Mathematics & Music: Symmetry & Symbiosis

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The Bray Heads U3A
Thursday 16 April 2026

Outline

- “The Two Cultures”
- Pythagoras
- Sinusoidal Waves
- Musical Notation
- Tuning
- Johann Sebastian Bach
- Canons & Fugues
- Symmetry
- Fascinating Rhythm
- The Tonnetz
- Finis

Outline

Three Sweeping Statements: No. 1

Everyone Loves Music

Three Sweeping Statements: No. 2

Everyone Hates Maths

Three Sweeping Statements: No. 3

Music and Mathematics
are
Inextricably Intertwined

Love/Hate Image

Given the close link between music and maths,
how can we love one and hate the other?

Love/Hate Image

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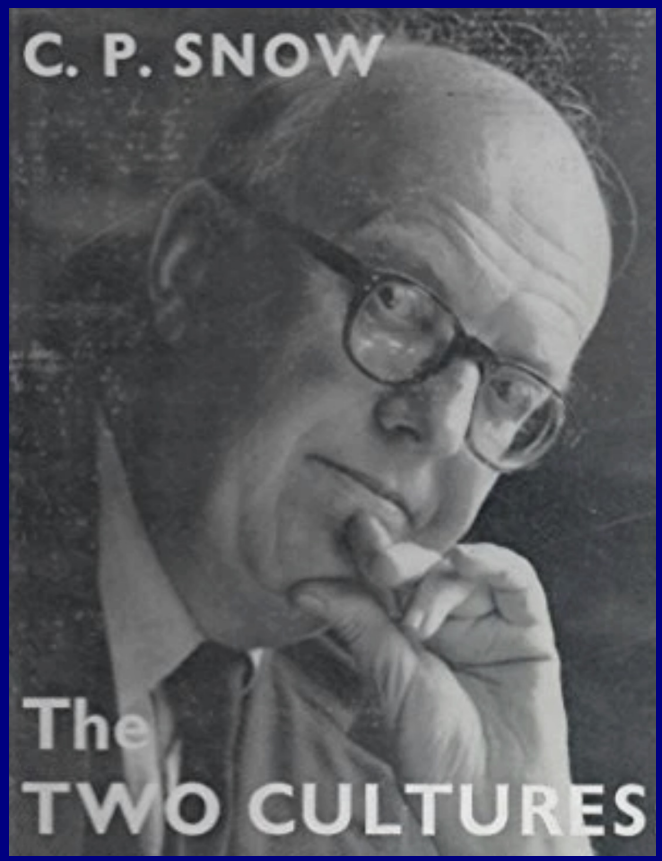

The concept of The Two Cultures was introduced by the British scientist and novelist C. P. Snow.

This concept remains relevant today.

1959 Rede Lecture: *The Two Cultures*

Who Can This Be? Who Can This Be?

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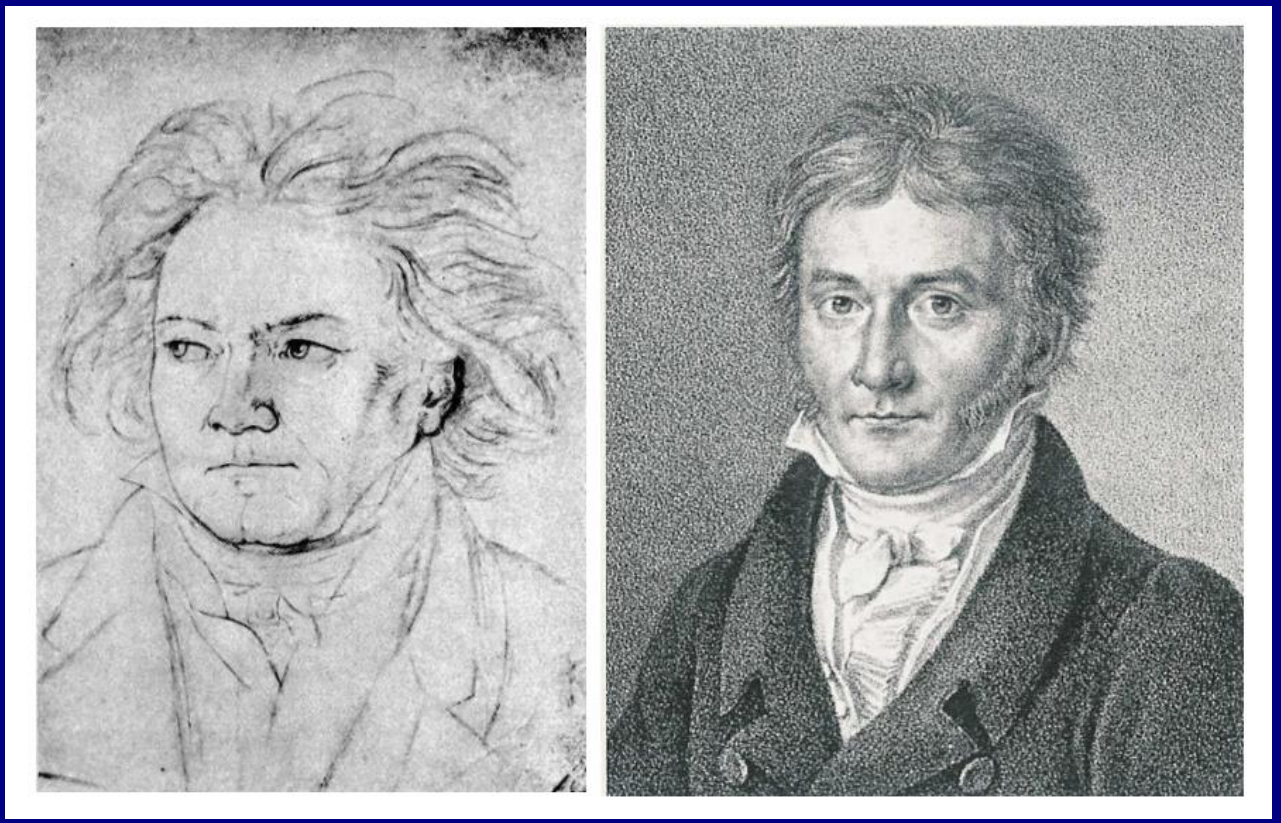
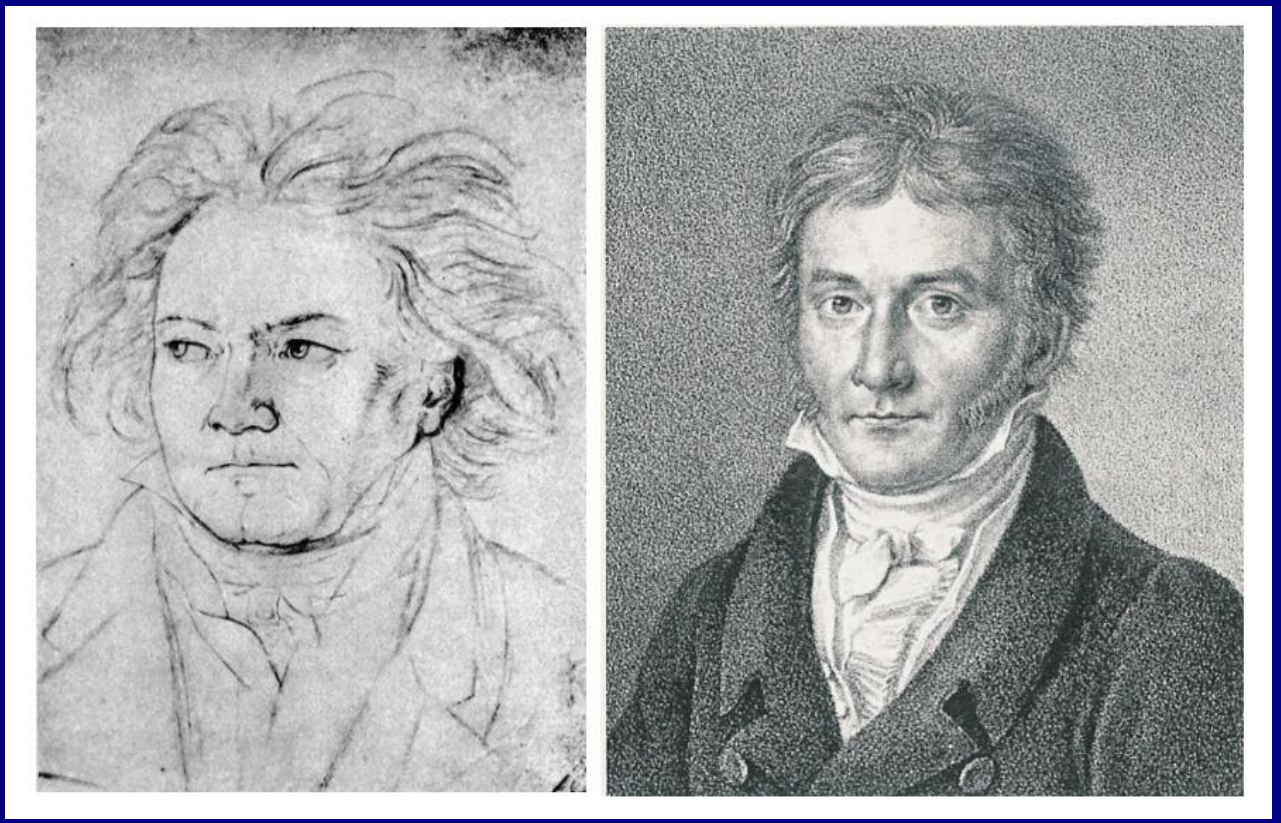


LEFT: Ludwig van Beethoven (1770–1827)

Who Can This Be? Who Can This Be?



LEFT: Ludwig van Beethoven (1770–1827)
RIGHT: Carl Friedrich Gauss (1777–1855)

Beethoven / Gauss

Beethoven and Gauss were at the height of their creativity in the early 19th century.

The work of Gauss has a greater impact on our daily lives than the magnificent creations of Beethoven.

Yet, Beethoven is known to all, Gauss to only a few!

"Of course I've heard of Beethoven, but who is this Gauss dude?"

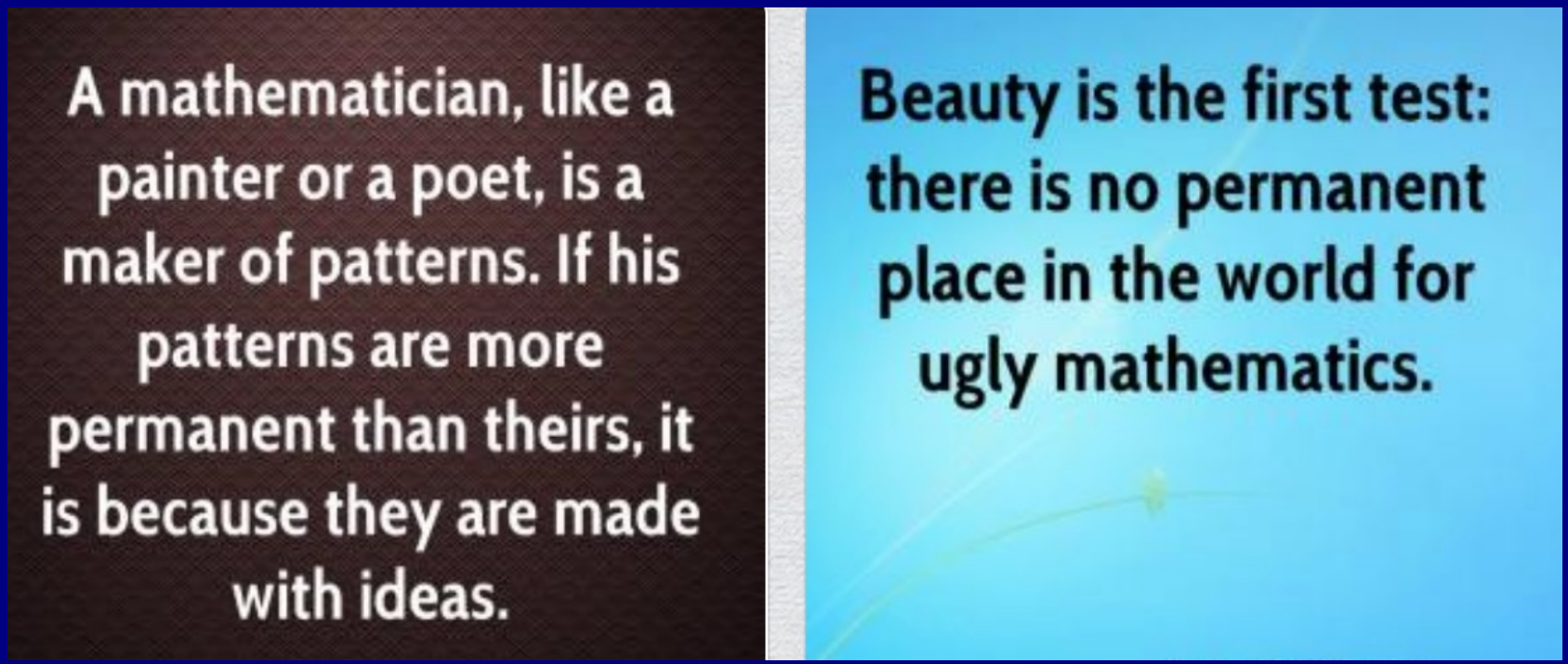
Meaning and Content of Mathematics

The word **Mathematics** comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

- Quantity [Numbers: Arithmetic]
- Structure [Patterns: Algebra]
- Space [Geometry: Topology]
- Change [Analysis: Calculus]

British Mathematician G H Hardy:



Two Parallel Languages

MUSIC	MATHS
Pitch	Frequency
Scales	Modular Arithmetic
Intervals	Logarithms
Overtones	Integers
Octave Identification	Equivalence Relation
Equal Temperament	Exponents

You know more mathematics than you realize!

Music & Maths

There are many parallels between music and maths:

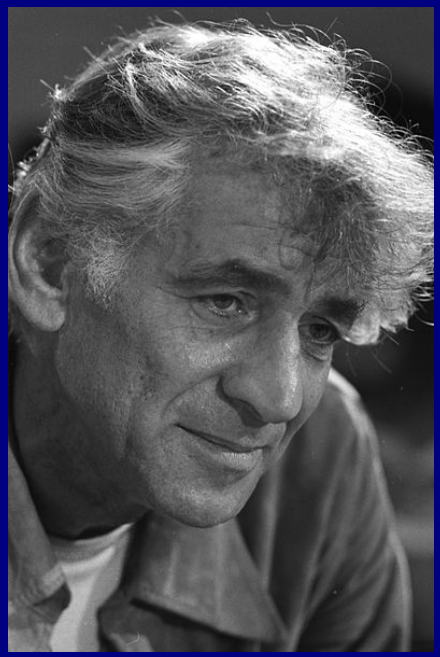
Structure **Symmetry** **Pattern**

But music is accessible to all while maths is not.
Music gets into the soul through the emotions.

Maths is understood through the intellect.
Appreciation comes via a rational route.

Music has instant appeal. Maths takes time.

Leonard Bernstein



"Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?"

Outline

The Quadrivium

The Pythagoreans organized their studies into the Quadrivium, comprising four disciplines:

- **Arithmetic**
- **Music**
- **Geometry**
- **Astronomy**

Static/Dynamic Number. Pure/Applied

- **Arithmetic:** Static number
- **Music:** Dynamic number

Arithmetic represents numbers at rest.

Music is numbers in motion.

Arithmetic is **pure or abstract** in nature.

Music is **applied or concrete** in nature.

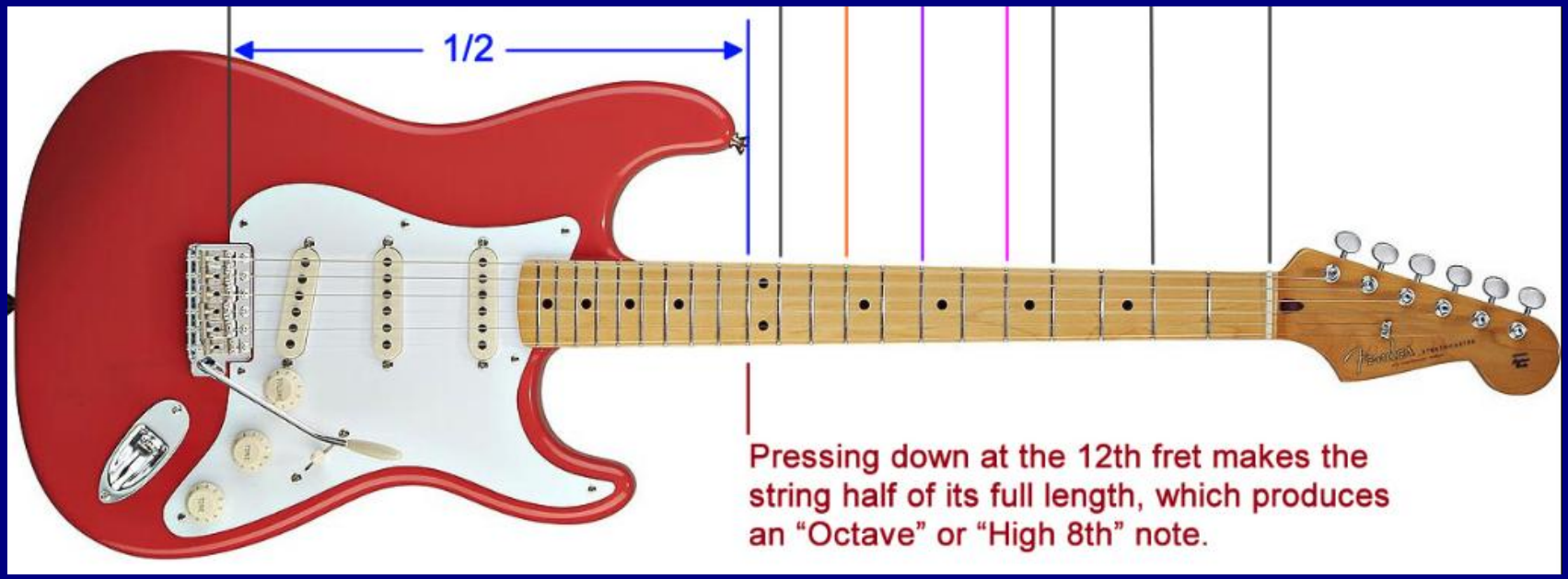
Discovery of Pythagoras

The Pythagoreans discovered a remarkable connection:

Ratios of small whole numbers are directly linked with consonant or harmonically pleasing chords.

