Simple Mechanical Models & Complex Physical Systems

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Springs & Triads

In a Nutshell

A mathematical equivalence with a simple mechanical system sheds light on the dynamics of resonant Rossby waves in the atmosphere.

The Swinging Spring

Two distinct oscillatory modes with distinct restoring forces:

- Elastic or ‘springy’ modes
- Pendular or ‘swingy’ modes

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- Elastic or ‘springy’ modes
- Pendular or ‘swingy’ modes

Take a peek at the Java Applet

http://mathsci.ucd.ie/∼plynch/

In a paper in 1981, ***Breitenberger and Mueller*** **made the following comment:**

**"This simple system looks like a toy at best,
but its behaviour is astonishingly complex,
with many facets of more than academic lustre."**

I hope to convince you of the validity of this remark.

Lagrange's Equations of Motion

Joseph Louis Lagrange had a brilliant realization:

The dynamics of a wide range of mechanical systems are encapsulated in a simple function of the coordinates:

$$L = T - V = \textbf{K.E.} - \textbf{P.E.}$$

We now call L the Lagrangian.

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We now call L the Lagrangian.

The Lagrange equations of motion may be written:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\rho} = \frac{\partial L}{\partial q_\rho}$$

The Exact Equations for the Spring

In Cartesian coordinates the Lagrangian is

$$L = T - V = \underbrace{\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{Z}^2\right)}_{K.E} - \underbrace{\tfrac{1}{2}k(r - \ell_0)^2}_{E.P.E} - \underbrace{mgZ}_{G.P.E}$$

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The equations of motion are (with $\omega_Z^2 \equiv k/m$):

$$\begin{aligned}
\ddot{x} &= -\omega_Z^2\left(\frac{r-\ell_0}{r}\right)x \\
\ddot{y} &= -\omega_Z^2\left(\frac{r-\ell_0}{r}\right)y \\
\ddot{Z} &= -\omega_Z^2\left(\frac{r-\ell_0}{r}\right)Z - g
\end{aligned}$$

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Two constants, energy and angular momentum:

$$E = T + V \qquad h = x\dot{y} - y\dot{x}\,.$$

Regular and Chaotic Motion

Two invariants, three DOF:
The system is <u>not integrable.</u>

We consider the phenomenon of Resonance.

For the spring, resonance occurs for

$$\omega_Z = 2\omega_R \, , \qquad \epsilon = \frac{1}{2} \, .$$

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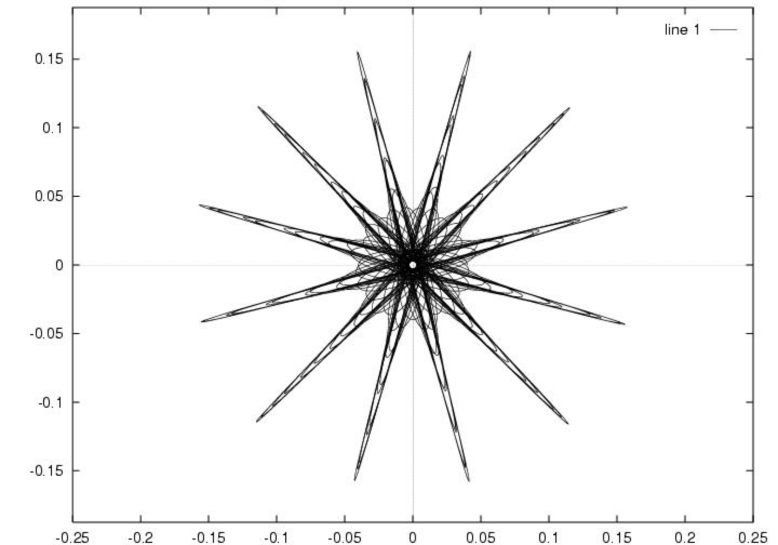
$$\omega_Z = 2\omega_R \,, \qquad \epsilon = \frac{1}{2} \,.$$

