

Mathematics & Music: Symmetry & Symbiosis

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The Bray Heads U3A
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Outline

“The Two Cultures”

Pythagoras

Sinusoidal Waves

Musical Notation

Tuning

Johann Sebastian Bach

Canons & Fugues

Symmetry

Fascinating Rhythm

The Tonnetz

Finis



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Three Sweeping Statements: No. 1

Everyone Loves Music



Three Sweeping Statements: No. 2

Everyone Hates Maths



Three Sweeping Statements: No. 3

**Music and Mathematics
are
Inextricably Intertwined**



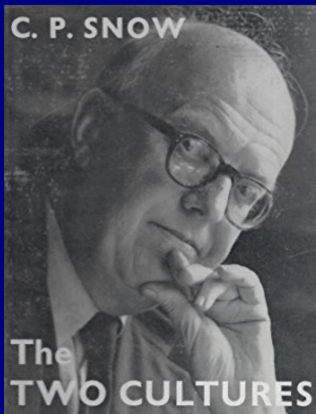
Love/Hate Image

Given the close link between music and maths,
how can we love one and hate the other?



Love/Hate Image

Given the close link between music and maths,
how can we love one and hate the other?



The concept of
The Two Cultures
was introduced by the
British scientist and novel-
ist **C. P. Snow**.

This concept remains
relevant today.

1959 Rede Lecture: *The Two Cultures*



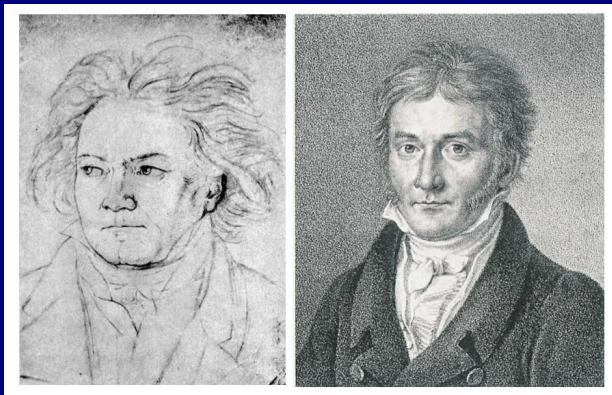
Who Can This Be?

Who Can This Be?



Who Can This Be?

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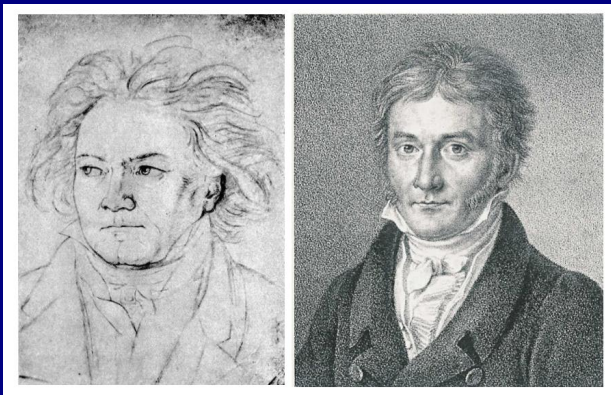


LEFT: Ludwig van Beethoven (1770–1827)



Who Can This Be?

Who Can This Be?



LEFT: Ludwig van Beethoven (1770–1827)

RIGHT: Carl Friedrich Gauss (1777–1855)



Beethoven / Gauss

Beethoven and Gauss were at the height of their creativity in the early 19th century.

The work of Gauss has a greater impact on our daily lives than the magnificent creations of Beethoven.

Yet, Beethoven is known to all, Gauss to only a few!

*“Of course I’ve heard of Beethoven,
but who is this Gauss dude?”*



Meaning and Content of Mathematics

The word **Mathematics** comes from Greek *μαθημα* (máthéma), meaning “knowledge” or “lesson” or “learning”.

It is the study of topics such as

- ▶ Quantity [Numbers: Arithmetic]
- ▶ Structure [Patterns: Algebra]
- ▶ Space [Geometry: Topology]
- ▶ Change [Analysis: Calculus]



British Mathematician G H Hardy:

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

Beauty is the first test: there is no permanent place in the world for ugly mathematics.



Two Parallel Languages

MUSIC	MATHS
Pitch	Frequency
Scales	Modular Arithmetic
Intervals	Logarithms
Overtones	Integers
Octave Identification	Equivalence Relation
Equal Temperament	Exponents

You know more mathematics than you realize!



Music & Maths

There are many parallels between music and maths:

Structure

Symmetry

Pattern

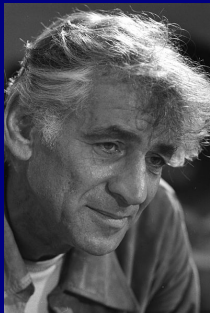
But music is **accessible to all** while maths is not.
Music gets into the soul through the **emotions**.

Maths is understood through the intellect.
Appreciation comes via a **rational route**.

Music has instant appeal. Maths takes time.



Leonard Bernstein



“Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?”



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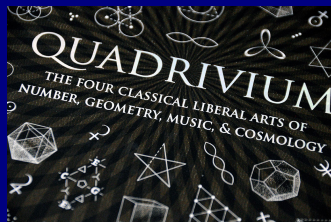
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The Quadrivium



The Pythagoreans organized their studies into the **Quadrivium**, comprising four disciplines:

- ▶ Arithmetic
- ▶ Music
- ▶ Geometry
- ▶ Astronomy



Static/Dynamic Number. Pure/Applied

- ▶ **Arithmetic:** Static number
- ▶ **Music:** Dynamic number

Arithmetic represents numbers at rest.