"There is geometry in the humming of the strings, There is music in the spacing of the spheres."

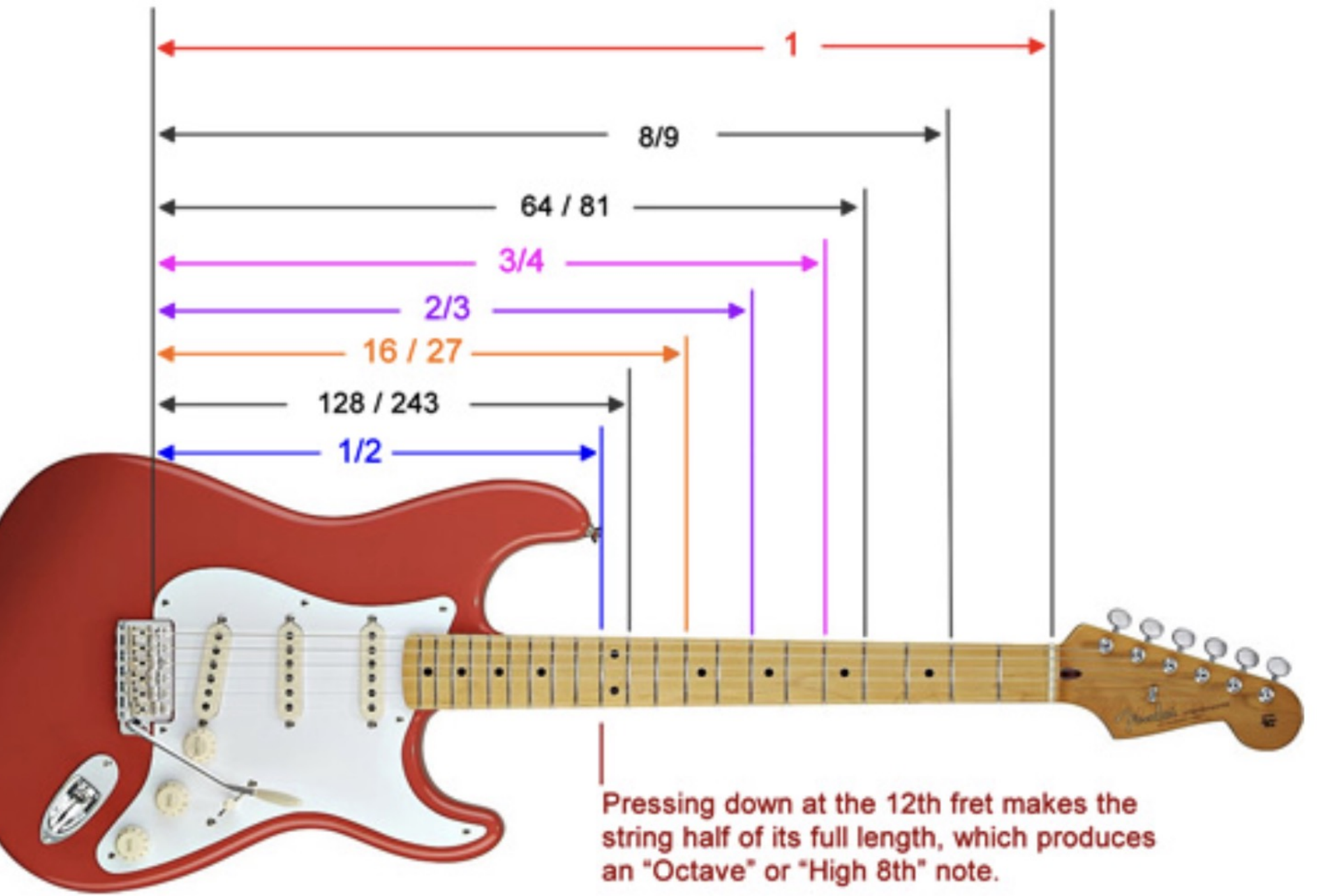
Guitar Strings



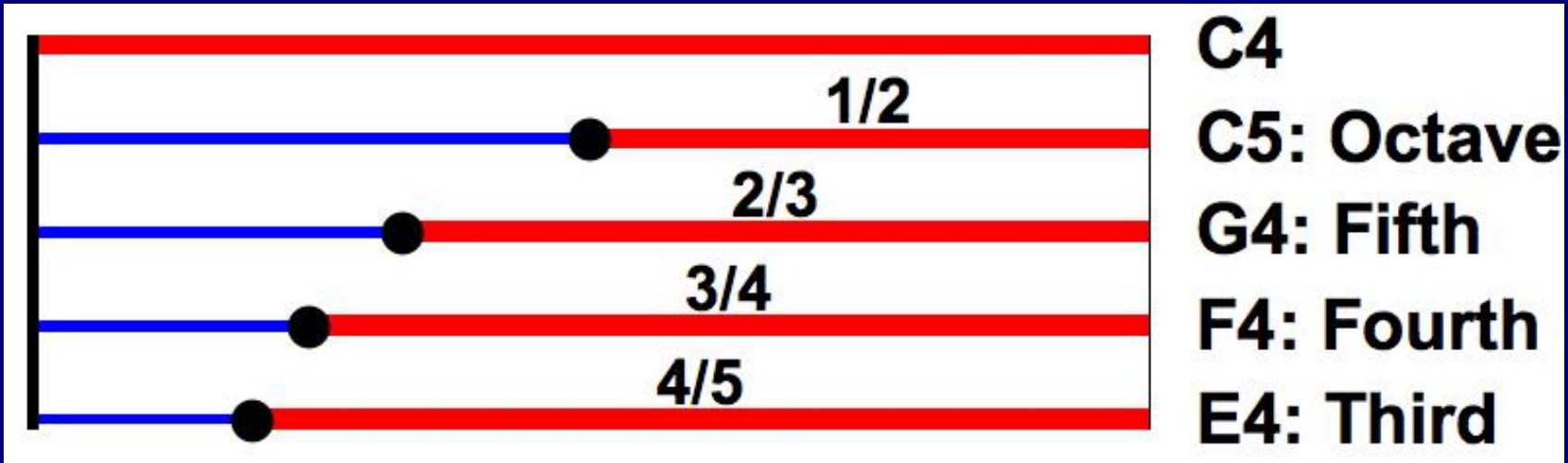
Open string vibrates at 264 Hz. Call this f.

- **String of length $\frac{1}{2}$ vibrates at $\frac{2}{1}f$.**
- **String of length $\frac{2}{3}$ vibrates at $\frac{3}{2}f$.**
- **String of length $\frac{3}{4}$ vibrates at $\frac{4}{3}f$.**

Demonstrate the pitches on the uke.



The Monochord



Small-number ratios and harmonic intervals

Feynman on Pythagoras' Discovery

Richard Feynman

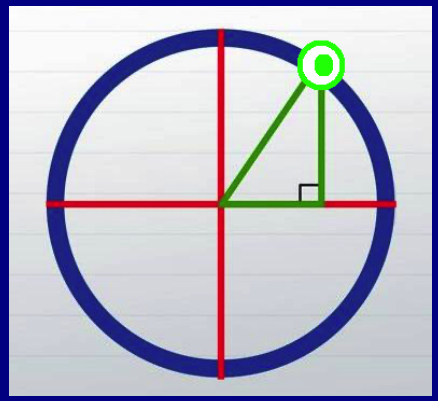
"The first example, outside geometry, of a numerical relationship in nature."

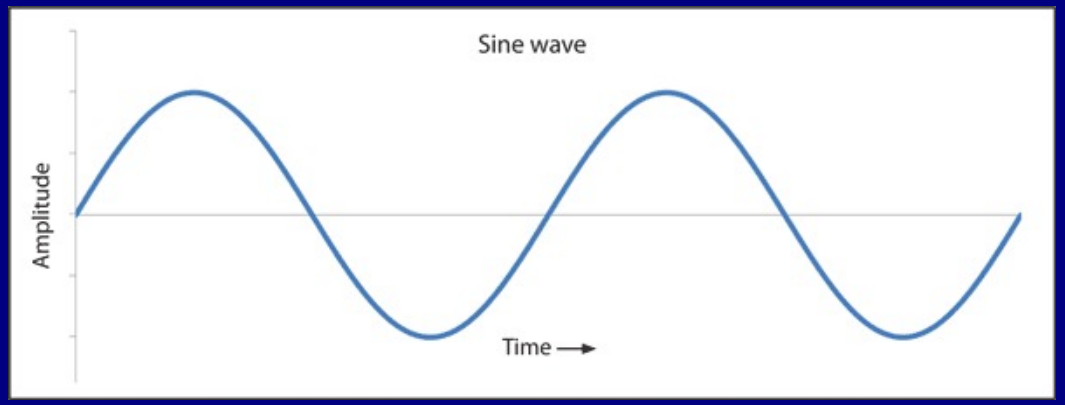
Pythagoras made his discovery through observation. This aspect does not seem to have impressed him.

Had Pythagoras followed up on this idea,
"Physics might have had a much earlier start."

Outline

Sine Waves and Circles





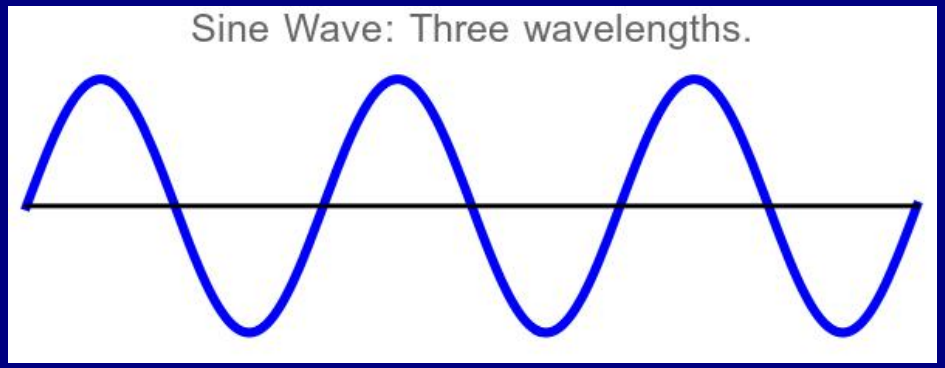
As the green dot moves around the circle, its height traces a sine wave in time.

Up and down and up and down and up and down

and up and down and up and down and . . .

Sine waves are a kind of circular functions.

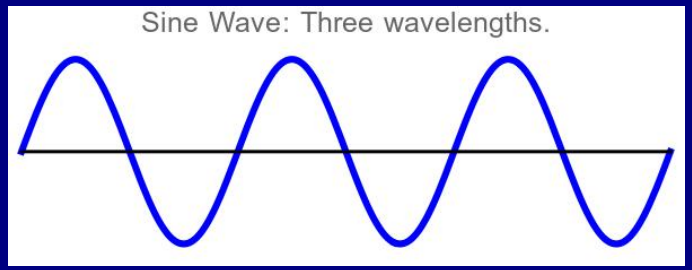
Terminology



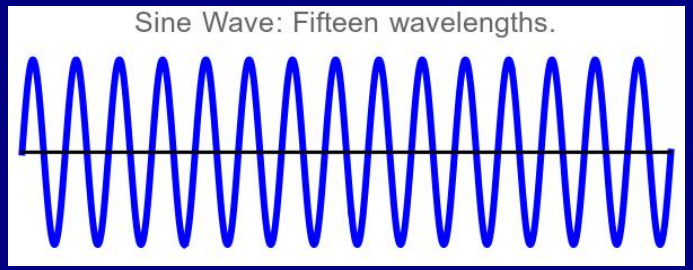
Music **Maths**

Loudness <=> Amplitude

Pitch <=> Frequency

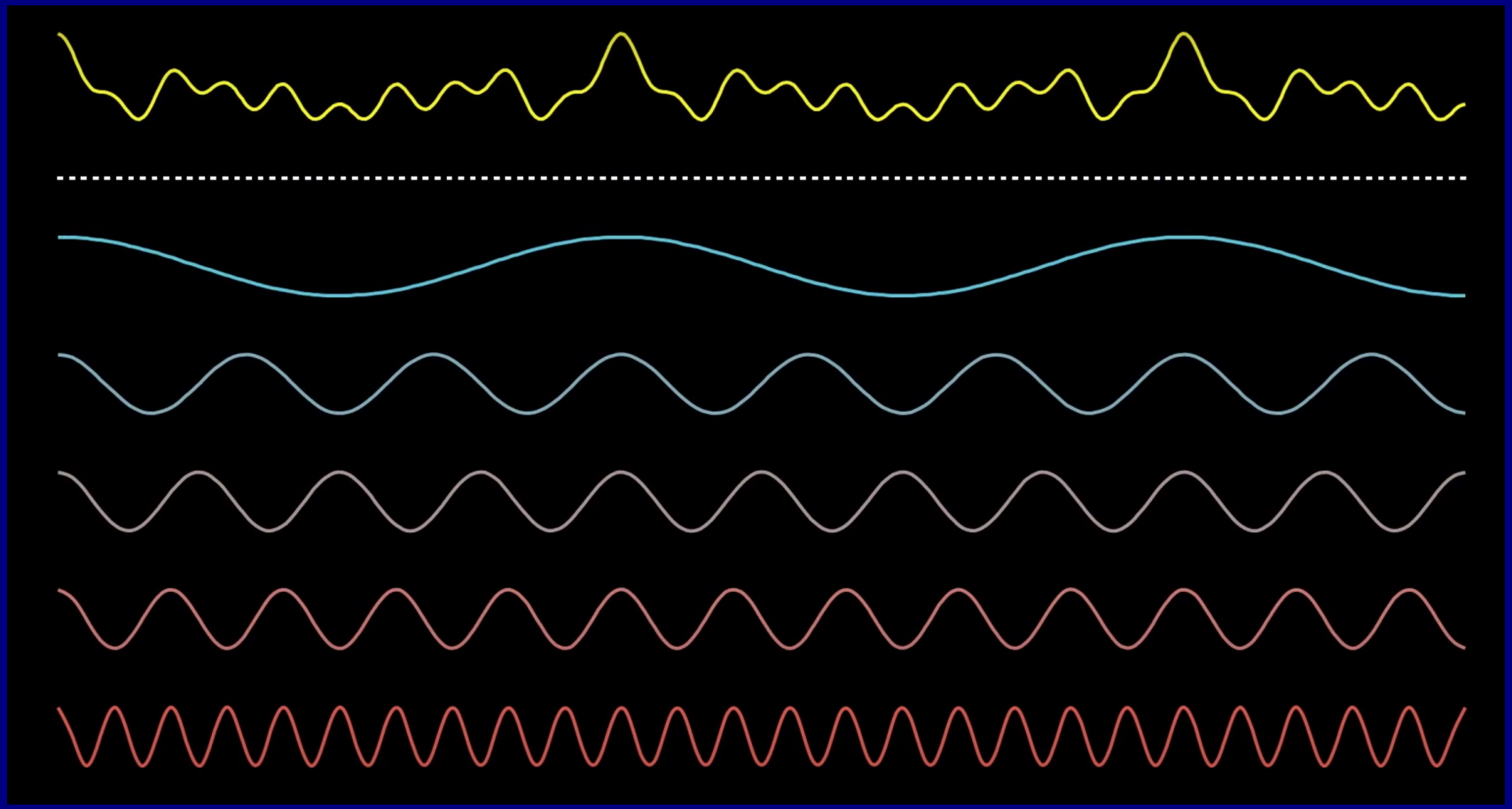


Low Pitch



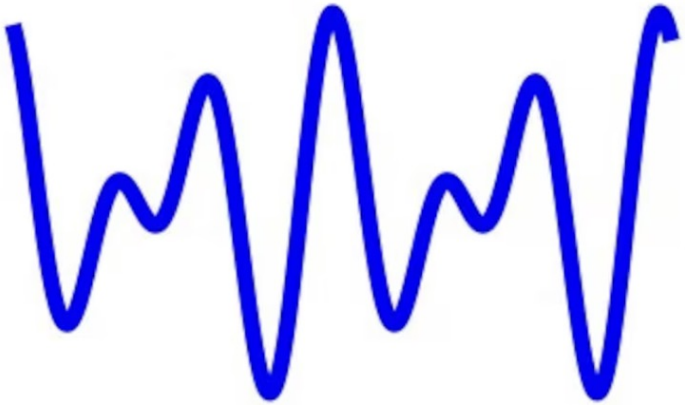
High Pitch

Fourier Components — (or Harmonics)

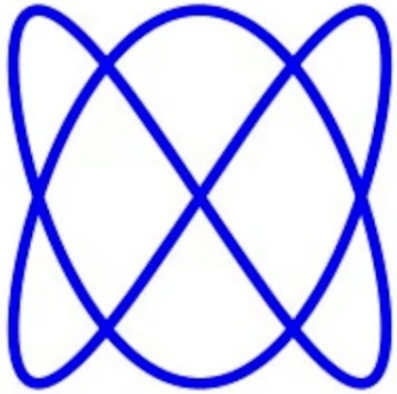


Lissajous Figures for Intervals

2:3. C–G

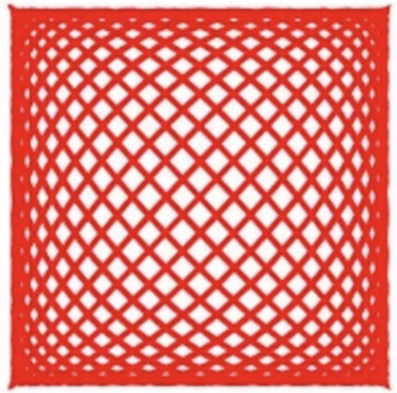


2:3. Perfect Fifth



16:17. C–C♯

16:17. Semitone



A Harmonic Generator

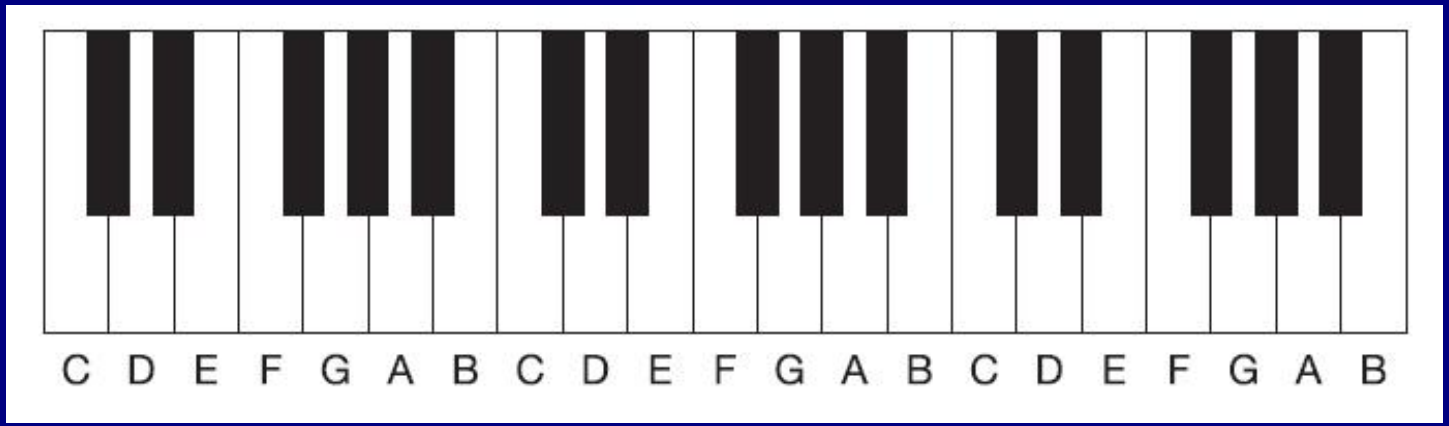
https://meettechniek.info/additional/additive-synthesis.html

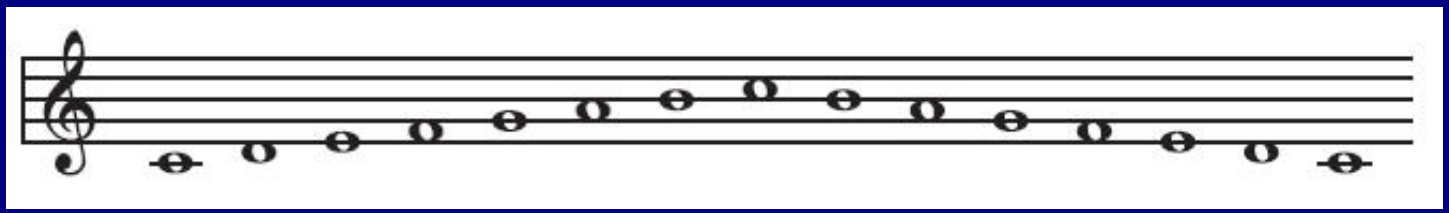
A website for generating sine waves and harmonics.

