ON THE EXISTENCE OF SPECIAL TYPES OF p-BLOCKS IN p-SOLVABLE GROUPS

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1. *Introduction*

Let G be a finite group and let p be a prime divisor of $|G|$. Let $O_{p'}(G)$ denote the largest normal p'-subgroup of G and $O_p(G)$ the largest normal p-subgroup of G. Let F be an algebraically closed field of characteristic p and let FG denote the group algebra of G over F. Let $Z(FG)$ denote the centre of FG.

We recall that the p-blocks of FG may be identified with the primitive central idempotents of FG. Given a p-block B of FG, determined by the primitive idempotent $e \in Z(FG)$, there is an associated algebra homomorphism $\lambda = \lambda_B : Z(FG) \to F$ satisfying $\lambda(e) = 1$. We call λ the central character of B. Let K be a conjugacy class of G and let $K^+ \in Z(FG)$ be the sum of the elements in K. We say that K is a defect class for B if $\lambda(K^+) \neq 0$ and K^+ occurs with non-zero coefficient in the expression for e in terms of the basis of $Z(FG)$ consisting of conjugacy class sums. If K is a defect class for B and D is a Sylow p-subgroup of the centralizer of any element of K, we call D a defect group of B. It is a basic theorem of block theory that a different choice of defect class leads to a conjugate defect group. If $|D| = p^d$, the integer d is called the defect of B.

Suppose that $O_{p'}(G)$ is non-trivial and let K be a conjugacy class of G contained in $O_{p'}(G)$. It is a theorem of Tsushima, [], that K is a defect class for a p-block of FG. More precisely, if $O_{p'}(G)$ contains exactly r conjugacy classes of G, FG has r p-blocks with the property that each of these blocks has a different defect class contained in $O_{p'}(G)$. Furthermore, in the case that

G is p-solvable, FG has a unique p-block, namely the principal block, if and only if $O_{p'}(G)$ is trivial. It follows then that if G is p-solvable and FG has a non-principal p-block B, FG has at least one non-principal p-block B' with a defect class contained in $O_{p'}(G)$. It is the intention of this paper to give an upper bound for the defect of B' in terms of the defect of B (we do not know of any canonical association between B and B' and thus our results are largely existential).

In a previous paper, [], we have shown that if $p = 2$ and G is 2-solvable and if FG has a 2-block whose defect is less than maximal, then FG has a real non-principal 2-block. The results of this paper enable us to obtain an estimate of the defect of such a real 2-block provided FG has a 2-block whose defect is no bigger than half the maximal possible defect. We remark that real 2-blocks have additional properties, including the concept of an extended defect group, which are not enjoyed by non-real 2-blocks. (A real block is one whose defining central idempotent is fixed by the involutory automorphism of FG generated by the map $x \to x^{-1}$ in G.)

2. *Construction of certain p-blocks*

We will make use of the following theorem of Knörr, [], to find elements of bounded defect in $O_{p'}(G)$.

Lemma 1. *Let G be a finite group and let N be a normal subgroup of G. Let B be a p-block of FG with defect group D and let b be a p-block of FN covered by B (in the sense of block theory). Then $N \cap D$ is a defect group of b. In particular, the defect of b does not exceed the defect of B.*

Using the standard notation, we define the normal subgroup $O_{p'p}(G)$ by

$$O_{p'p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G)).$$

We require the following consequence of the Hall-Higman Lemma 1.2.3, proved in Lemma 3 of []. (We note in the statement of Lemma 3 in [], the hypothesis that $O_{p'p}(G)$ should be isomorphic to $O_{p'}(G) \times O_p(G)$ is irrelevant.)

Lemma 2. *Let G be a p-solvable group and let p^a be the p-part of $|G|$. Let p^b be the p-part of $O_{p'p}(G)$. Then we have*

$$a - b \leq f(p, b),$$

where

$$\begin{aligned} f(p,b) &= b - 1 \text{ if } p = 2; \\ &= bp/(p-1)^2 \text{ if } p \text{ is a Fermat prime}; \\ &= b/p - 1 \text{ otherwise.} \end{aligned}$$

We can now proceed to the proof of our main result.