Music is numbers in motion.

Arithmetic is **pure or abstract** in nature.

Music is **applied or concrete** in nature.



Discovery of Pythagoras

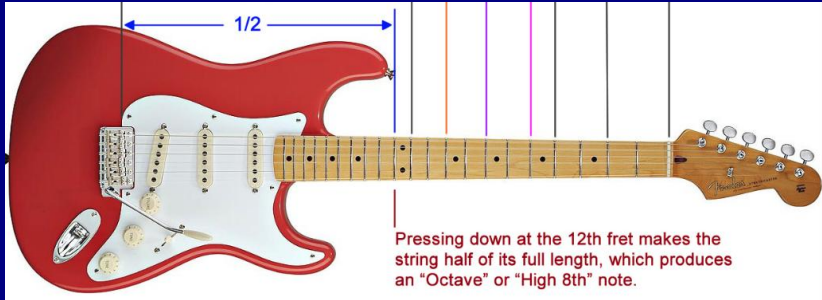
The Pythagoreans discovered
a remarkable connection:

**Ratios of small whole numbers are directly linked
with consonant or harmonically pleasing chords.**

*“There is geometry in the humming of the strings,
There is music in the spacing of the spheres.”*



Guitar Strings

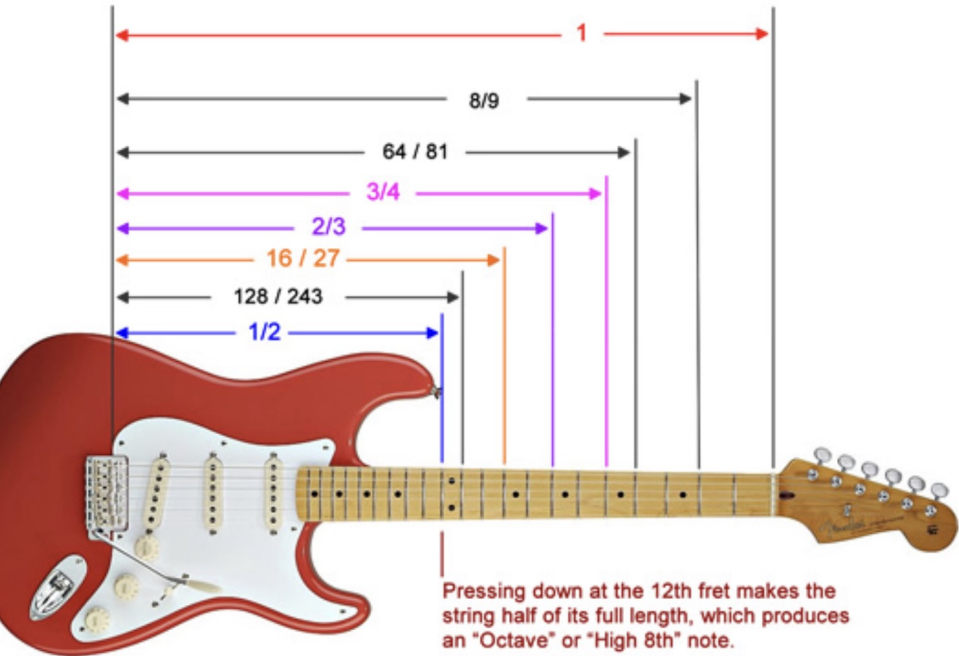


Open string vibrates at 264 Hz. Call this f .

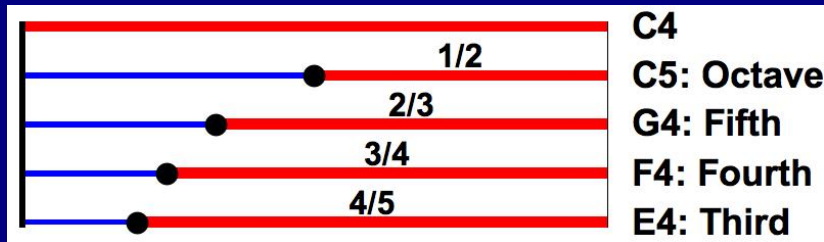
- ▶ String of length $\frac{1}{2}$ vibrates at $\frac{2}{1}f$.
- ▶ String of length $\frac{2}{3}$ vibrates at $\frac{3}{2}f$.
- ▶ String of length $\frac{3}{4}$ vibrates at $\frac{4}{3}f$.

Demonstrate the pitches on the uke.





The Monochord



Small-number ratios and harmonic intervals



Feynman on Pythagoras' Discovery



Richard Feynman

“The first example, outside geometry, of a numerical relationship in nature.”

Pythagoras made his discovery through **observation**. This aspect does not seem to have impressed him.

Had Pythagoras followed up on this idea,
“Physics might have had a much earlier start.”



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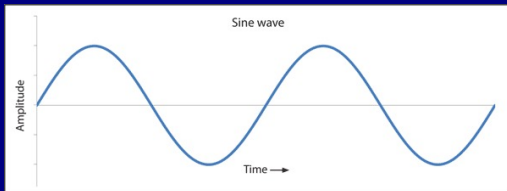
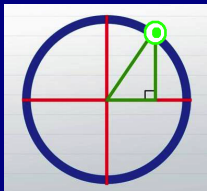
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Sine Waves and Circles



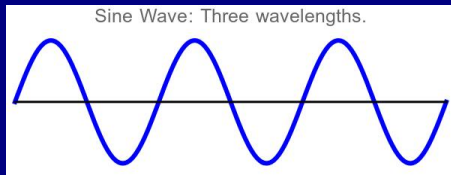
As the **green dot** moves around the circle,
its **height** traces a **sine wave** in time.