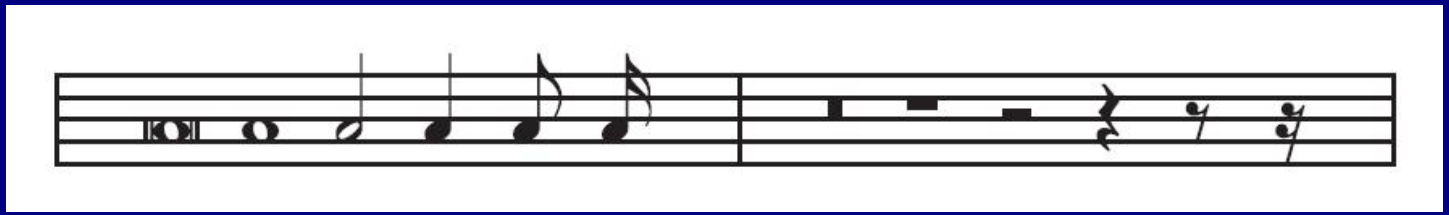
*** * * Click: Examples / Extra / Waveform Generator * * ***

Outline

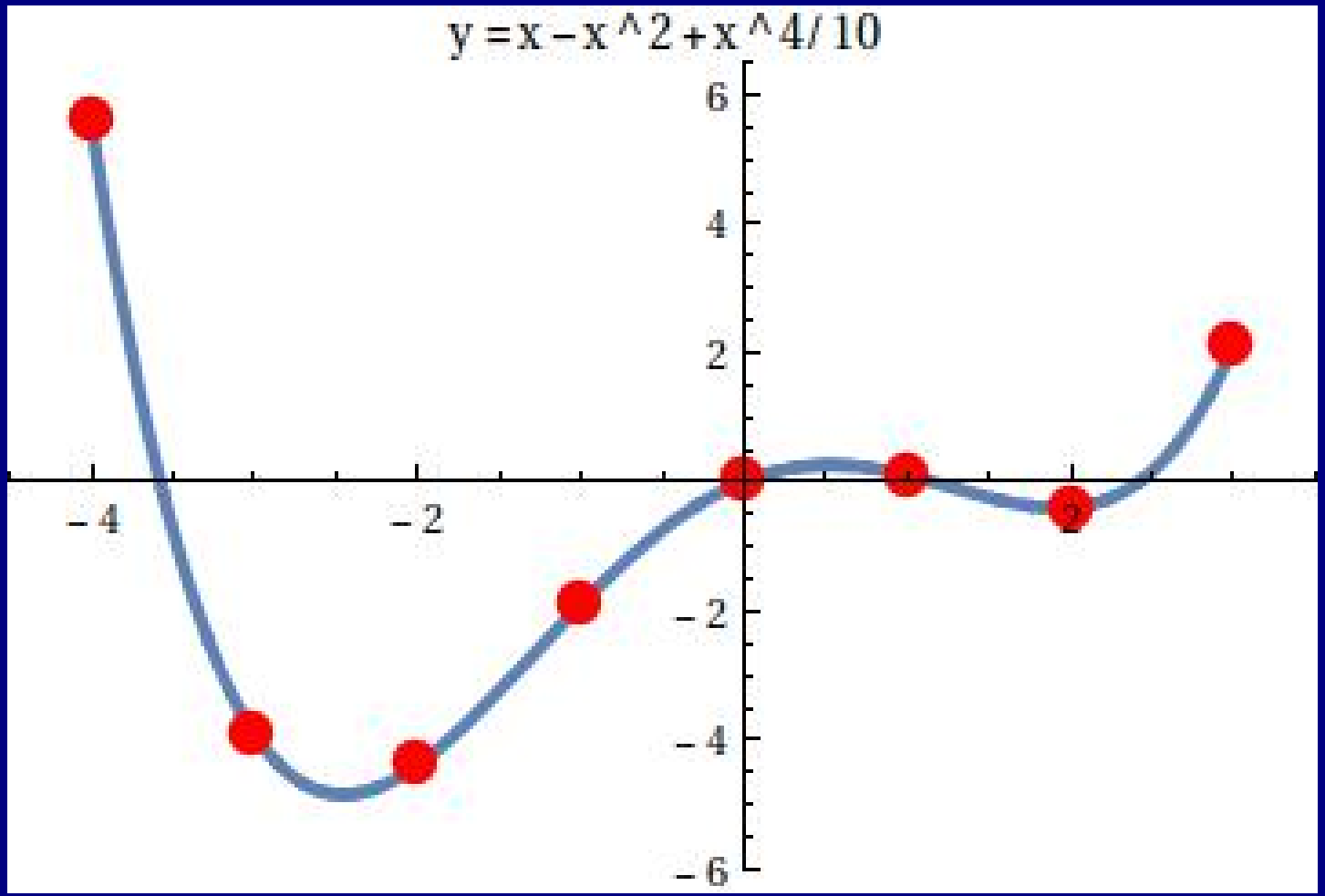
Notating Musical Notes





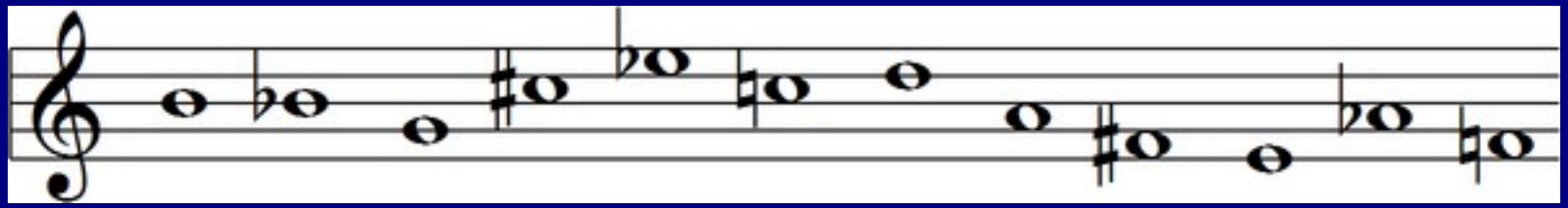


A Mathematical Graph: Joining the Dots

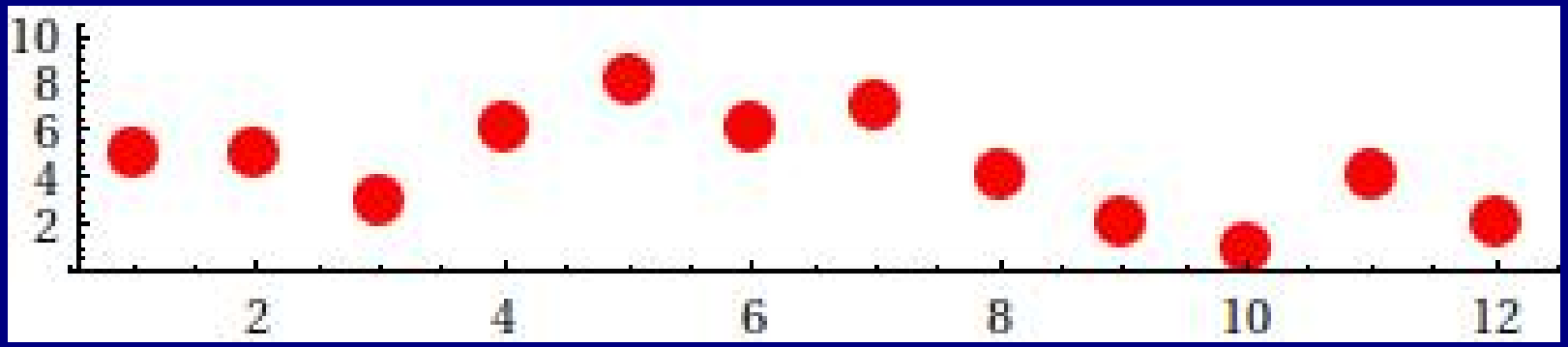


A plot of y versus x.

Music as a Graph

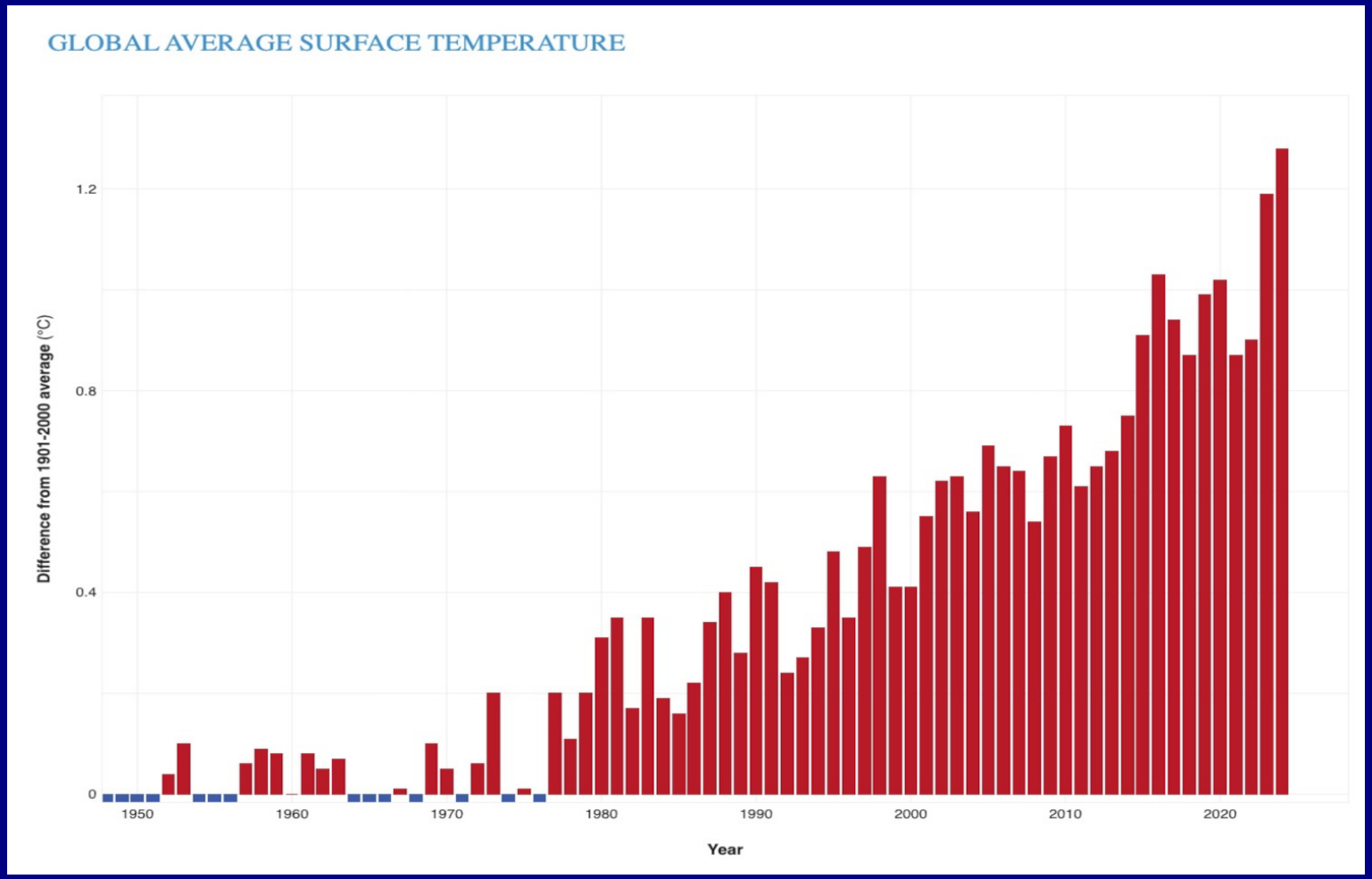


A sequence of musical notes: a simple tone row.



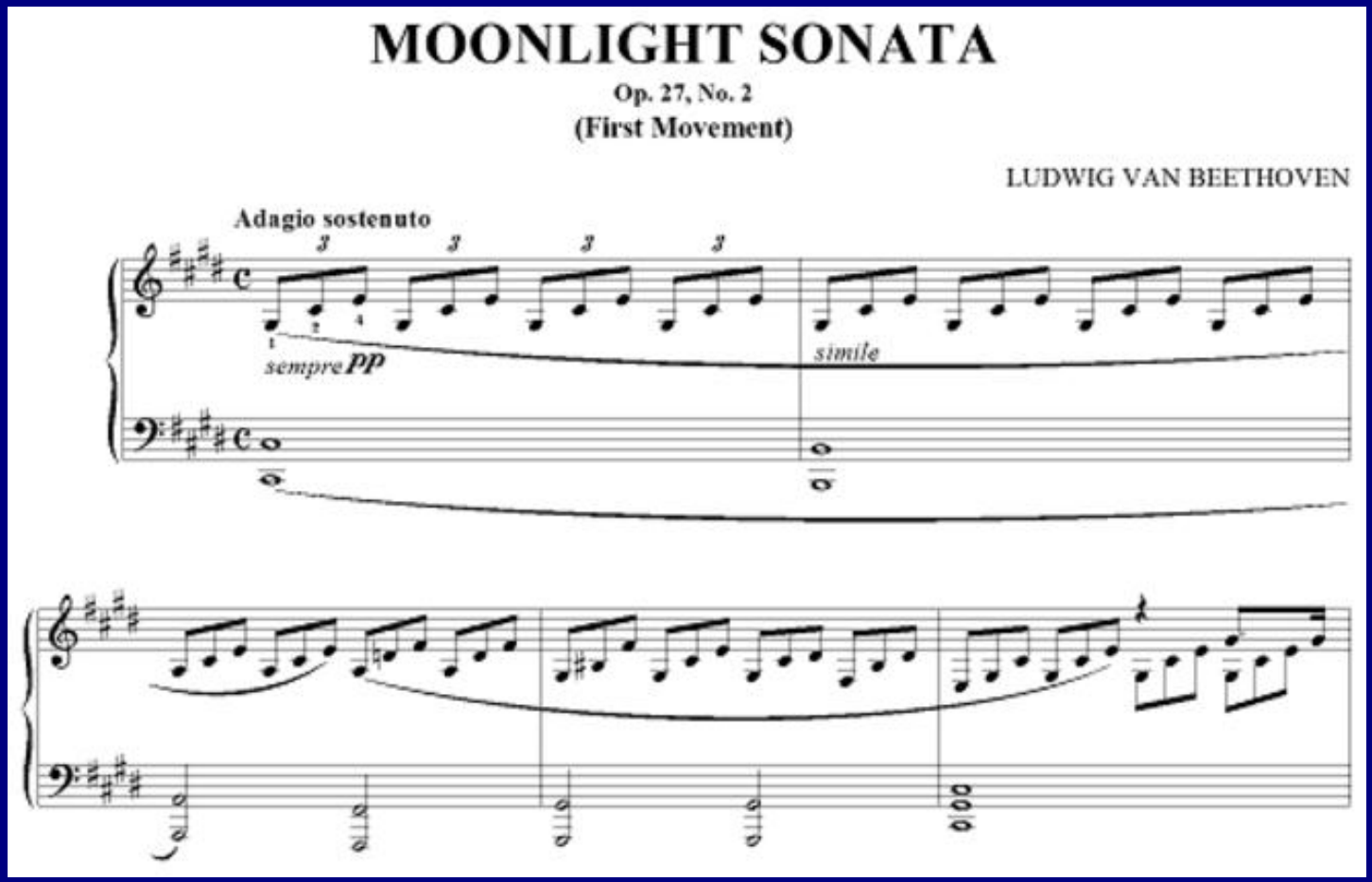
A musical score is just a graph of pitch *versus* time!

Symbiosis: Musing on a Graph



A new composition: *The Heat is On.*

Beethoven's *Moonlight Sonata*



An ingenious and delightful combination of precision (pitch) and vagueness (pp).

