times 48, \quad 3 = 17 \times 69 - 10 \times 117.$$

Thus we can take $s = 17$ and $t = -10$.

6. The gcd of 312 and 1084 is 4. For working out s and t, we summarize the calculations as follows:

$$4 = 148 - 9 \times 16, \quad 16 = 312 - 2 \times 148, \quad 148 = 1084 - 3 \times 312.$$

Putting this together, we get

$$4 = 19 \times 148 - 9 \times 312, \quad 4 = 19 \times 1084 - 66 \times 312,$$

so that $s = -66$ and $t = 19$ here.

7. The gcd of 594 and 781 is 11. Omitting the details of the calculation, we obtain

$$11 = 25 \times 594 - 19 \times 781.$$

8. $111,111$ is not a prime, as it is divisible by 11. In fact,

$$111,111 = 11 \times 10,101.$$

The number 11,111 is a prime but it is quite slow check this.

9. Suppose that 3 divides $c - 1$. Then we can write $c = 1 + 3d$ for some integer d and hence

$$\begin{aligned} c^2 + c + 1 &= 1 + 6d + 9d^2 + 1 + 3d + 1 \\ &= 3 + 9(d + d^2). \end{aligned}$$

This means that $c^2 + c + 1 \equiv 3 \bmod 9$, so that 3 divides $c^2 + c + 1$ but 9 does not.

10. Let $d = \gcd(c + 1, c^2 + 1)$. Note that

$$c^2 + 1 = (c + 1)(c - 1) + 2.$$

Since d divides $c^2 + 1$ and $c + 1$, it divides 2. Thus $d = 1$ or $d = 2$. However, as c is assumed to be even, $c + 1$ is odd, and hence d must be odd. Thus $d = 1$.