*Up and down and up and down and up and down
and up and down and up and down and ...*

Sine waves are a kind of **circular functions**.



Terminology

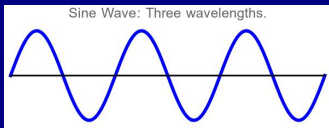


Music

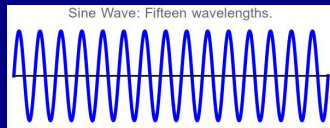
Maths

Loudness $\Leftrightarrow$ Amplitude

Pitch $\Leftrightarrow$ Frequency



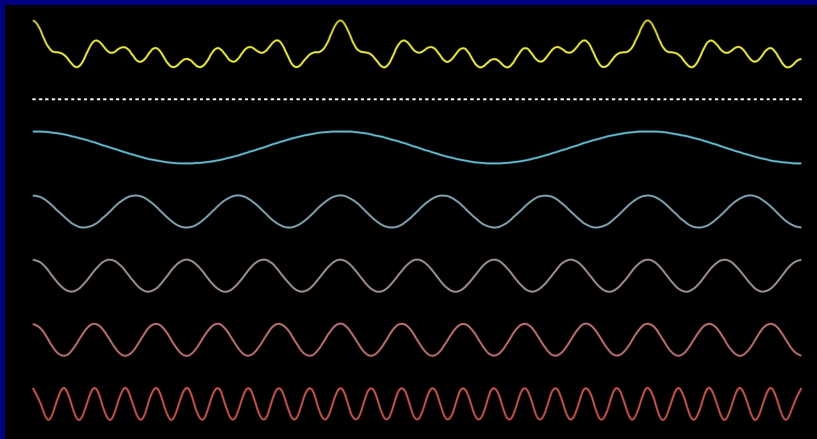
Low Pitch



High Pitch



Fourier Components — (or Harmonics)

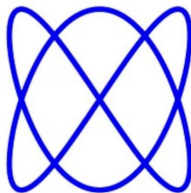


Lissajous Figures for Intervals

2:3. C-G



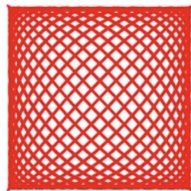
2:3. Perfect Fifth



16:17. C-C#



16:17. Semitone



A Harmonic Generator



[https://meettechniek.info/
additional/additive-synthesis.html](https://meettechniek.info/additional/additive-synthesis.html)

A website for generating sine waves and harmonics.

***** Click: Examples / Extra / Waveform Generator *****



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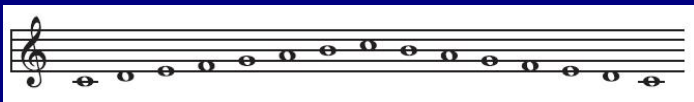
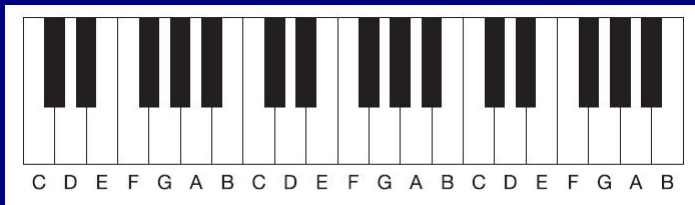
Fascinating Rhythm

The Tonnetz

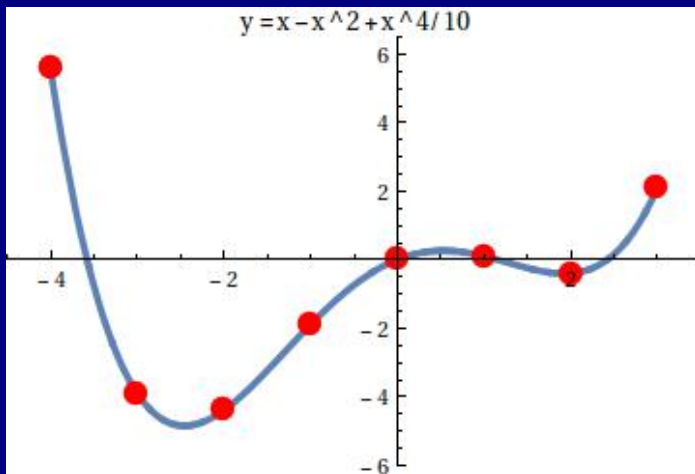
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Notating Musical Notes

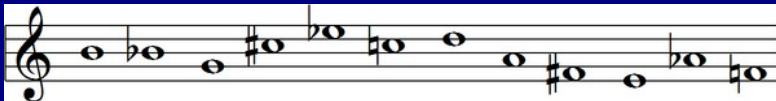


A Mathematical Graph: Joining the Dots

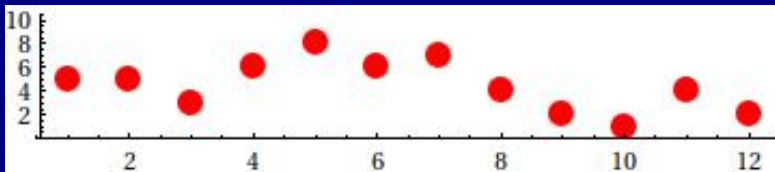


A plot of y versus x .

