First Arts Modular Degree

Mathematical Studies 2004–2005

Combinatorics and Number Theory Solution Sheet 2

2. We want to show that

$$m\binom{n}{m} = n\binom{n-1}{m-1}.$$

Now

$$m\binom{n}{m} = m\frac{n!}{m!(n-m)!} = \frac{n!}{(m-1)!(n-m)!}$$

and

$$n\binom{n-1}{m-1} = \frac{n(n-1)!}{(m-1)!(n-1-(m-1))!} = \frac{n!}{(m-1)!(n-m)!}$$

As these two expressions are the same, we have the desired equality.

3. We have to solve

$$\binom{n}{3} = 7 \times \binom{n}{2}$$

and this gives

$$\frac{n(n-1)(n-2)}{6} = 7 \times \frac{n(n-1)}{2}.$$

We can cancel $n(n-1)$ from each side and then obtain $n-2=21$. Hence $n=23$.

4.

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$$

and

$$2\binom{2n-1}{n} = \frac{2\times(2n-1)!}{n!\,(n-1)!} = \frac{2n\times(2n-1)!}{n\times(n-1)!\,n!} = \frac{(2n)!}{n!\,n!}.$$

As these two formulae are the same, we have the desired equality. For the final part, as $\binom{2n-1}{n}$ is an integer,

$$\binom{2n}{n} = 2\binom{2n-1}{n}$$

is an even integer.

5. For the first part, the number of rearrangements is

$$\frac{11\,!}{2\,!}.$$

For the second part, we treat the two T's as one letter. Then we effectively have 10 different letters and hence $10\,!$ of the required rearrangements. Finally, choose 5 numbers between 1 and 11. This can be done in $\binom{11}{5}$ ways. Place the A in the position corresponding to the smallest number, E in the position corresponding to the next smallest number, and so on. There are 6 positions still to fill. Choose 2 numbers from the 6 remaining and put the 2 T's in these positions. This can be done in $\binom{6}{2}$ ways. Then we can assign the remaining 4 different letters to the 4 available positions in $4\,!$ ways. Total number of rearrangements is

$$\binom{11}{6}\binom{6}{2} \times 4\,! = \frac{11\,!}{5\,!!6\,!} \times \frac{6\,!}{2\,!4\,!} \times 4\,! = \frac{11\,!}{2\times 5\,!}$$

The answer is 120 times smaller than the first answer.

6. There are 10! ways of arranging 10 different books. It is obvious that in exactly half of this number, A is to the left of B (and in the other half, B is to the left of A. For the second part, we treat A and B as tied together to form AB. Then we are effectively arranging 9 books and we have 9! arrangements of the second kind.

7. We first consider numbers whose digit are 7, 8 and X, where X is different from 7 and 8. (We can allow X to be 0, representing 78 as 078 and 87 as 087). There are 8 choices for X and then we can arrange 7, 8 and X in $3! = 6$ ways. This gives 48 numbers with 3 different digits. Finally, we have the additional 3 numbers

$$788, \qquad 878, \qquad 887$$

and so the total is $48 + 3 = 51$.

8. The first answer is

$$\binom{12}{4}.$$

Now if there are equal numbers of men and women on the committee, there must be 2 men and 2 women. We can choose such a committee in

$$\binom{6}{2} \times \binom{6}{2} = 225$$

ways. It is clear that as there are equal numbers of men and women, among the committees where there are not equal numbers of men and women, there are as many committees with more men present as there are committees with more women present. So if s is the required number,

$$\binom{12}{4} = 225 + 2s$$

and hence $s = 135$.

9. We have 3 1's, 3 2's, 2 3's, 2 4's and thus can form

$$\frac{10!}{3! \times 3! \times 2! \times 2!}$$

numbers. For the second part, we place 1's at the beginning and end of the number and then distribute one 1, 3 2's, 2 3's and 2 4's into the remaining 8 positions to obtain

$$\frac{8!}{3! \times 2! \times 2!}$$

numbers.

10. There are 8! ways of arranging the exams. Treating the two mathematics papers as one, there are $2 \times 7!$ ways of arranging the exams so that mathematics papers are consecutive (the factor of 2 comes from interchanging the two papers). Answer is $8! - 2 \times 7! = 6 \times 7!$.