Precision and Vagueness

**The standard pitch is A_4,
the A above Middle C.
Its frequency is 440 Hz.**

By contrast, the time-scale is written vaguely:

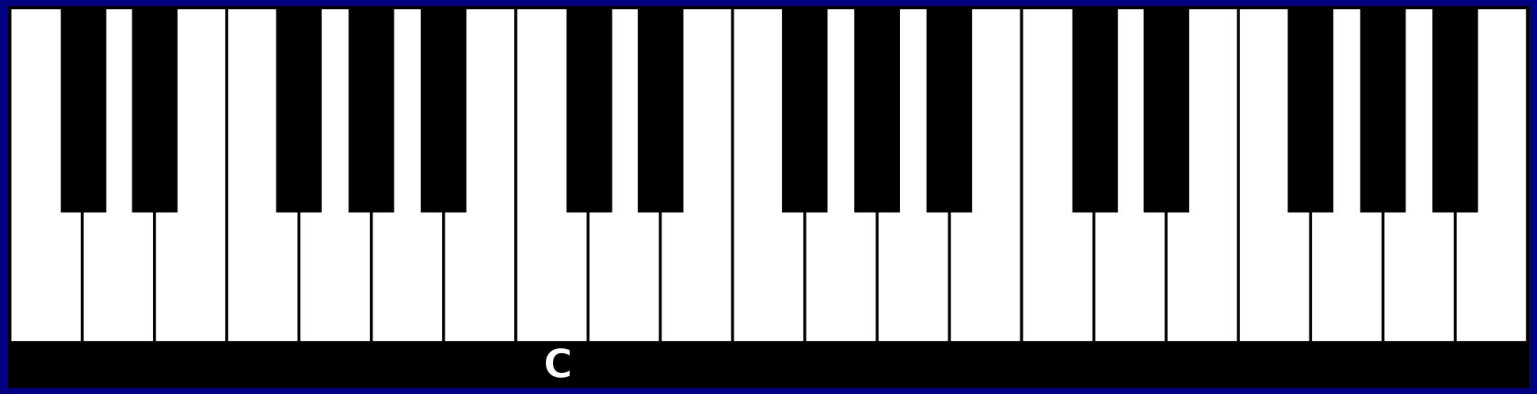
Adagio sostenuto.

The loudness is specified simply by *sempre* pp.

**The vagueness allows the performer
freedom of interpretation and expression.**

Outline

The Piano Keyboard



Where do all the notes come from?

Pythagorean Tuning

Pythagoras discovered that a perfect fifth — with frequency ratio 3:2 — is especially harmonious.

The entire musical scale can be constructed using only the ratios 2:1 (octaves) and 3:2 (fifths).

In the tonic sol-fa scale the eight notes of the major scale are Do, Re, Mi, Fa, So, La, Ti, Do.

Pythagorean and Just Intonation

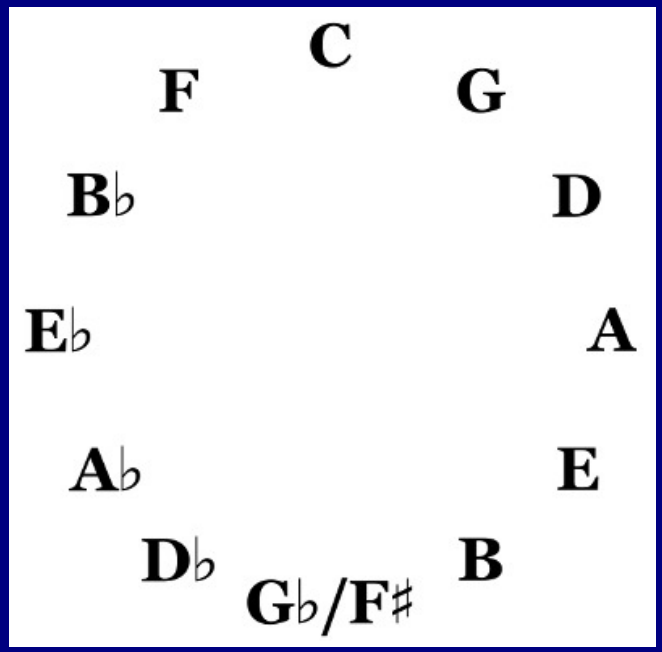
Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Pythagorean intonation.

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Just intonation.

Organizing Scheme: the Circle of Fifths

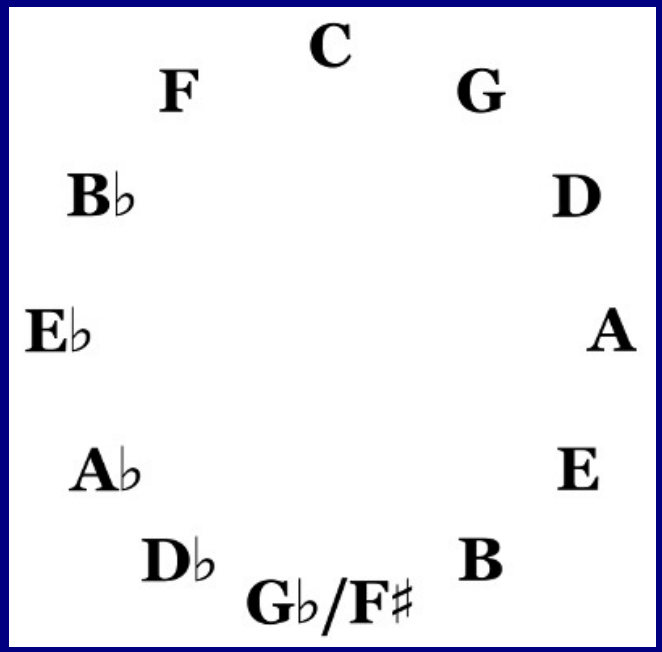


The Circle of Fifths represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

The Circle is useful in harmonising melodies and building chords.

Organizing Scheme: the Circle of Fifths



The Circle of Fifths represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

The Circle is useful in harmonising melodies and building chords.

PROBLEM: The Circle doesn't close!

The Well-Tempered Scale

As music developed, composers found it was not always possible to change smoothly between different keys without **dissonant consequences.**

Many tuning methods, improving on just intonation, were developed in the seventeenth century. These were known as **Well-Tempered Scales.**

* * *

Johann Sebastian Bach was the first major composer to employ the well-tempered scale.

The Well-Tempered Clavier (BWV 846-893) is a collection of preludes and fugues in all 24 keys.

Equal Temperament

Today, the Equal Tempered Tuning is universal.

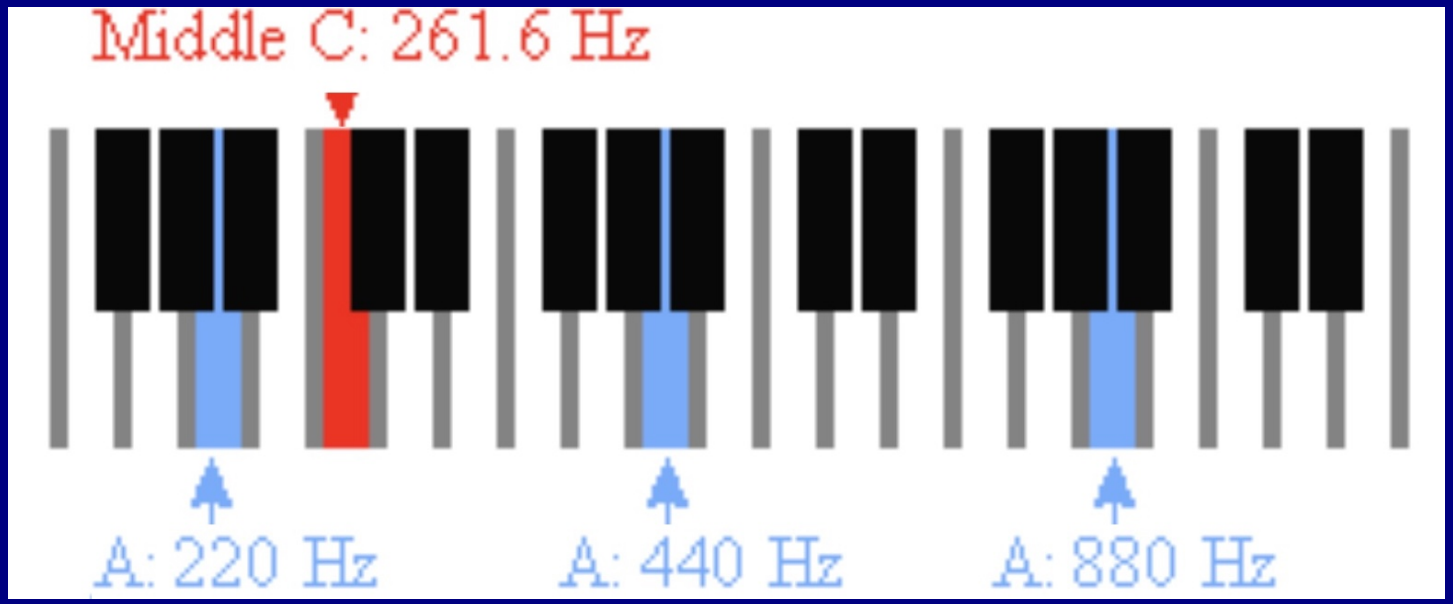
Big Idea: Make all semitone intervals equal.

An octave has ratio 2:1. We need a number that yields 2 when multiplied by itself 12 times:

$$X = \sqrt[12]{2} \approx 1.059$$

All intervals of the equal-tempered scale are imperfect, but close enough to be acceptable.

Middle C commonly pitched at 261.63 Hz.



Middle C or **C4** is the 'central note' on the piano.
The standard frequency of the note **A4** is 440 Hz.

$$261.63 = 440 \div 2^{3/4}$$

Where does the peculiar factor $2^{3/4}$ come from?
It arises from the **equal temperament scale.**

Outline


https://medium.com/the-quantastic-journal/

Johann Sebastian Bach & Mathematics

While not a mathematician, Bach thought like one.

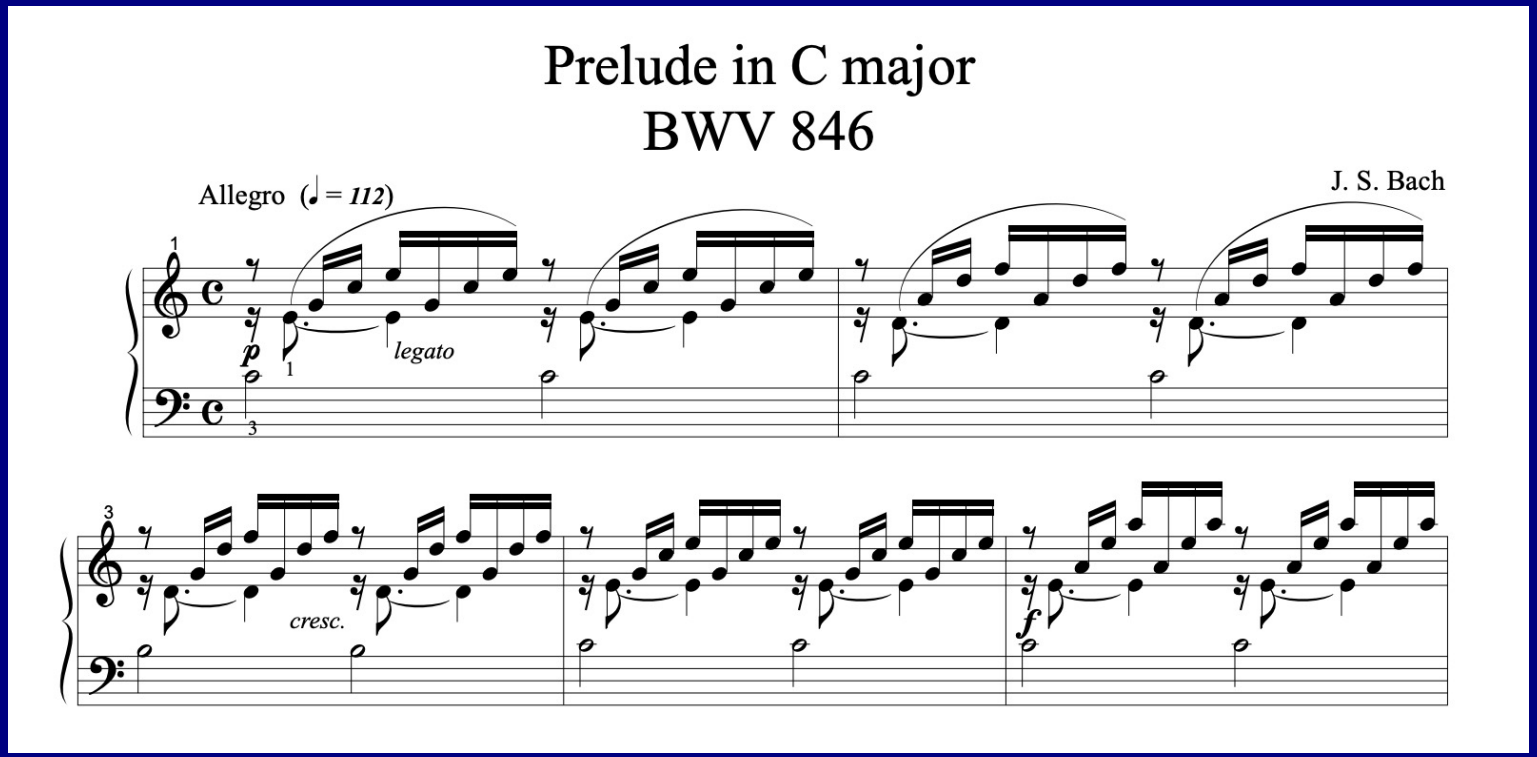
He was fond of using geometric transformations,
like transposition, inversion, and rotation,
and exploring their musical possibilities.

He combined these mathematical transforms
in creative and aesthetically pleasing ways . . .

. . . with structural integrity and aesthetic beauty.

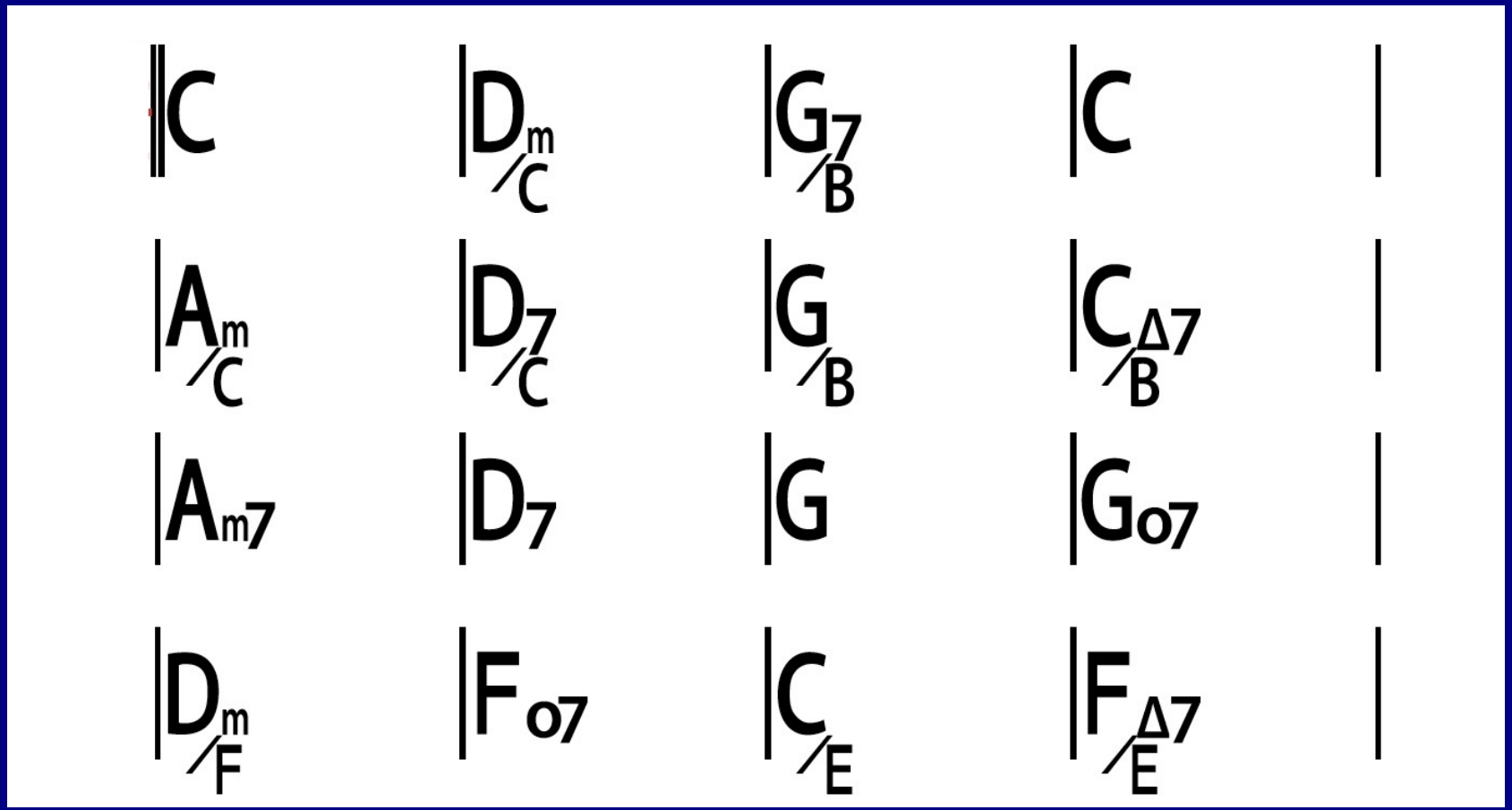
The Well-Tempered Clavier (1722)

Johann Sebastian Bach's Well-Tempered Clavier is a collection of preludes and fugues in all 24 keys.



https://www.youtube.com/watch?v=frxT2qB1POQ&list=RDfrxT2qB1POQ&start_radio=1

Bach C Major Prelude Lead Sheet



Bach's *Prelude* & Gounod's *Ave Maria*

Bach's Prelude No. 1 in C major, BWV 846, from Book I of *The Well-Tempered Clavier* . . .