Music as a Graph



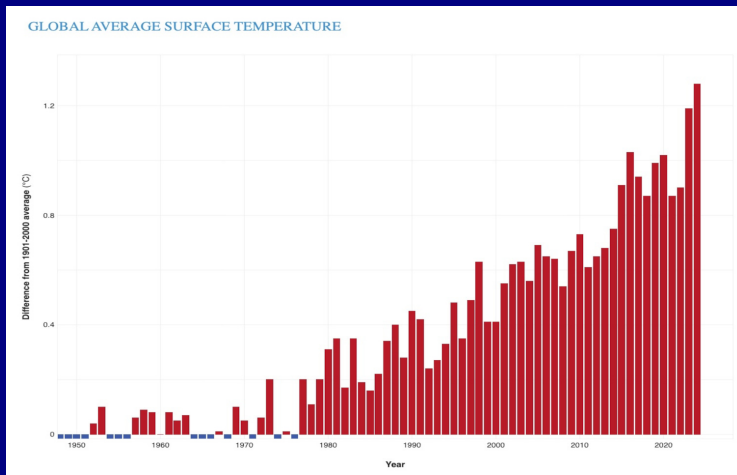
A sequence of musical notes: a simple tone row.



A musical score is just a graph of **pitch** *versus* time!



Symbiosis: Musing on a Graph



A new composition: *The Heat is On.*



Beethoven's *Moonlight Sonata*

MOONLIGHT SONATA
Op. 27, No. 2
(First Movement)
LUDWIG VAN BEETHOVEN

Adagio sostenuto

The image displays the first movement of Beethoven's Moonlight Sonata, Op. 27, No. 2. The score is in F# major (three sharps) and common time (C). It features a piano (pp) dynamic and a tempo marking of Adagio sostenuto. The first system shows the right hand playing a continuous triplet of eighth notes, with the left hand providing a simple harmonic accompaniment. The second system continues the right hand's triplet pattern, with the left hand playing a more active role, including a triplet of eighth notes. The score is written for piano and includes performance instructions like 'sempre pp' and 'simile'.

An ingenious and delightful combination
of precision (pitch) and vagueness (pp).



Precision and Vagueness



The standard **pitch** is A₄,
the A above Middle C.
Its frequency is **440 Hz**.

By contrast, the **time-scale** is written vaguely:
Adagio sostenuto.

The **loudness** is specified simply by *sempre pp*.

The **vagueness** allows the performer
freedom of interpretation and expression.



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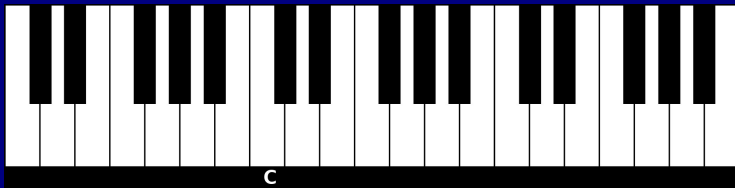
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The Piano Keyboard



Where do all the notes come from?



Pythagorean Tuning

Pythagoras discovered that a **perfect fifth** — with frequency ratio $3:2$ — is especially harmonious.

The entire musical scale can be constructed using only the ratios $2:1$ (octaves) and $3:2$ (fifths).

In the **tonic sol-fa** scale the eight notes of the major scale are **Do, Re, Mi, Fa, So, La, Ti, Do**.



Pythagorean and Just Intonation

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

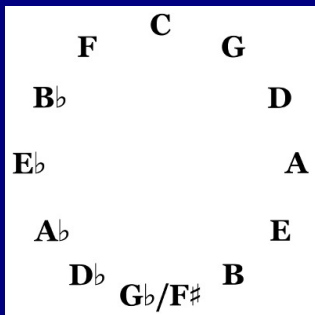
Pythagorean intonation.

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Just intonation.



Organizing Scheme: the Circle of Fifths



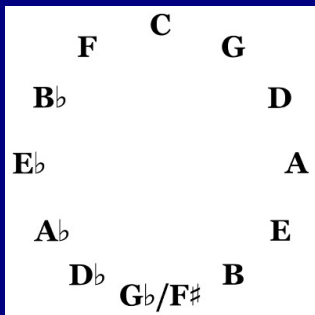
The **Circle of Fifths** represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

The Circle is useful in harmonising melodies and building chords.



Organizing Scheme: the Circle of Fifths



The **Circle of Fifths** represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

The Circle is useful in harmonising melodies and building chords.

PROBLEM: The Circle doesn't close!



The Well-Tempered Scale

As music developed, composers found it was not always possible to change smoothly between different keys without **dissonant consequences**.

Many tuning methods, improving on just intonation, were developed in the seventeenth century. These were known as **Well-Tempered Scales**.



Johann Sebastian Bach was the first major composer to employ the well-tempered scale.

The Well-Tempered Clavier (BWV 846-893) is a collection of preludes and fugues in all 24 keys.



Equal Temperament

Today, the **Equal Tempered Tuning** is universal.

Big Idea: Make all semitone intervals equal.