Theorem 1. *Let G be a p-solvable group and let p^a be the p-part of $|G|$. Suppose that FG has a p-block of defect d. Define a' by*

$$\begin{aligned} a' &= (a-1)/2 \text{ if } p = 2; \\ &= ap/(p^2 - p + 1) \text{ if } p \text{ is a Fermat prime}; \\ &= a/p \text{ otherwise.} \end{aligned}$$

Then FG has a p-block B' of defect at most $d + a'$ with the property that B' has a defect class contained in $O_{p'}(G)$.

Proof. We set $N = O_{p'p}(G)$. By Lemma 1, FN has a block of defect at most d. Thus N contains a p-regular conjugacy class K, say, which has defect at most d in N. Since N is p-nilpotent, K is contained in $O_{p'}(G)$. Let x be an element of K. Then the p-part of $|C_N(x)|$ is at most p^d. It follows that the p-part of $|C_G(x)|$ is at most p^{d+a-b}, where p^b is the p-part of $|N|$. If we use Lemma 2 to estimate $a - b$ in terms of b, we obtain the inequality $a - b \leq a'$, where a' is defined as above. Thus $O_{p'}(G)$ contains a conjugacy class of G

whose defect in G is at most $d+a'$. Tsushima's theorem now implies that FG has a p-block B' with defect class K and the result follows. ■

Corollary 1. *Let G be a 2-solvable group and let 2^a be the 2-part of $|G|$. Suppose that FG has a 2-block of defect d, where $d < (a+1)/2$. Then FG has a real non-principal 2-block of defect at most $d + (a-1)/2$ with a defect class contained in $O_{2'}(G)$.*

Proof. We have seen from the proof of Theorem 1 that under the given hypotheses, $O_{2'}(G)$ contains a conjugacy class K whose defect is at most $d+(a-1)/2$, which is less than a. Theorem 5.8 of [] now shows that FG has a real 2-block for which K is a defect class, and this block is not the principal block, as its defect is less than a. ■

It is clear that Corollary 1 is an existential theorem that gives no obvious connection between the original 2-block and the constructed real 2-block. It may, however, be of interest to look at an example which shows that our result has an appropriate qualitative aspect, even if it is quantitatively imprecise. Let G be the split extension of an elementary abelian group of order 9 by its full automorphism group $\mathrm{GL}_2(3)$. G is a solvable group of order $2^4 3^3$ and $O_{2'}(G)$ is elementary abelian of order 9. Each element of $O_{2'}(G)$ has 2-defect 1. It may be calculated that FG has exactly three 2-blocks, namely the principal block, a block of defect 1 containing two complex characters each of degree 8, and a block B of defect 0 containing an irreducible complex character χ of degree 16. Now let H denote the direct product of r copies of G. The 2-part of $|H|$ is then 2^{4r} and H has a (unique) 2-block of defect 0. Corollary 1 implies that FG has a real 2-block of defect at most $2r-1$ that is weakly regular with respect to $O_{2'}(G)$. As it is easy to see that each element of $O_{2'}(H)$ has 2-defect at least r, and there are elements $O_{2'}(H)$ that have 2-defect exactly r, it follows that any 2-block of FH that is weakly regular with respect to

$O_{2'}(H)$ has defect at least r and there is such a 2-block of defect exactly r.

Suppose now that G is an arbitary group and that B is a 2-block of FG with defect group D. The present author and J. C. Murray have shown in [] that, provided $N_G(D)/D$ has no subgroup isomorphic to a dihedral group of order 8, then FG has a real 2-block with the same defect group. We would like to finish this paper by showing how a related, more precise result can be obtained in the context of 2-solvable groups. The proof is based on the methods used earlier in this paper.

We first prove what must be well-known results.

