First Arts Modular Degree

Mathematical Studies 2004–2005

Combinatorics and Number Theory Solution Sheet 1

1. The formula is true when $n = 1$, since

$$1^2 = 1 = \frac{1 \times 2 \times 3}{6}.$$

Assume now that the sum formula is true when $n = r$ and then try to prove that

$$1^2 + 2^2 + \cdots + (r+1)^2 = \frac{(r+1)(r+1+1)(2(r+1)+1)}{6} = \frac{(r+1)(r+2)(2r+3)}{6}$$

Now by induction, the left hand side above is

$$\frac{r(r+1)(2r+1)}{6} + (r+1)^2 = (r+1)\frac{(2r^2+r+6r+6)}{6} = \frac{(r+1)(r+2)(2r+3)}{6},$$

as required.

2. We wish to prove that

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}.$$

First check that the formula holds for $n = 1$. In this case, the left hand side is $1 + x$ and the right hand side is $(x^2-1)/x-1$, which also equals $1+x$, as required. Assume now that the sum formula is true when $n = r$ and then try to prove that

$$1 + x + x^2 + \cdots + x^{r+1} = \frac{x^{r+2}-1}{x-1}.$$

Now by induction, the left hand side above is

$$\frac{x^{r+1}-1}{x-1} + x^{r+1} = \frac{x^{r+1}-1+x^{r+2}-x^{r+1}}{x-1} = \frac{x^{r+2}-1}{x-1},$$

which equals the required expression on the right hand side. Thus the formula is true for all $n \geq 1$.

3. Note that $a_1 = 1$ is an odd integer. Assume by induction that a_r is an odd integer. Now

$$\frac{a_{r+1}}{a_r} = \frac{(2r+2)!}{2^{r+1}(r+1)!} \times \frac{2^r r!}{(2r)!} = 2r+1.$$

Therefore, $a_{r+1} = (2r+1)a_r$ is an odd positive integer, as it is the product of two odd positive integers.

4. Write

$$a_n = \left(1 - \frac{1}{4}\right) \times \left(1 - \frac{1}{9}\right) \times \cdots \times \left(1 - \frac{1}{n^2}\right)$$

for $n \geq 2$. Now the formula for a_n is correct when $n = 2$, since

$$a_2 = \frac{3}{4} = \frac{2+1}{2 \times 2}$$

Now assume that $a_r = \frac{r+1}{2r}$ and try to prove that $a_{r+1} = \frac{r+2}{2(r+1)}$. But it is clear that

$$a_{r+1} = a_r \times \left(1 - \frac{1}{(r+1)^2}\right) = \frac{r+1}{2r} \times \frac{(r+1)^2 - 1}{(r+1)^2} = \frac{r+2}{2(r+1)},$$

as required.

5. The proposition is true when $n = 1$, since $a_1 = 1 = \frac{1}{2}(3^{1-1} + 1)$. Assume now that $a_r = \frac{1}{2}(3^{r-1} + 1)$ and try then to prove that $a_{r+1} = \frac{1}{2}(3^r + 1)$. But

$$a_{r+1} = 3a_r - 1 = \frac{3}{2}(3^{r-1} + 1) - 1 = \frac{1}{2}(3^r + 1),$$

as required.

6. Let us set $a_n = 2^{2n} - 3n - 1$. We want to prove that 9 exactly divides a_n. When $n = 1$, $a_1 = 0$ and 9 certainly exactly divides 0. Assume by induction that 9 exactly divides a_r and then try to prove that 9 exactly divides a_{r+1}. Now

$$a_{r+1} - 4a_r = 2^{r+2} - 3(r+1) - 1 - 2^{r+2} + 12r + 4 = 9r.$$

Hence $a_{r+1} = 4a_r + 9r$. Since 9 divides a_r exactly, it also divides $4a_r$, and 9 clearly divides $9r$ exactly. Hence we see that 9 also divides $4a_r + 9r = a_{r+1}$, as required.

7. We can form $7! = 5040$ integers by permuting the 7 digits. To do the second part, we proceed more slowly. If we make the first digit 5, we can form 6! integers. The same is true if the first digit is 6 or 7. This gives us $3 \times 6!$ integers between 5,000,000 and 7,999,999. Now we want to count those integers with first digit 8 that are less than 8,700,000. This can be done as follows. We can form 6! integers whose first digit is 8. Of these, the integers greater than 8,700,000 have second digit equal to 7 or to 9. With first digit 8 and second 7, we can form 5! integers, and the same is true for those integers with first digit 8 and second digit 9. Excluding these $2 \times 5!$ integers, we have $6! - 2 \times 5!$ integers in the correct range. The total number is

$$3 \times 6! + 6! - 2 \times 5! = 2,640.$$

8. We treat a, b, c, and d as a single letter. Then we are effectively permuting four letters and we have $4! = 24$ permutations in which the four letters are in the given order. If we allow a, b, c and d to be in a group but not in any prescribed order in that group, we obtain $4! = 24$ rearrangements of the letters in the group and hence $24 \times 24 = 576$ possible rearrangements.

9. There are 24 different integers which we can form. Let's look at the last digit of each of the numbers. It's fairly clear that each digit occurs 6 times as last digit of these 24 integers. The sum of the last digits is

$$6 \times (1 + 2 + 3 + 4) = 60.$$

The same holds for the second last digit, second digit and first digit. So, taking into account units, tens, hundreds and thousands, the sum of all the 24 integers is

$$60 + 10 \times 60 + 100 \times 60 + 1000 \times 60 = 66,660.$$