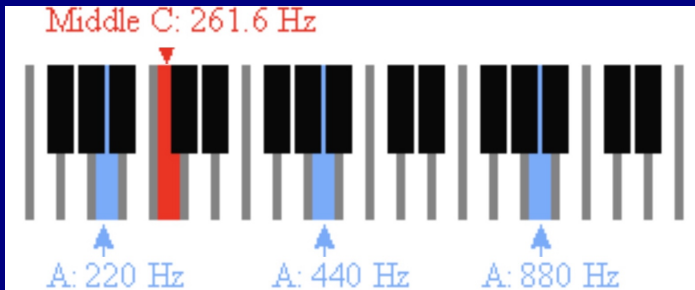
An octave has ratio 2:1. We need a number that yields 2 when multiplied by itself 12 times:

$$X = \sqrt[12]{2} \approx 1.059$$

All intervals of the equal-tempered scale are imperfect, but **close enough to be acceptable.**



Middle C commonly pitched at 261.63 Hz.



Middle C or **C4** is the 'central note' on the piano.
The standard frequency of the note **A4** is 440 Hz.

$$261.63 = 440 \div 2^{3/4}$$

Where does the peculiar factor $2^{3/4}$ come from?
It arises from the **equal temperament scale**.



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Johann Sebastian Bach & Mathematics

While not a mathematician, Bach thought like one.

He was fond of using geometric transformations, like **transposition, inversion, and rotation, and exploring their musical possibilities.**

He combined these mathematical transforms in creative and aesthetically pleasing ways ...

... with structural integrity and aesthetic beauty.



The Well-Tempered Clavier (1722)

Johann Sebastian Bach's **Well-Tempered Clavier** is a collection of preludes and fugues in all 24 keys.

Prelude in C major
BWV 846

Allegro ($\text{♩} = 112$)

J. S. Bach

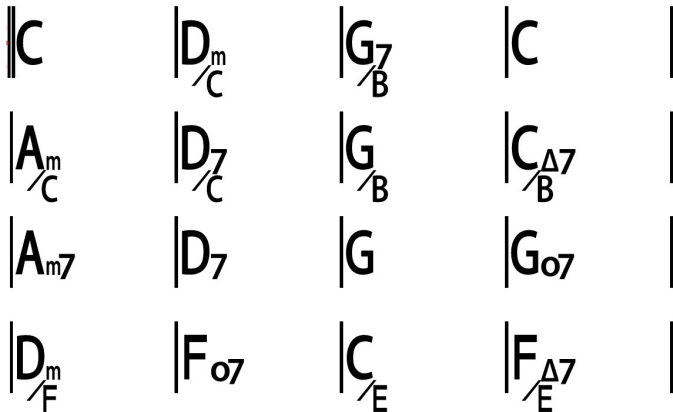
legato

cresc.

https://www.youtube.com/watch?v=frxT2qB1POQ&list=RDfrxT2qB1POQ&start_radio=1



Bach C Major Prelude Lead Sheet



Bach's *Prelude* & Gounod's *Ave Maria*

Bach's Prelude No. 1 in C major, BWV 846,
from Book I of *The Well-Tempered Clavier* ...

... a study of harmony in broken chords.

In 1853, some 130 years later, the French Romantic
composer Charles Gounod **superimposed a melody**,
a setting of the Latin prayer **Ave Maria**.

Recording by Fritz Kreisler and John McCormack

<https://www.youtube.com/watch?v=US41r9nlkq8&t=125s>



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The Geometry of Canons

The image shows a musical score for a canon in G major, 2/4 time. It consists of four staves, each representing a different voice. The first staff begins with the melody: Frè - re Jac - ques, Dor - mez - vous ? Son - nez les ma - ti - nes, Ding, daing, dong ! etc. The second staff enters with the same melody after a two-measure rest: Frè - re Jac - ques, Dor - mez - vous ? Son - nez les ma - ti - nes, etc. The third staff enters after a four-measure rest: Frè - re Jac - ques, Dor - mez - vous ? etc. The fourth staff enters after a six-measure rest: Frè - re Jac - ques, etc. The score illustrates the staggered entry of voices, a key feature of a canon.

A manuscript at the French National Library contains 86 canons by J.-P. Rameau, including Frère Jacques.



The Geometry of Canons

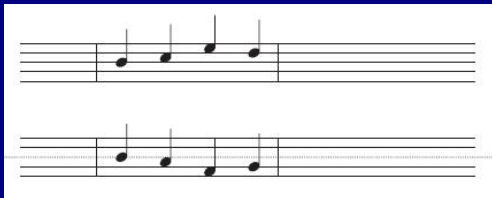
- ▶ Canon at the Unison
- ▶ Canon at an Interval
- ▶ Canon *contrario motu*
- ▶ Retrograde Canon
- ▶ Perpetual Canon

**Johann Sebastian Bach was the
grand master of canon form.**

**The transformations used in Canon Form
are described by mathematical **groups**.**

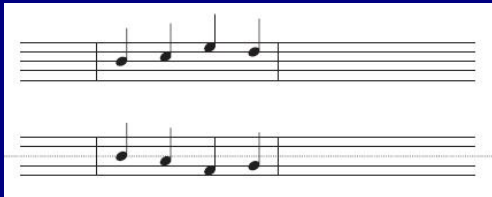


Simple Transformations

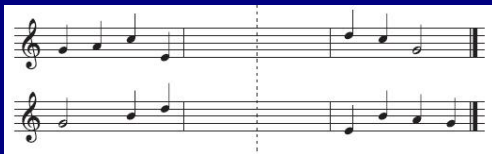


Inversion, or rotation about a horizontal line.