LEMMA 3. *Let G be a finite group and let p be a prime divisor of $|G|$. Suppose that a Sylow p-subgroup P of G is normal in G. Then we have $C_G(P) = Z(P) \times O_{p'}(G)$, where $Z(P)$ denotes the centre of P.*

Proof. We set $C = C_G(P)$. C is certainly normal in G and it contains both $Z(P)$ and $O_{p'}(G)$. Now a Sylow p-subgroup of C is contained in P and hence equals $Z(P)$. Let H be a Sylow p-complement of $Z(P)$ in C. H centralizes $Z(P)$, as it centralizes P, and thus H is normal in C. It follows that H is also normal in G and thus is contained in $O_{p'}(G)$. Since, however, $O_{p'}(G)$ is also contained in H, we have the equality $H = O_{p'}(G)$. ∎

LEMMA 4. *Let G be a finite group and let D be a 2-subgroup of G. Let h be a real non-identity 2-regular element of G with defect group D. Suppose that $N_G(D)/D$ is 2-solvable of 2-length 1. Then $h \in O_{2'}(N_G(D))$.*

Proof. We set $N = N_G(D)$. We claim that h is real in N. For D has index 2 in a Sylow 2-subgroup E, say, of the extended centralizer of h in G and thus E is contained in N and the elements of $E \setminus D$ invert h. This establishes our claim. Now hD is a real 2-regular element of N/D and as N/D has 2-length 1, it follows that $hD \in O_{2'}(N/D)$. Set $H/D = O_{2'}(N/D)$. Then H

is normal in N and has a normal Sylow 2-subgroup D. Since h is in H and centralizes D, Lemma 2 implies that $h \in O_{2'}(H)$. But as H is normal in N, it follows that $O_{2'}(H) \leq O_{2'}(N)$ and thus $h \in O_{2'}(N)$, as required. ■

We now use the Brauer's First Main Theorem to prove the existence of real 2-blocks with prescribed defect group D, provided $N_G(D)/D$ has the structure described in Lemma 3.

Theorem 2. *Let G be a finite group and let D be a 2-subgroup of G. Suppose that $N_G(D)/D$ is 2-solvable of 2-length 1. Then the number of real 2-blocks of FG with defect group D equals the number of real 2-regular conjugacy classes of G with defect group D.*

Proof. We set $N = N_G(D)$. Let $h_1, \ldots, h_r$ be representatives of all the real 2-regular conjugacy classes of G which have D as defect group. By Lemma 4, $h_1, \ldots, h_r$ are non-conjugate real 2-regular elements of $O_{2'}(N)$. It follows from Theorem 6.4 of [] that FN has r real 2-blocks with defect group D. As the Brauer correspondence is easily seen to map real 2-blocks of FN into real 2-blocks of FG, Brauer's First Main Theorem implies that FG has r real 2-blocks with defect group D. On the other hand, Lemma 3.1 of [] shows that FG has at most r real 2-blocks with defect group D. It follows that the number of real 2-blocks of FG with defect group D is exactly r. ■

We remark that the hypotheses of Theorem 2 automatically hold if D has index 2 in a Sylow 2-subgroup of G. They also hold if G is 2-solvable and $N(D)/D$ has an abelian Sylow 2-subgroup (or more generally if $N(D)/D$ is 2-solvable and has an abelian Sylow 2-subgroup). We conclude this paper by noting that the reality hypotheses which occur in Theorem 2 are quite natural in the case that $N(D)/D$ has 2-length 1, as the following consequence of [] shows.

THEOREM 3. *Let G be a finite group and let D be a 2-subgroup of G. Suppose that $N_G(D)/D$ is 2-solvable of 2-length 1. Then if FG has a 2-block with defect group D, it has a real 2-block with defect group D.*

Proof. Suppose that FG has a 2-block with defect group D. By Theorem 3.* of [], G has a real 2-regular class with defect group D. The result now follows from Theorem 2. ■