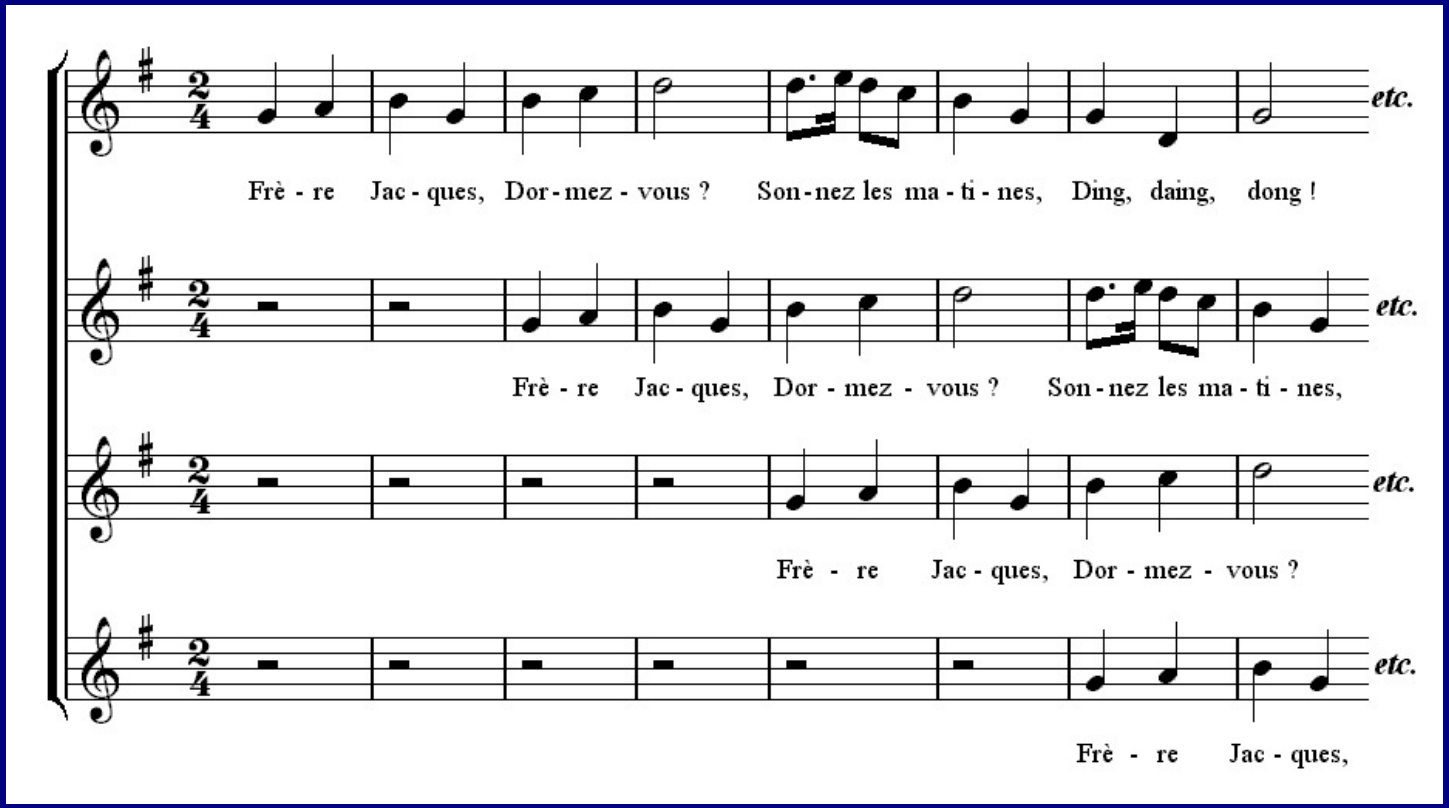
. . . a study of harmony in broken chords.

In 1853, some 130 years later, the French Romantic composer Charles Gounod superimposed a melody, a setting of the Latin prayer Ave Maria.

Recording by Fritz Kreisler and John McCormack
https://www.youtube.com/watch?v=US41r9nlkq8&t=125s

Outline

The Geometry of Canons



A manuscript at the French National Library contains 86 canons by J.-P. Rameau, including Frère Jacques.

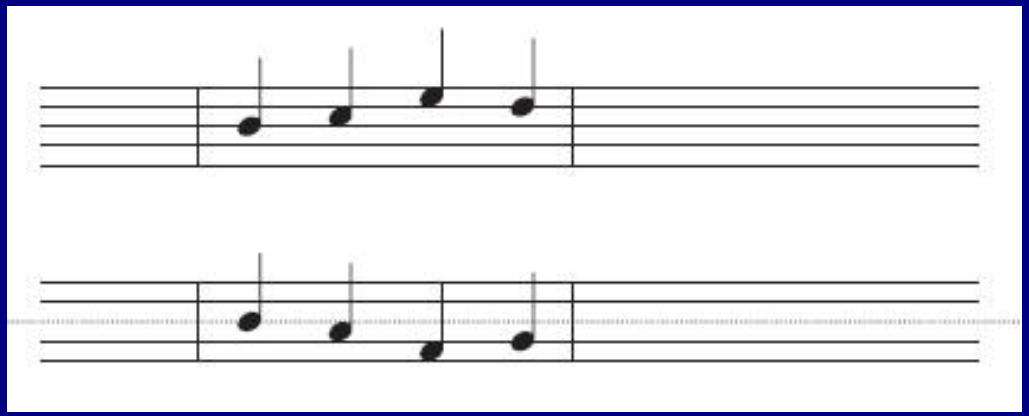
The Geometry of Canons

- Canon at the Unison
- Canon at an Interval
- Canon *contrario motu*
- Retrograde Canon
- Perpetual Canon

Johann Sebastian Bach was the grand master of canon form.

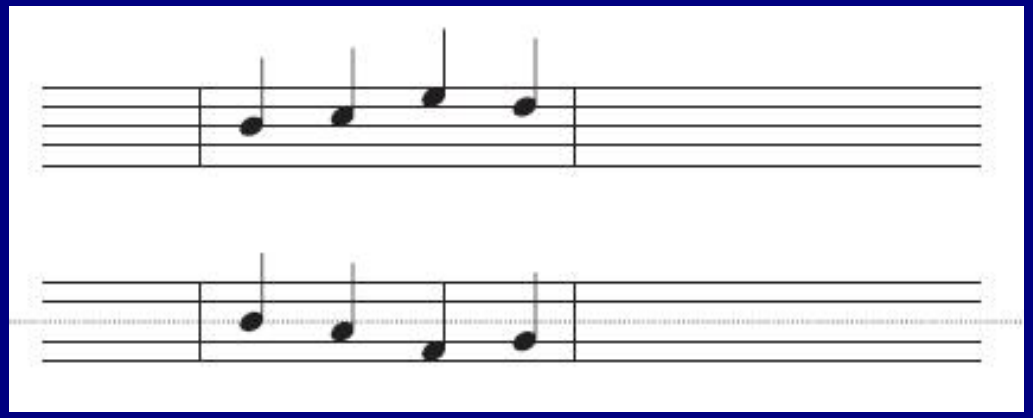
The transformations used in Canon Form are described by mathematical groups.

Simple Transformations

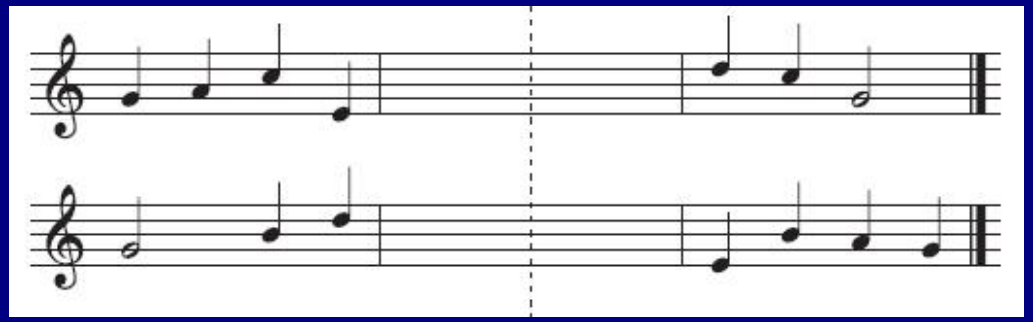


Inversion, or rotation about a horizontal line.

Simple Transformations



Inversion, or rotation about a horizontal line.



Retrogression, or rotation about a vertical line.

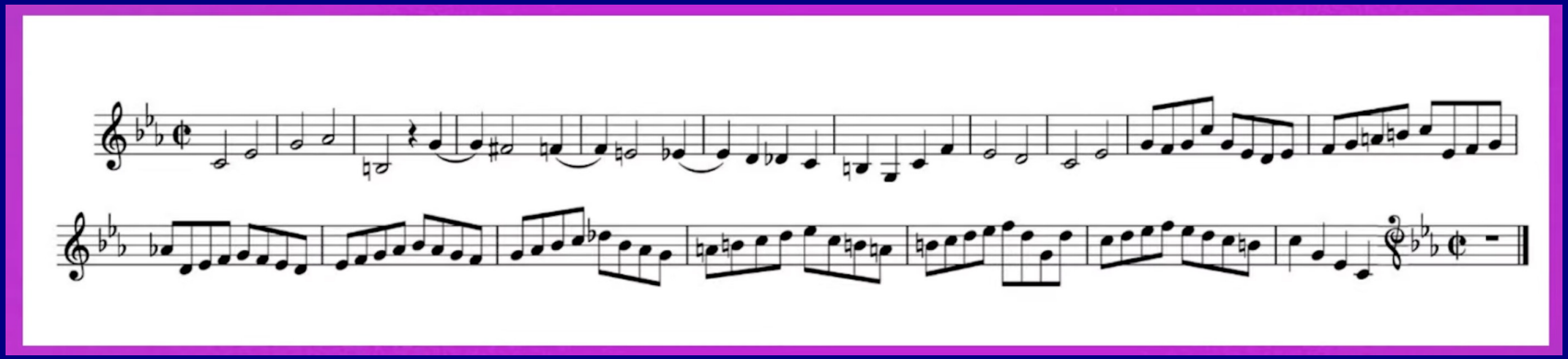
Bach's "Great" Fugue in G minor

Bach's Great Fantasia and Fugue for Organ in G minor, BWV 542 is among his most complex and marvellous compositions.

https://www.youtube.com/watch?v=4WhPUqpaRp4&t=4s

A comment on the YouTube Video:
Bach ist der größte aller Zeiten.

Bach's Crab Canon (if time allows)



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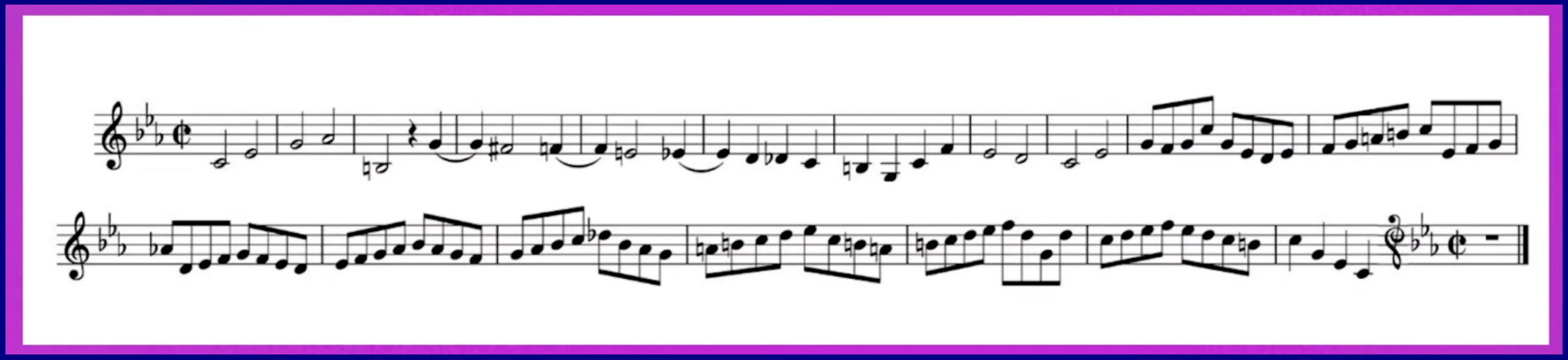
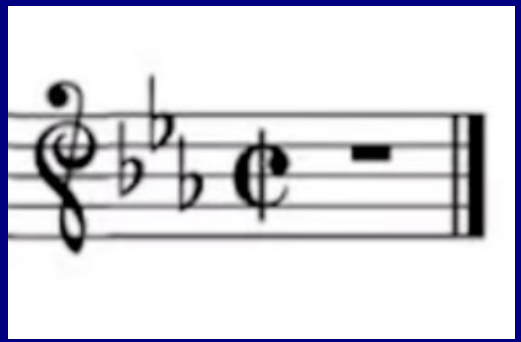
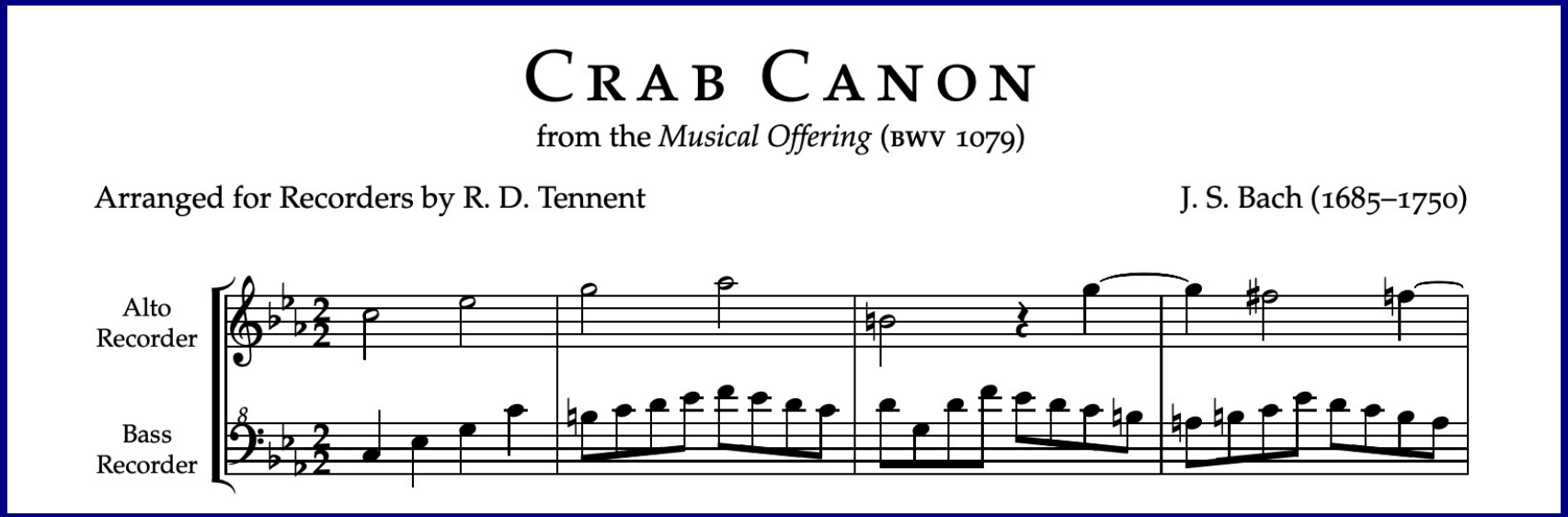
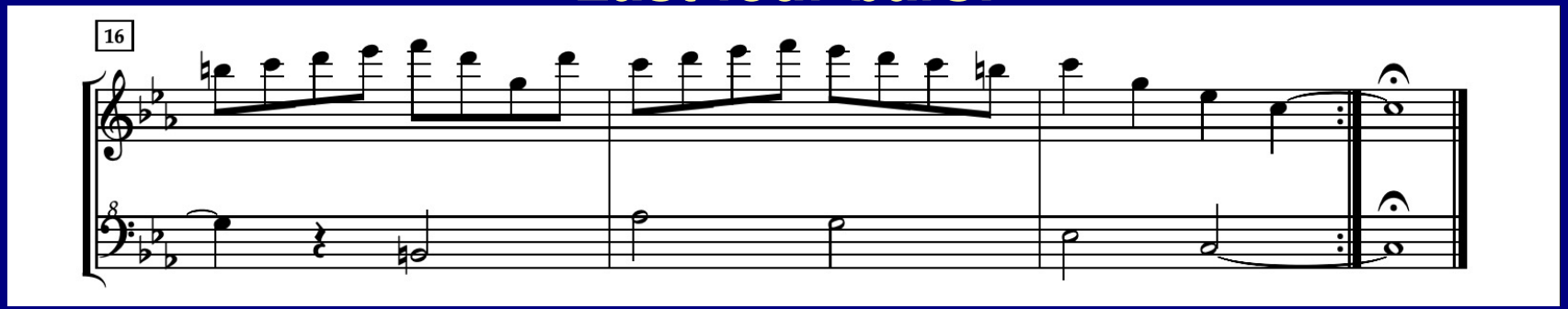




Crab Canon (for Alto & Bass Recorders)

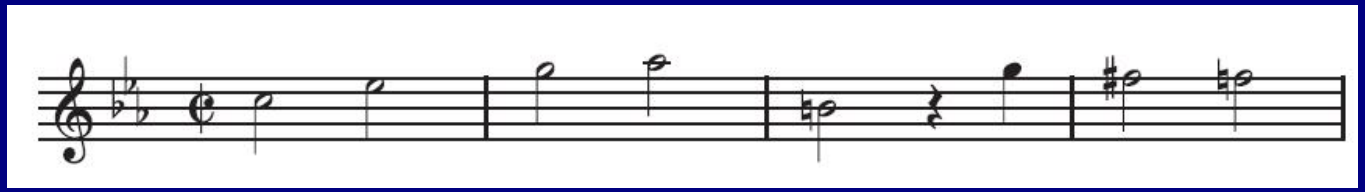


Last four bars:



“A Musical Offering”

Frederick the Great provided Bach with a theme:



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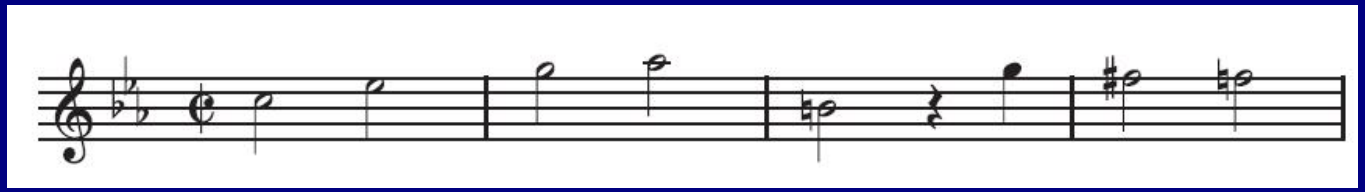
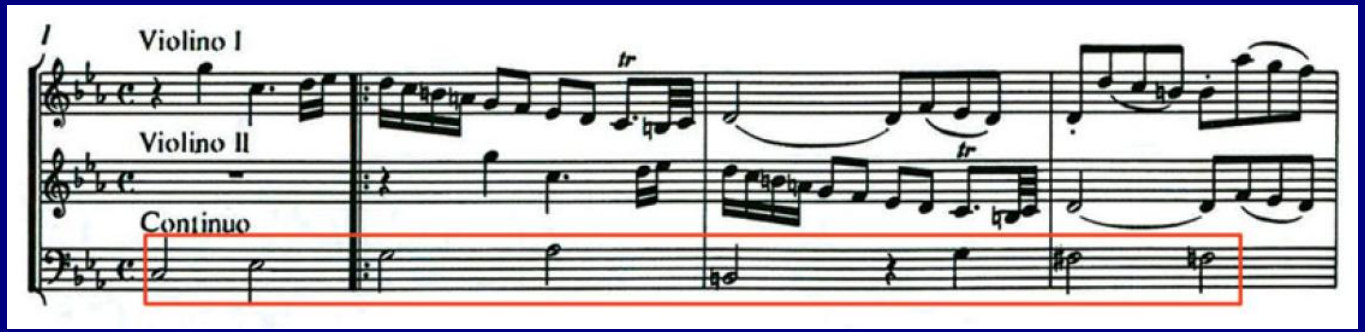


Bach worked this into *A Musical Offering*, a collection of ten canons, a sonata and two fugues.