Simple Transformations



Inversion, or rotation about a horizontal line.



Retrogression, or rotation about a vertical line.



Bach's "Great" Fugue in G minor

Bach's Great Fantasia and Fugue for Organ in G minor, BWV 542 is among his most complex and marvellous compositions.

<https://www.youtube.com/watch?v=4WhPUqpaRp4&t=4s>

A comment on the YouTube Video:
Bach ist der größte aller Zeiten.



Bach's Crab Canon (if time allows)



Bach's Crab Canon (if time allows)



Crab Canon (for Alto & Bass Recorders)

CRAB CANON

from the *Musical Offering* (BWV 1079)

Arranged for Recorders by R. D. Tennent

J. S. Bach (1685–1750)

Alto Recorder

Bass Recorder



The image shows the first four measures of the Crab Canon for Alto and Bass Recorders. The Alto Recorder part is in the treble clef, and the Bass Recorder part is in the bass clef. Both are in 2/2 time and B-flat major (two flats). The Alto part starts with a half note B-flat, followed by a half note D, then a half note F, and a half note G. The Bass part starts with a half note B-flat, followed by a half note D, then a half note F, and a half note G. The Alto part has a repeat sign after the first measure, and the Bass part has a repeat sign after the first measure. The Alto part has a repeat sign after the second measure, and the Bass part has a repeat sign after the second measure. The Alto part has a repeat sign after the third measure, and the Bass part has a repeat sign after the third measure. The Alto part has a repeat sign after the fourth measure, and the Bass part has a repeat sign after the fourth measure.

Last four bars:

16



The image shows the last four measures of the Crab Canon for Alto and Bass Recorders. The Alto Recorder part is in the treble clef, and the Bass Recorder part is in the bass clef. Both are in 2/2 time and B-flat major (two flats). The Alto part starts with a half note B-flat, followed by a half note D, then a half note F, and a half note G. The Bass part starts with a half note B-flat, followed by a half note D, then a half note F, and a half note G. The Alto part has a repeat sign after the first measure, and the Bass part has a repeat sign after the first measure. The Alto part has a repeat sign after the second measure, and the Bass part has a repeat sign after the second measure. The Alto part has a repeat sign after the third measure, and the Bass part has a repeat sign after the third measure. The Alto part has a repeat sign after the fourth measure, and the Bass part has a repeat sign after the fourth measure.



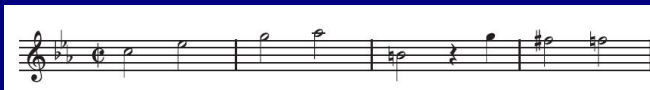
“A Musical Offering”

Frederick the Great provided Bach with a theme:



“A Musical Offering”

Frederick the Great provided Bach with a theme:



Bach worked this into *A Musical Offering*, a collection of ten canons, a sonata and two fugues.

The image shows a musical score for three instruments: Violino I, Violino II, and Continuo. The Continuo part is highlighted with a red box. The score is in 3/4 time and features a key signature of one flat (B-flat). The Continuo part consists of a single line of music, while the Violino I and Violino II parts are staves with multiple lines of music. The Continuo part is marked with a red box, indicating it is the focus of the analysis.

The work has several riddles and hidden jokes.



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Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- ▶ At the **sub-atomic scale**, symmetry reigns.
- ▶ **Viruses** can display stunning symmetries.
- ▶ **Crystals** have great symmetry.
- ▶ **Our bodies** have bilateral symmetry.
- ▶ Many **buildings** are symmetric.
- ▶ **Galaxies** have many symmetries.



The Taj Mahal



A Face with Symmetry: Halle Berry



Halle Berry

Berry Halle



Bach's Love of Symmetry

J S Bach was deeply familiar with symmetry.



This is illustrated by the seal he designed in 1722.



Bach's Love of Symmetry

J S Bach was deeply familiar with symmetry.



This is illustrated by the seal he designed in 1722.



Symmetry and Group Theory

Symmetry is an essentially **geometric** concept.

The mathematical theory of symmetry is **algebraic**.
The key concept is that of a **group**.

A group is a set of elements such that any two elements can be combined to produce another.

Instead of giving the mathematical **definition**,
I give an **example** to make things clear.



The Klein 4-Group

The four orientations of a book can be described in terms of four simple rotations:

- ▶ **P**: Place book upright with front cover upright
- ▶ **R**: Rotate 180° about vertical through centre
- ▶ **I**: Rotate 180° about horizontal through centre
- ▶ **RI**: Rotate 180° about perp. through centre

These operations make up the **Klein 4-Group**.



Twelve-tone Music

Table: Klein 4-Group.

	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

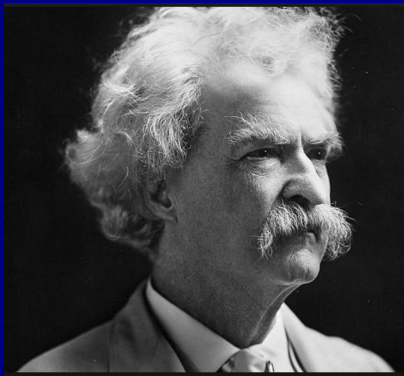
The Klein 4-group is the basic group of transformations in **twelve tone music**.

The operations are retrogression (R), inversion (I) and the rotation (RI).