The work has several riddles and hidden jokes.

Outline

Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- At the sub-atomic scale, symmetry reigns.
- Viruses can display stunning symmetries.
- Crystals have great symmetry.
- Our bodies have bilateral symmetry.
- Many buildings are symmetric.
- Galaxies have many symmetries.

The Taj Mahal

A Face with Symmetry: Halle Berry

Halle Berry

Berry Halle

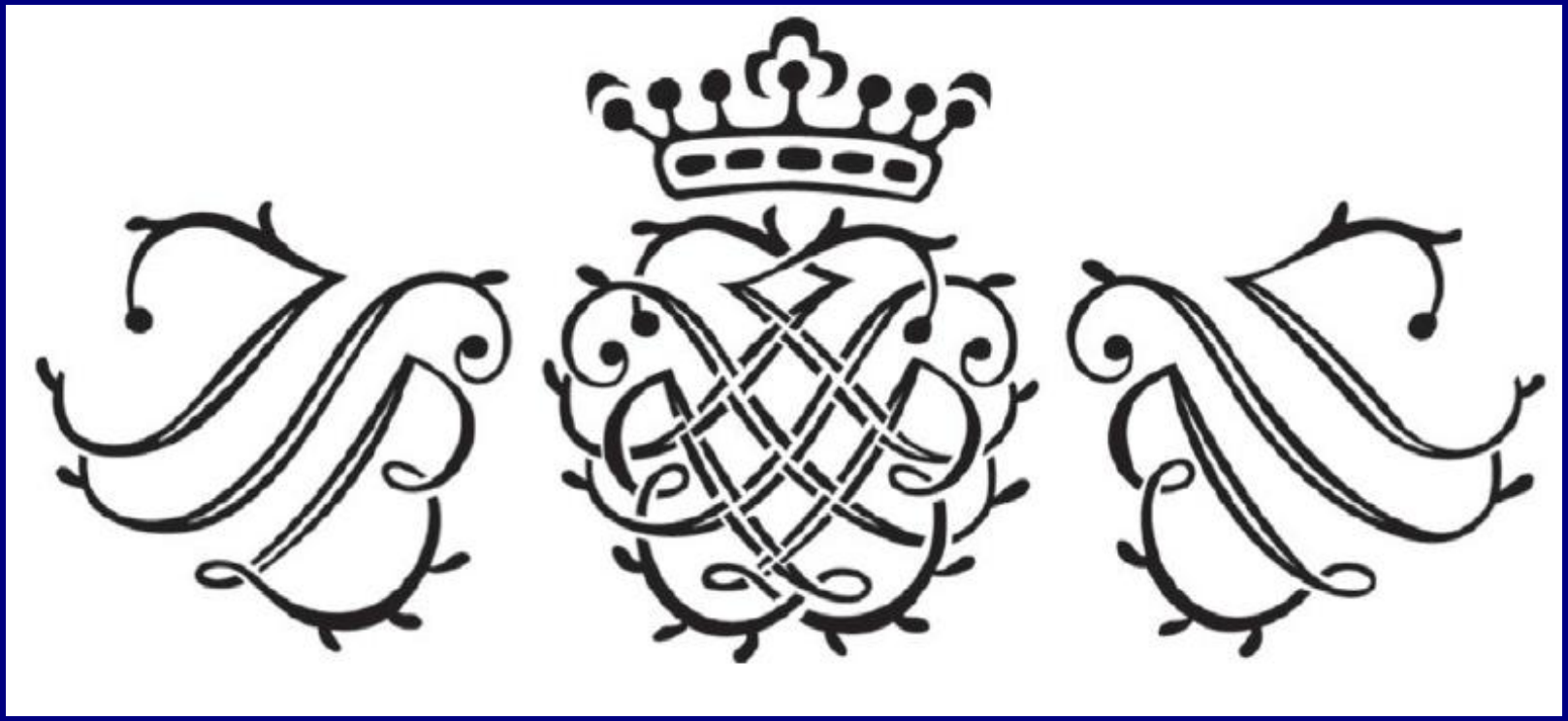
Bach's Love of Symmetry

J S Bach was deeply familiar with symmetry.

This is illustrated by the seal he designed in 1722.

Bach's Love of Symmetry

J S Bach was deeply familiar with symmetry.



This is illustrated by the seal he designed in 1722.

Symmetry and Group Theory

Symmetry is an essentially geometric concept.

The mathematical theory of symmetry is algebraic.
The key concept is that of a group.

A group is a set of elements such that any two elements can be combined to produce another.

Instead of giving the mathematical definition,
I give an example to make things clear.

The Klein 4-Group

The four orientations of a book can be described in terms of four simple rotations:

- **P:** Place book upright with front cover upright
- **R:** Rotate 180° about vertical through centre
- **I:** Rotate 180° about horizontal through centre
- **RI:** Rotate 180° about perp. through centre

These operations make up the **Klein 4-Group.**

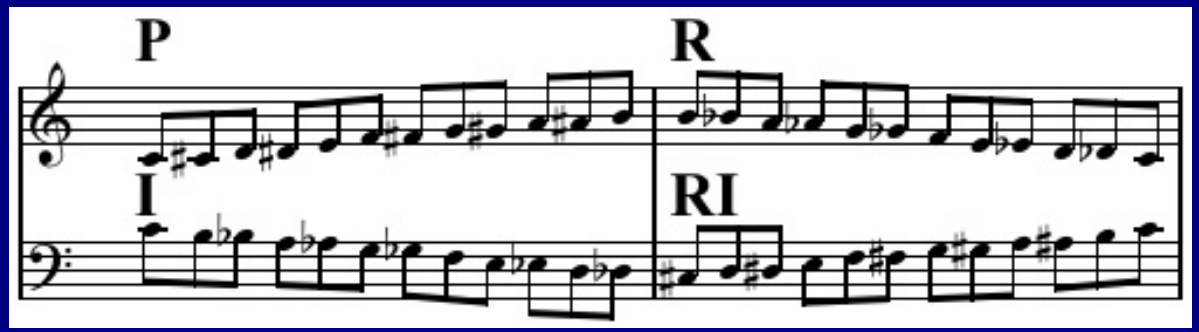
Twelve-tone Music

Table: Klein 4-Group.

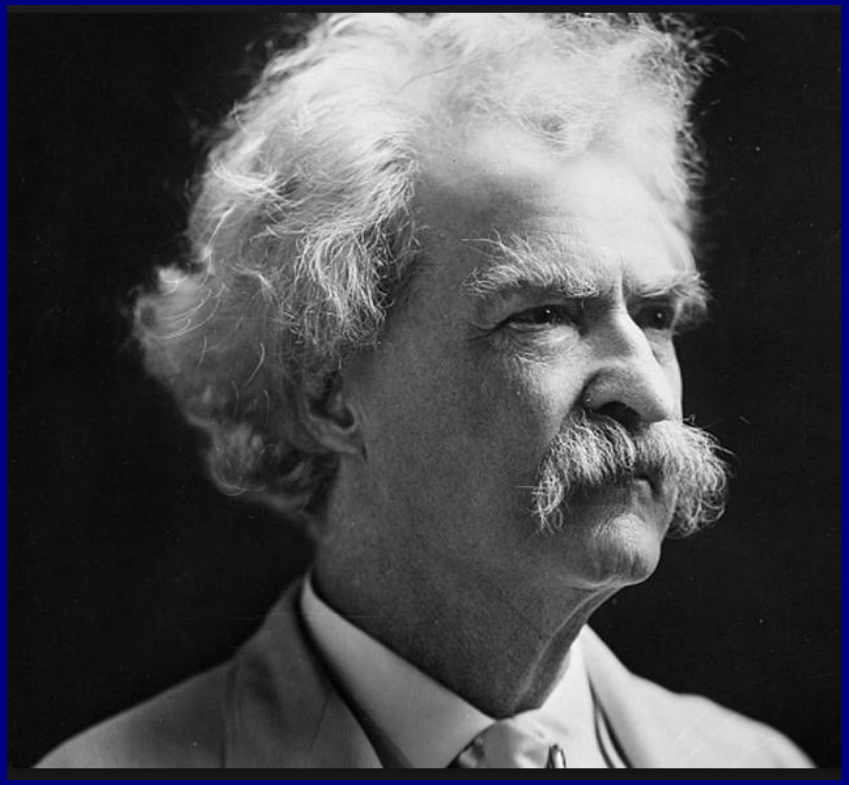
	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

The Klein 4-group is the basic group of transformations in twelve tone music.

The operations are retrogression (R), inversion (I) and the rotation (RI).



Mark Twain on Wagner



"Wagner's music is much better than it sounds."

I am not aware whether Mark Twain ever commented on Arnold Schoenberg's music !

Outline

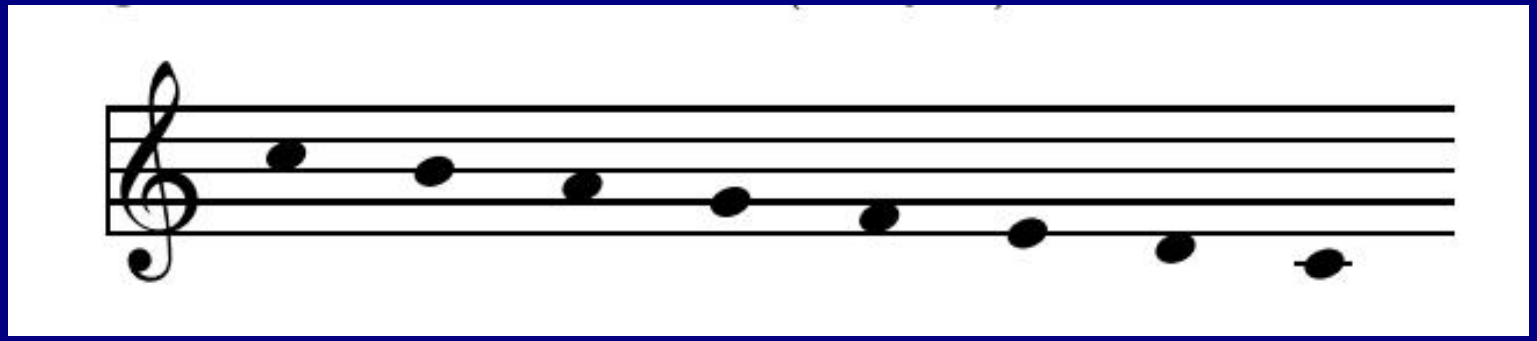
Leibniz on Music

Gottfried Wilhelm Leibniz

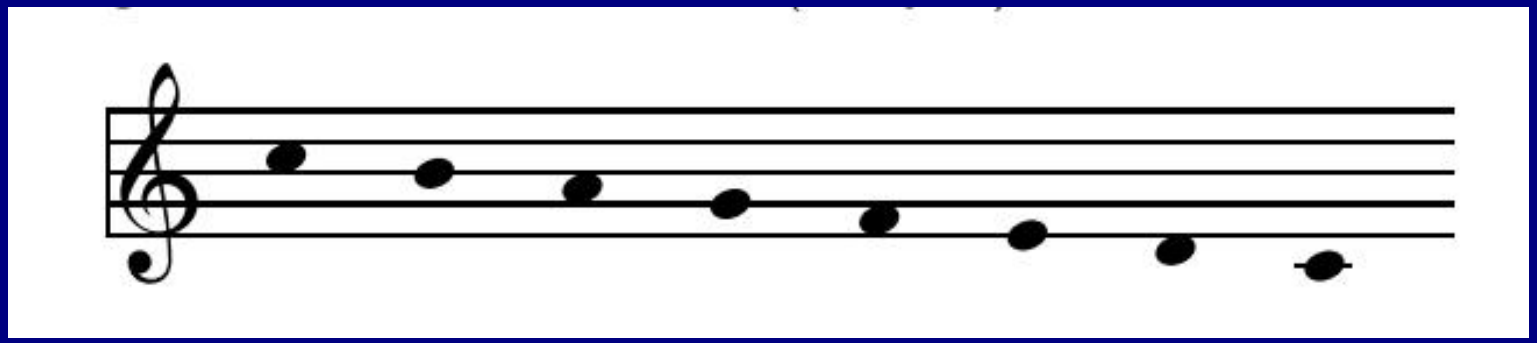
(1646–1716)

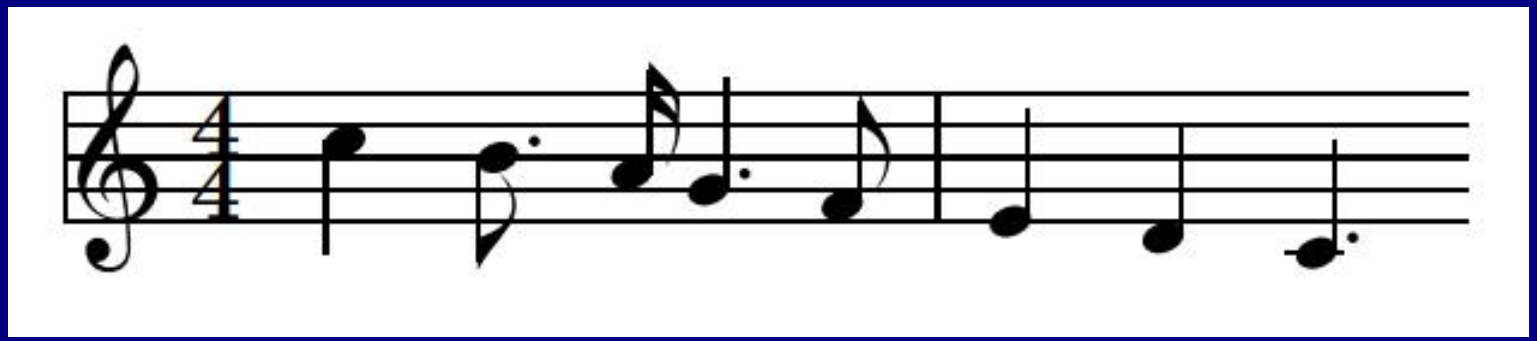
"Music is the pleasure the human mind experiences from counting without being aware that it is counting."