The image displays a musical score for a twelve-tone row and its transformations. The score is written on a grand staff with two staves. The first staff is labeled 'P' (Prime) and the second staff is labeled 'I' (Inversion). The third staff is labeled 'R' (Retrogression) and the fourth staff is labeled 'RI' (Rotation). The notation shows the original row and its transformations using various musical symbols, including notes, rests, and accidentals.



Mark Twain on Wagner



***“Wagner’s music
is much better
than it sounds.”***

**I am not aware whether Mark Twain ever
commented on Arnold Schoenberg’s music !**



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Leibniz on Music

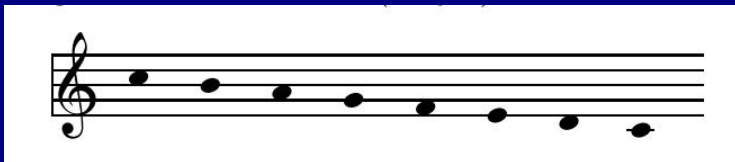


Gottfried Wilhelm Leibniz
(1646–1716)

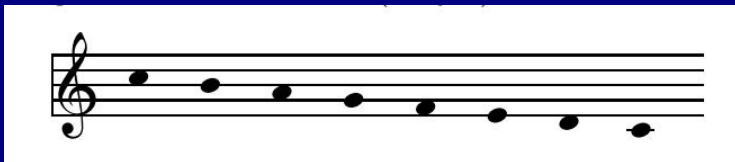
“Music is the pleasure the human mind experiences from counting without being aware that it is counting.”



Rhythm in Music



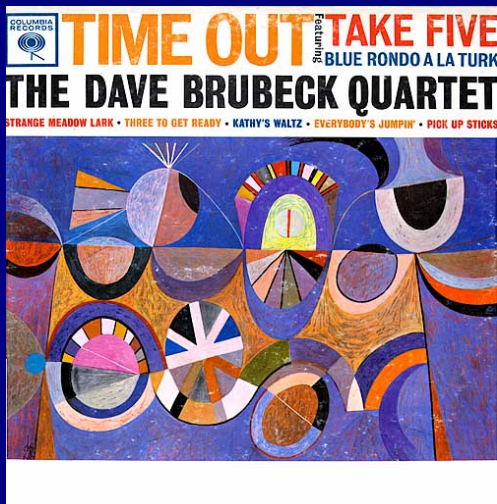
Rhythm in Music



Joy to the World, the Lord is Come.



Time Out (1959)



Take Five: 5/4 Time

Take Five

Music by Paul Desmond

Moderately fast $\text{♩} = 176$



**Based upon use of time signatures
that were unusual for jazz: $\frac{5}{4}$, $\frac{9}{8}$, etc.**



Blue Rondo a la Turk: 9/8 Time

Blue Rondo A La Turk

Music by Dave Brubeck

Vivace $\text{♩} = 148$

Con spirito

The musical score is written for piano in 9/8 time, featuring a complex 2+2+2+3 eighth-note pattern. The piece is in F major and consists of two systems of music. The first system contains two measures, and the second system contains four measures. The notation includes treble and bass staves with various chords and melodic lines. The chords are labeled as follows:

- Measure 1: Fmaj⁷, F⁷, F⁶, Faug
- Measure 2: F, Faug, F⁶, F⁷
- Measure 3: Fmaj⁷, F⁷, F⁶, Faug
- Measure 4: F, Faug, F⁶, Am⁷
- Measure 5: D⁷, Dm⁷, Am



The Stranglers: *Golden Brown*

THE STRANGLERS - EMI

GOLDEN BROWN

Tekst & muziek:
JEAN JACQUES BURNELL/HUGH CORNWELL/
JET BLACK & DAVID GREENFIELD

Moderately
Play 3 times



Bbm Fm Gb

Can you work out the time signature?

[Listen to the video clip.](#)



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Moderately
Play 3 times



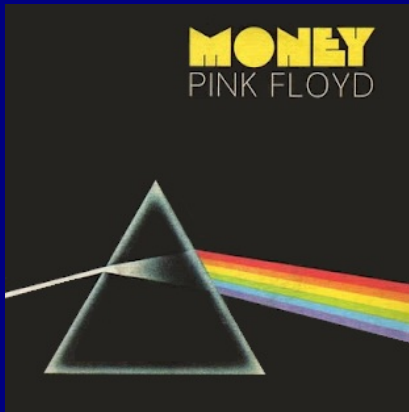
Bbm Fm Gb Db

It looks like $(3+3+3+4)/4 = 13/4$.

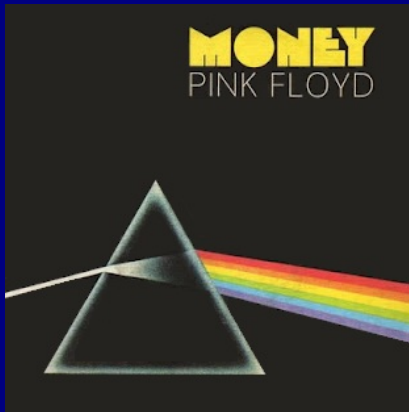
[Listen to the video clip.](#)



Pink Floyd: *Money*



Pink Floyd: *Money*



***Money* is composed mainly in 7/4 time.**



Tchaikovsky's *Pathétique Symphony*



The second movement, a dance form in 5/4 time, has been described as a “limping waltz”.



Outline

“The Two Cultures”

Pythagoras

Sinusoidal Waves

Musical Notation

Tuning

Johann Sebastian Bach

Canons & Fugues

Symmetry

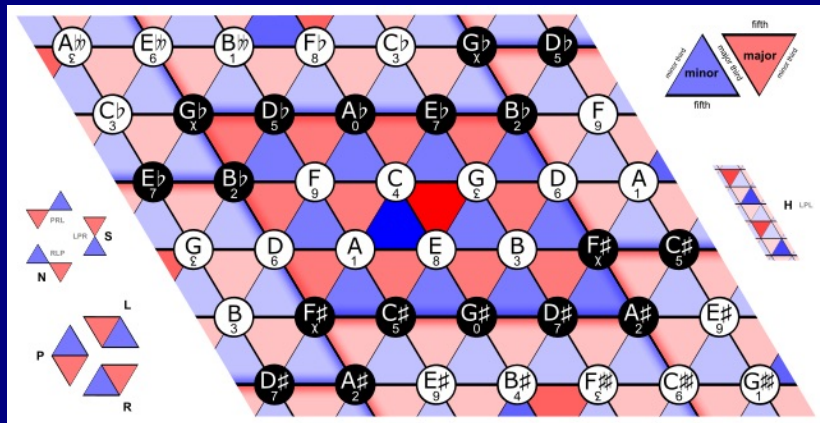
Fascinating Rhythm

The Tonnetz

Finis



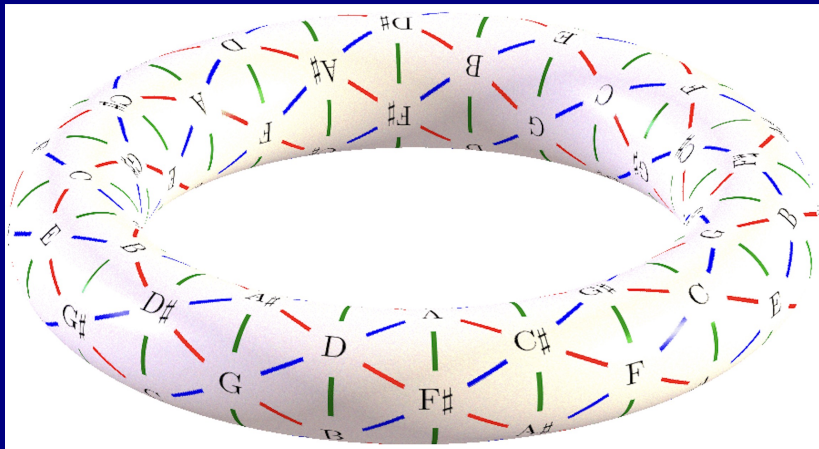
The Tonnetz



This has the topology of a torus or doughnut.



The Tonnetz Topology is the Torus

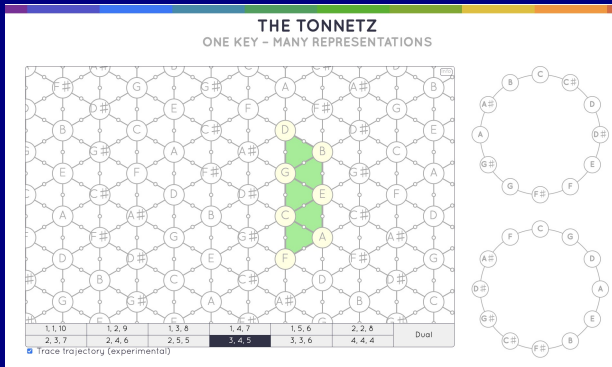


Animated gif of Tonnetz on a Torus.

<file:///Users/peter/Dropbox/Music/U3A-MusicClips/TonnetzTorus.gif>



Online Interactive Tonnetz



<https://thetonnnetz.com/>

Leonard Cohen: Hallelujah

file:///Users/peter/Dropbox/Music/U3A-MusicClips/LeonardCohen-Hallelujah.mp4



Outline

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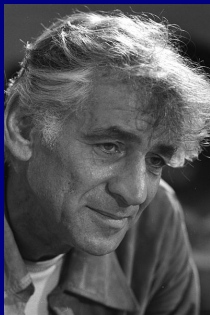
Fascinating Rhythm

The Tonnetz

Finis



Leonard Bernstein



“Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?”

“... music is not only a mysterious and metaphorical art; it is also born of science.

“It is made of mathematically measurable elements.

*... any explication of music must
combine mathematics with aesthetics.”*



Paganini: Caprice 24

Caprice no. 24 in A minor for violin

NICCOLO PAGANINI (1782-1840)

THEME



Paganini: Caprice 24

Caprice no. 24 in A minor for violin

NICCOLO PAGANINI (1782-1840)

THEME



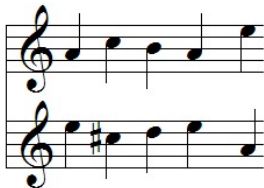
Rachmaninov's *Rhapsody*

Stephen Hough, British-Australian classical pianist

file:///Users/peter/Dropbox/Music/U3A-MusicClips/Rachmaninov-Clip1.mp4

Paganini (part of 24th capriccio theme)

Rahcmaninoff (inversion: 18th variation theme)



file:///Users/peter/Dropbox/Music/U3A-MusicClips/Rachmaninov-Clip2.mp4

Rachmaninoff: Rhapsody on a Theme of Paganini — Anna Fedorova

<https://www.youtube.com/watch?v=ppJ5uITLECE&t=980s>



Thank you