Rhythm in Music



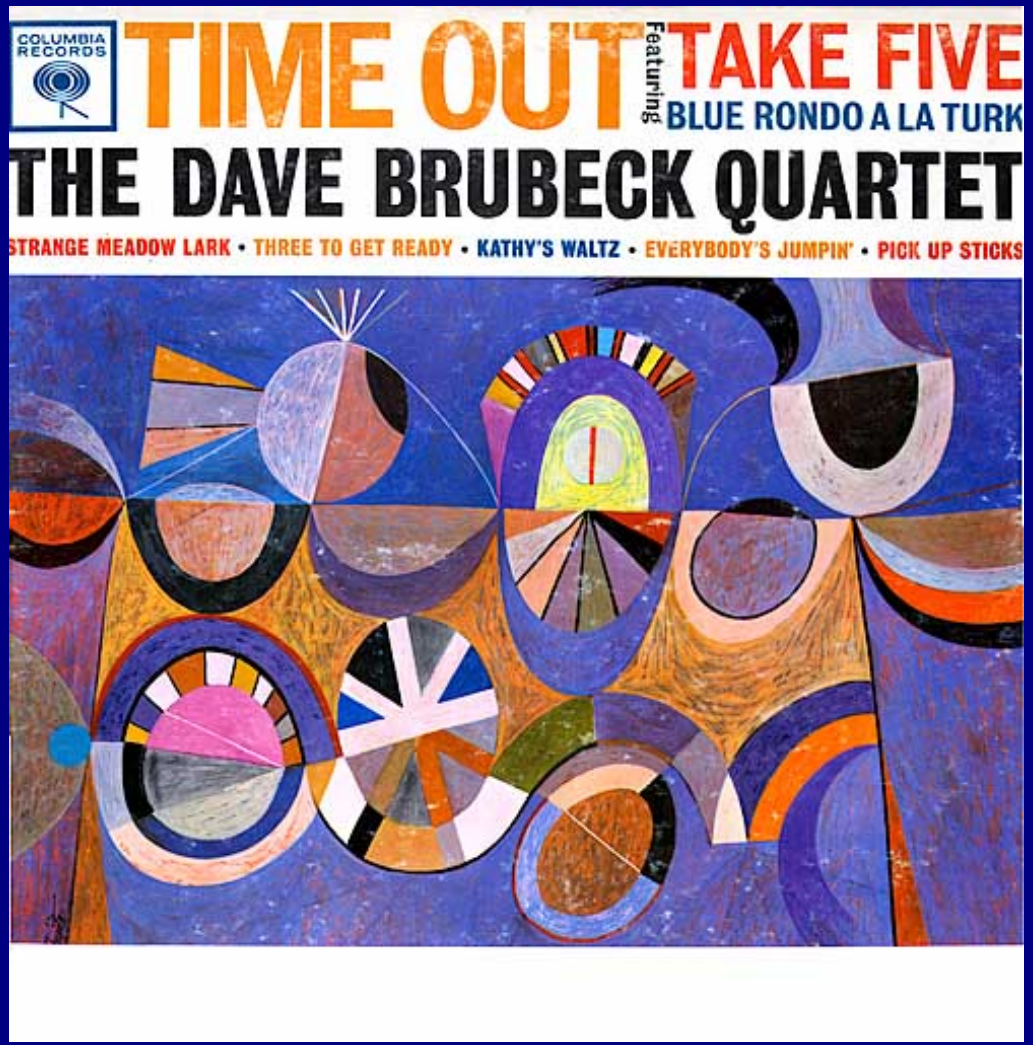
Rhythm in Music



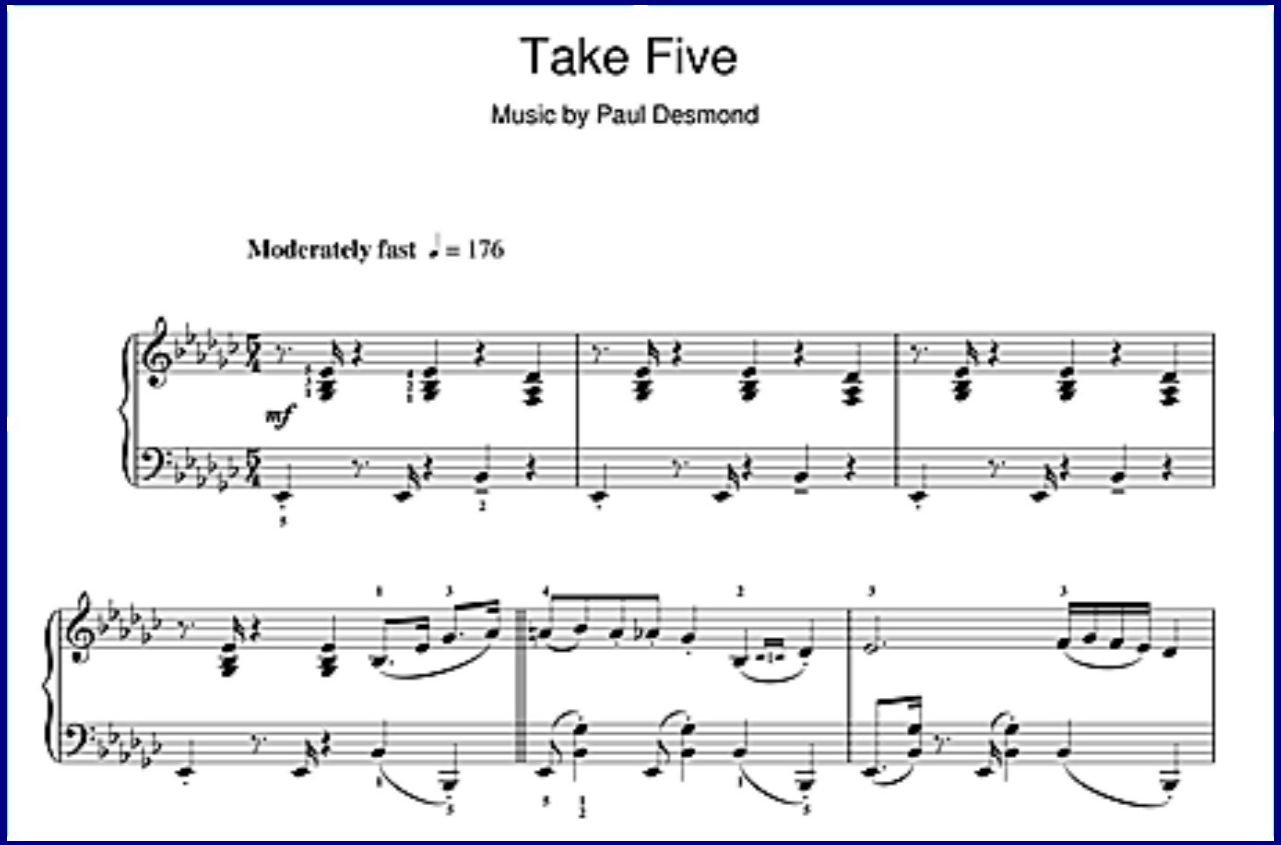


Joy to the World, the Lord is Come.

Time Out (1959)

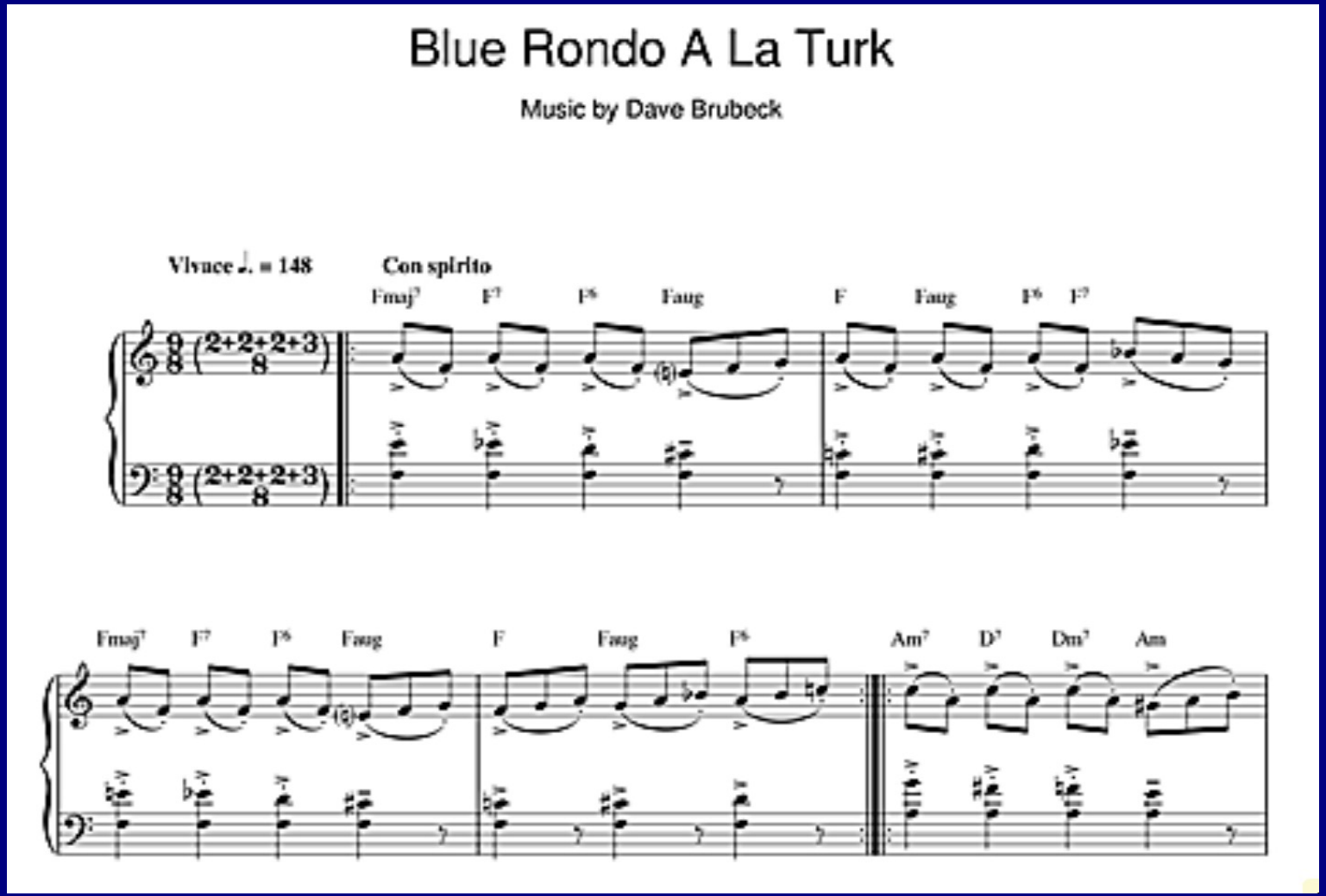


Take Five: 5/4 Time

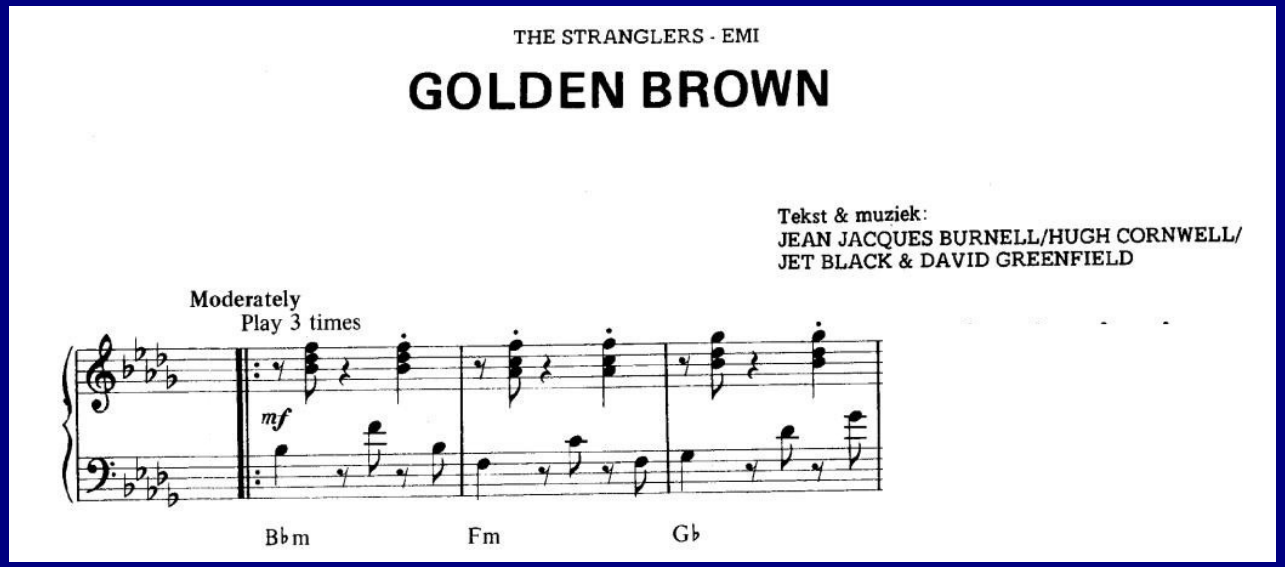


Based upon use of time signatures that were unusual for jazz: $\frac{5}{4}$, $\frac{9}{8}$, etc.

Blue Rondo a la Turk: 9/8 Time



The Stranglers: *Golden Brown*


THE STRANGLERS - EMI

GOLDEN BROWN

Tekst & muziek:
JEAN JACQUES BURNELL/HUGH CORNWELL/
JET BLACK & DAVID GREENFIELD

Moderately
Play 3 times

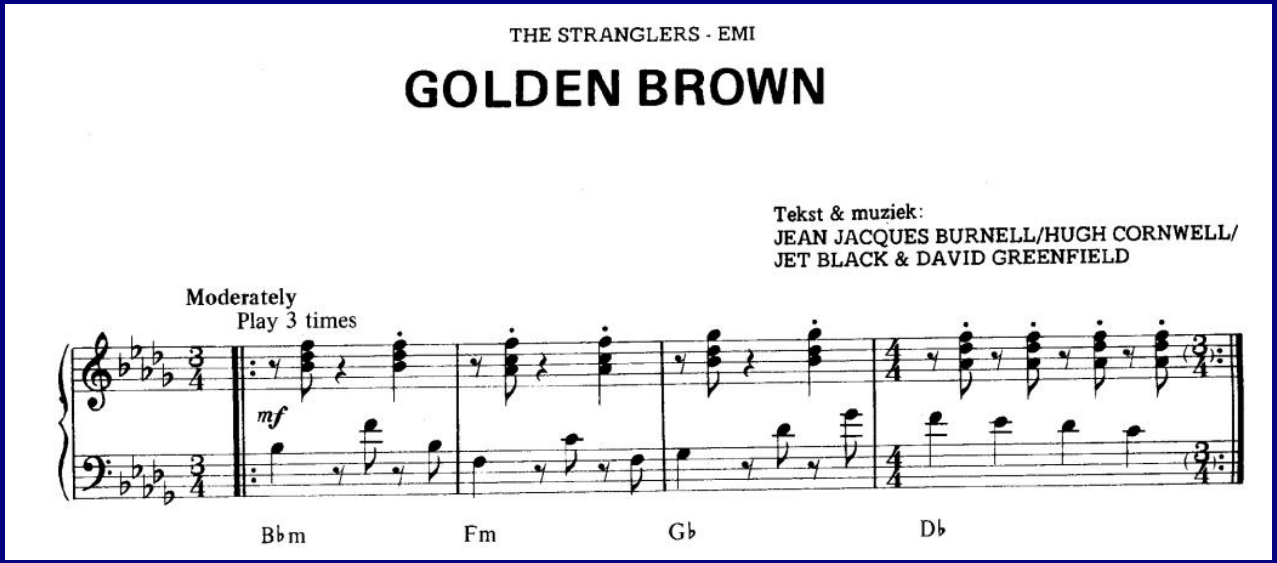
mf

B♭m Fm G♭

Can you work out the time signature?

Listen to the video clip.

The Stranglers: *Golden Brown*



It looks like (3+3+3+4)/4 = 13/4.

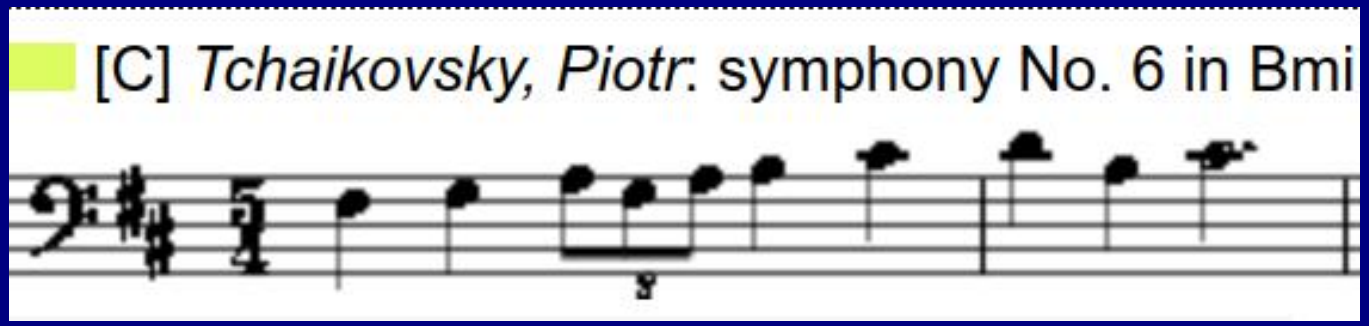
Listen to the video clip.

Pink Floyd: *Money*

Pink Floyd: *Money*

***Money* is composed mainly in 7/4 time.**

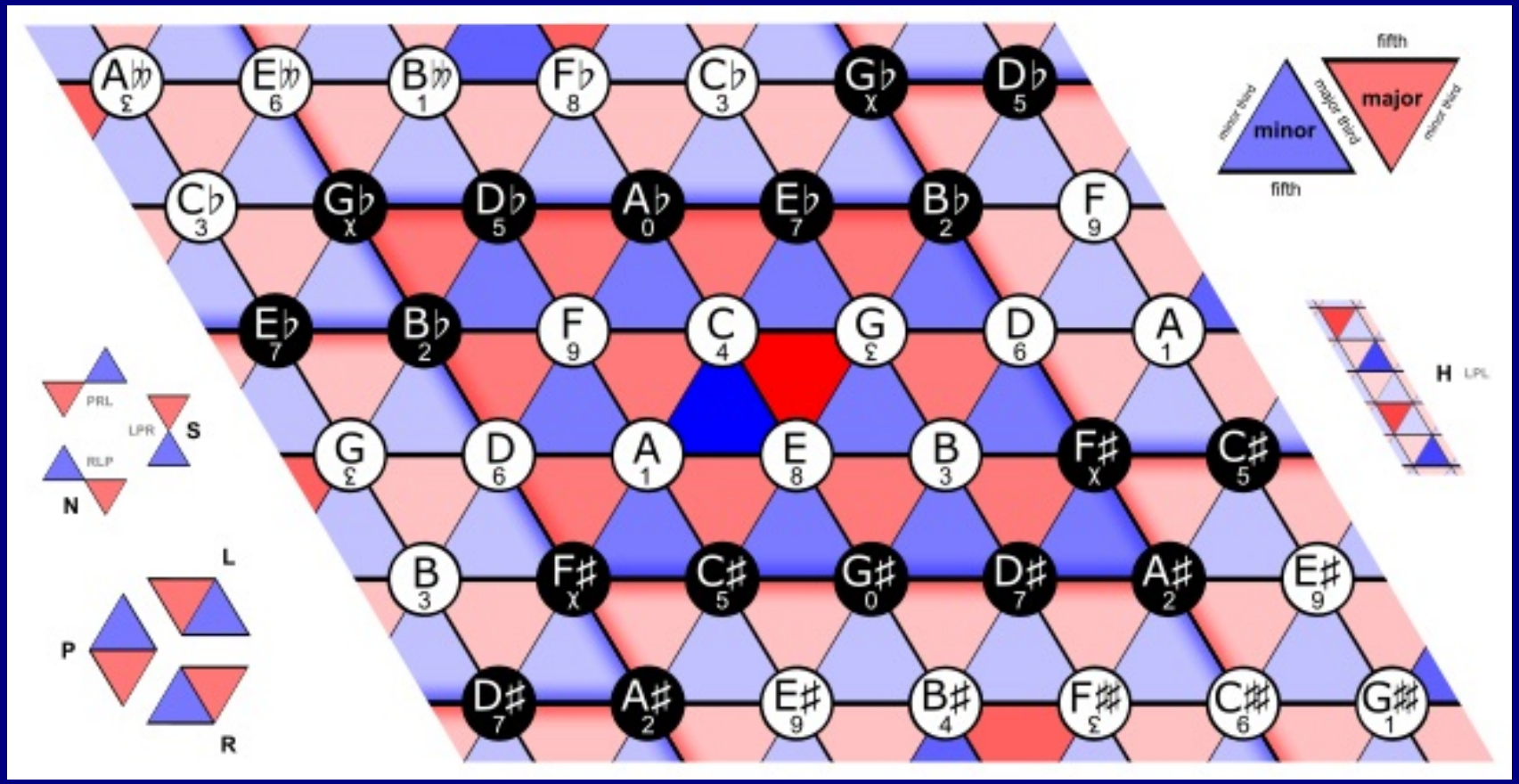
Tchaikovsky's *Pathétique Symphony*



The second movement, a dance form in 5/4 time, has been described as a "limping waltz".

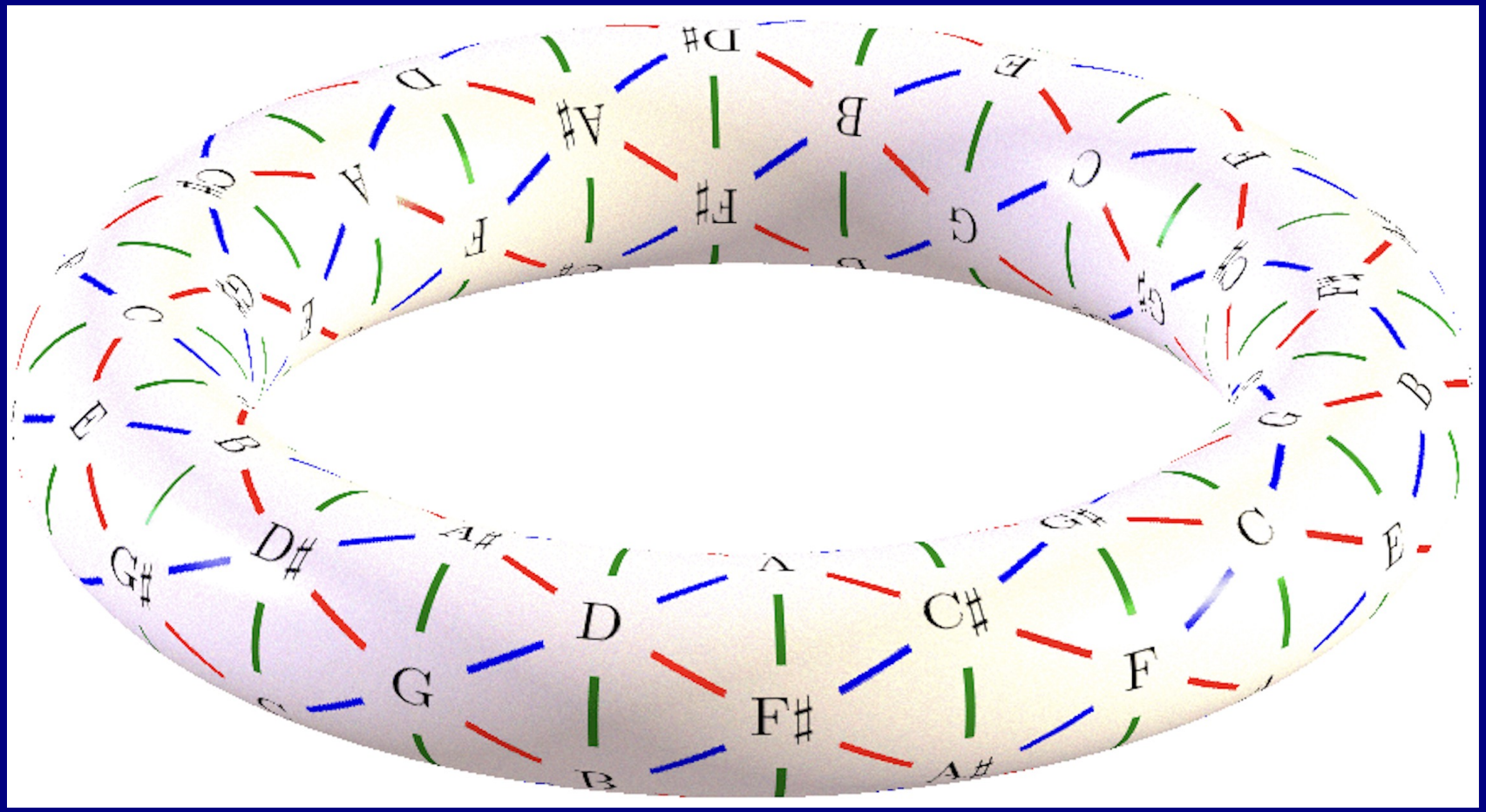
Outline

The Tonnetz



This has the topology of a torus or doughnut.

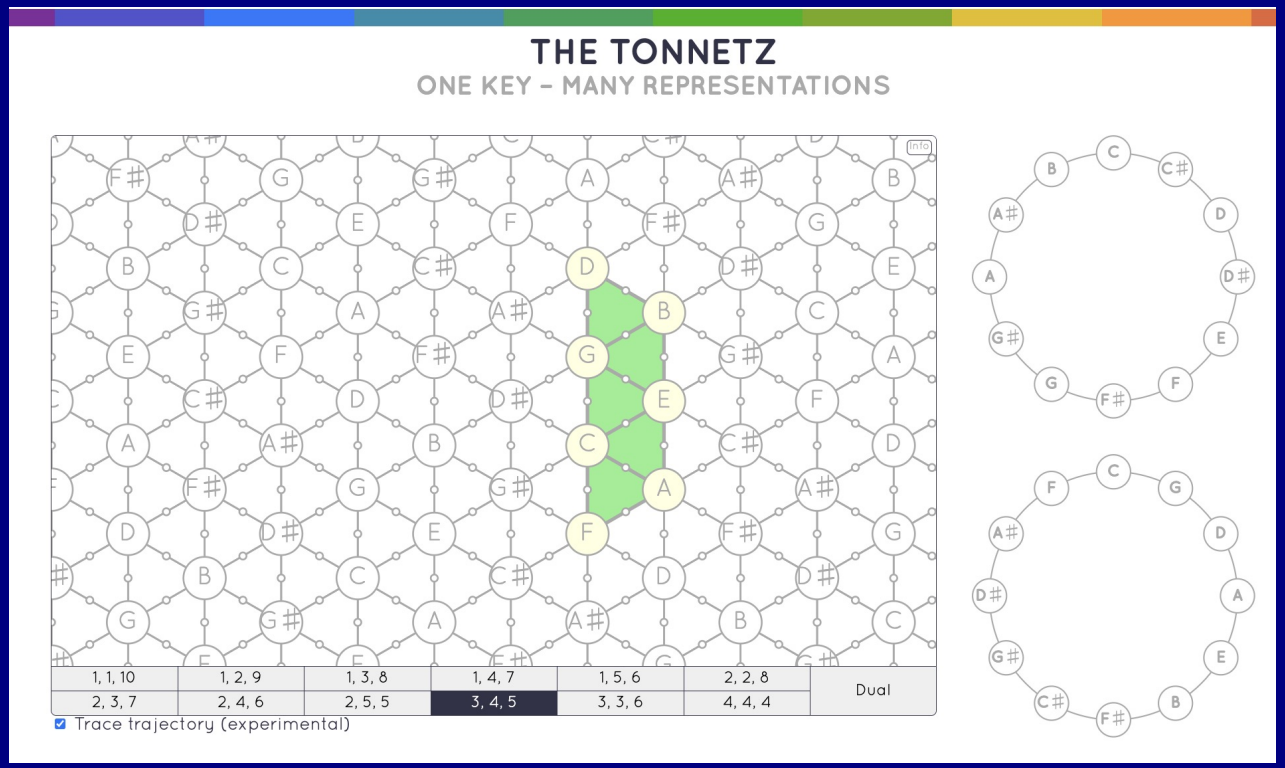
The Tonnetz Topology is the Torus



Animated gif of Tonnetz on a Torus.

file:///Users/peter/Dropbox/Music/U3A-MusicClips/TonnetzTorus.gif

Online Interactive Tonnetz



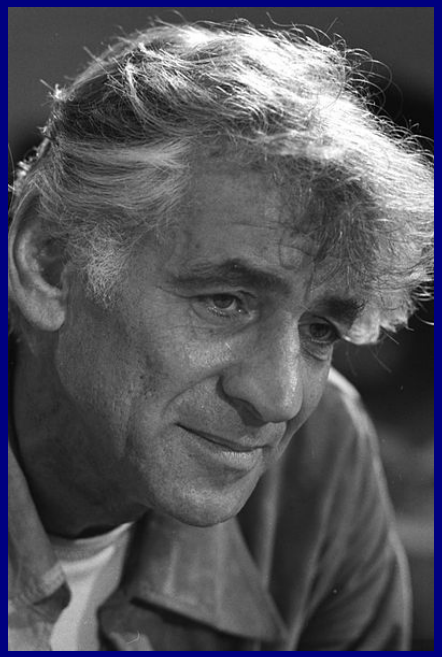
https://thetonnetz.com/

Leonard Cohen: Hallelujah

file:///Users/peter/Dropbox/Music/U3A-MusicClips/LeonardCohen-Hallelujah.mp4

Outline

Leonard Bernstein



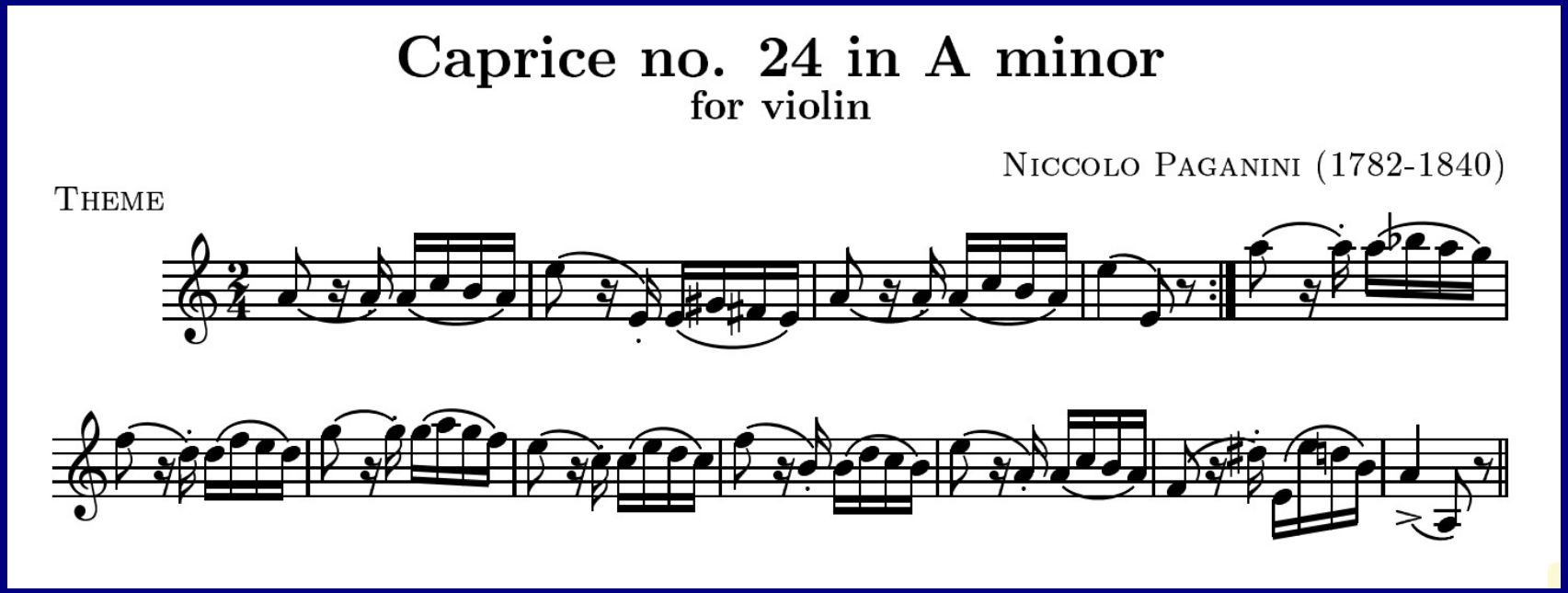
"Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?

" . . . music is not only a mysterious and metaphorical art; it is also born of science.

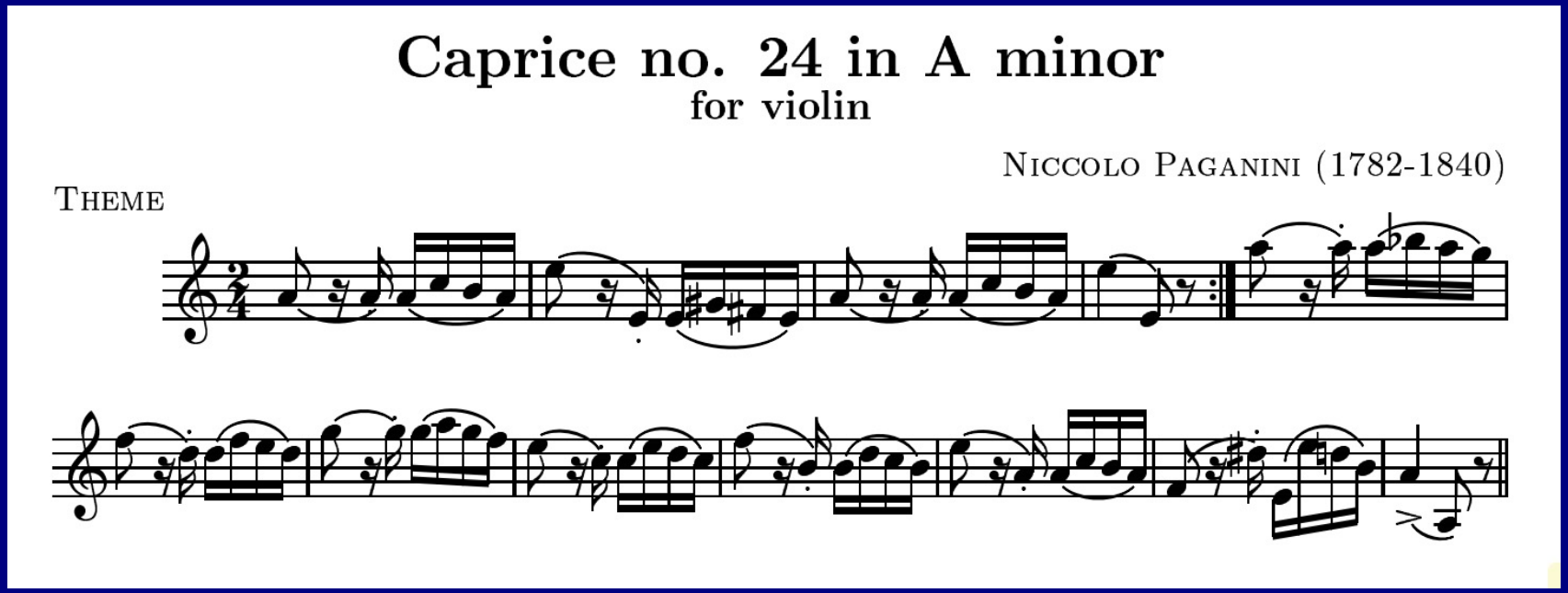
"It is made of mathematically measurable elements.

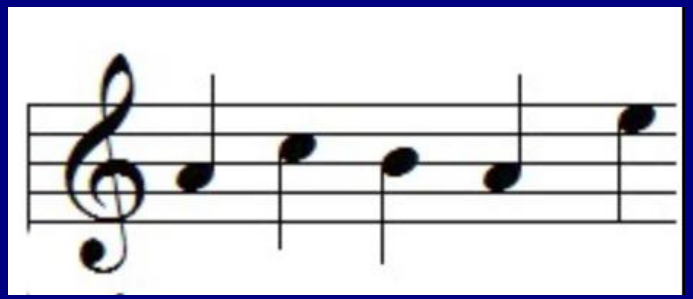
. . . any explication of music must combine mathematics with aesthetics."

Paganini: Caprice 24



Paganini: Caprice 24

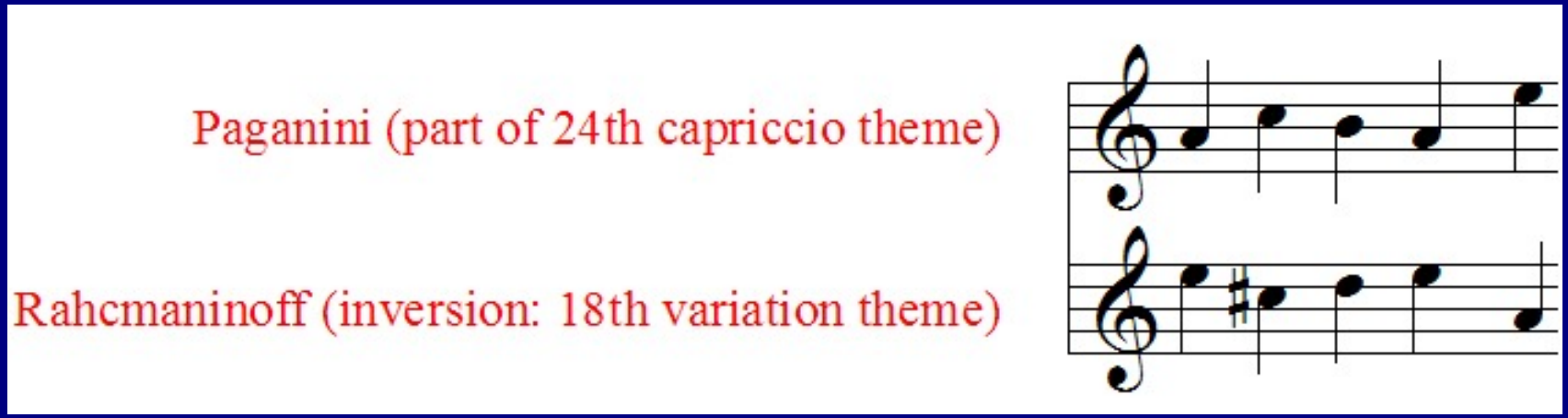




Rachmaninov's *Rhapsody*

Stephen Hough, British-Australian classical pianist

file:///Users/peter/Dropbox/Music/U3A-MusicClips/Rachmaninov-Clip1.mp4



file:///Users/peter/Dropbox/Music/U3A-MusicClips/Rachmaninov-Clip2.mp4

Rachmaninoff: Rhapsody on a Theme of Paganini — Anna Fedorova

`https://www.youtube.com/watch?v=ppJ5uITLECE&t=980s`

Thank you