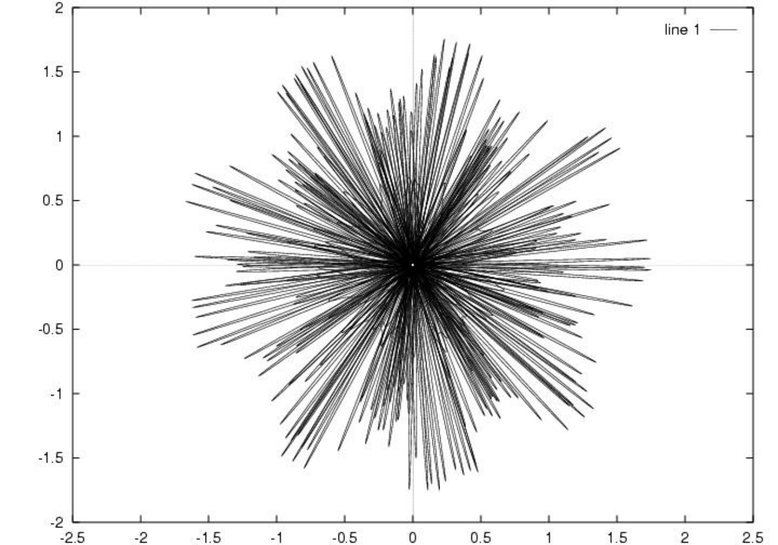
For *small amplitudes*, the motion is quasi-integrable.

We look at two numerical solutions, one with small amplitude, one with large.

Horizontal plan: Low energy case



Horizontal plan: High energy case



The Resonant Case

The Lagrangian, to cubic order is:

$$L = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \tfrac{1}{2}\left(\omega_R^2(x^2+y^2) + \omega_Z^2 z^2\right) + \tfrac{1}{2}\lambda(x^2+y^2)z\,,$$

We study the resonant case:

$$\omega_Z = 2\omega_R\,.$$

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We study the resonant case:

$$\omega_Z = 2\omega_R\,.$$

A, B and C are amplitudes in x, y and Z directions.

Using the **Averaged Lagrangian Technique**, the equations for the modulation amplitudes are:

$$i\dot{A} = B^* C\,,$$
$$i\dot{B} = C A^*\,,$$
$$i\dot{C} = AB\,,$$

These are the ***three-wave interaction equations***.

Ubiquity of Three-Wave Equations

- Modulation equations for wave interactions in fluids and plasmas.
- Three-wave equations govern envelop dynamics of light waves in an inhomogeneous material; and phonons in solids.
- Maxwell-Schrödinger envelop equations for radiation in a two-level resonant medium in a microwave cavity.
- Euler's equations for a freely rotating rigid body (when $H = 0$).

Analytical Solution of 3-Wave Equations

We can derive complete analytical expressions for the amplitudes and phases.

The amplitudes are expressed as elliptic functions.
The phases are expressed as elliptic integrals.

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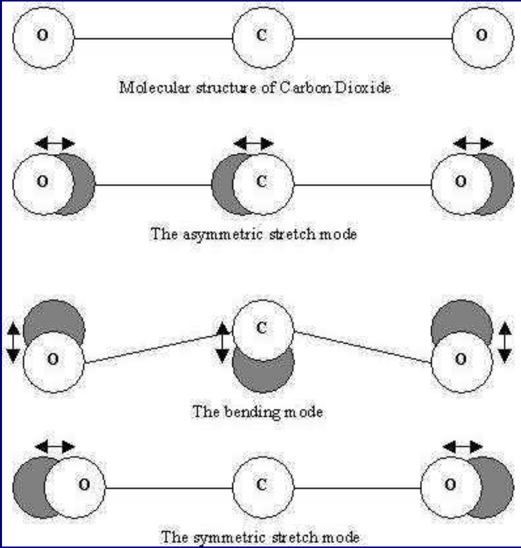
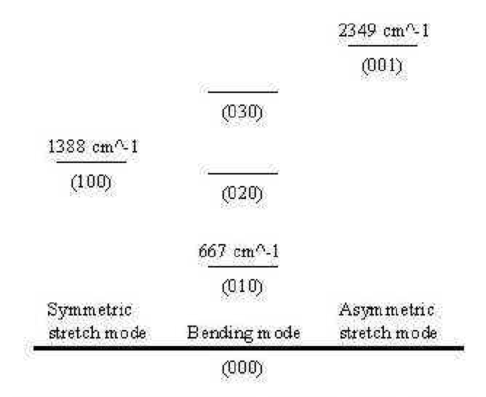
The phases are expressed as elliptic integrals.

The complete details are given in:

Lynch, Peter, and Conor Houghton, 2004:
Pulsation and Precession of the Resonant Swinging Spring.
Physica D, 190,1-2, 38-62
(See `http://www.maths.tcd.ie/~plynch`)

Vibrations of CO_2 Molecule





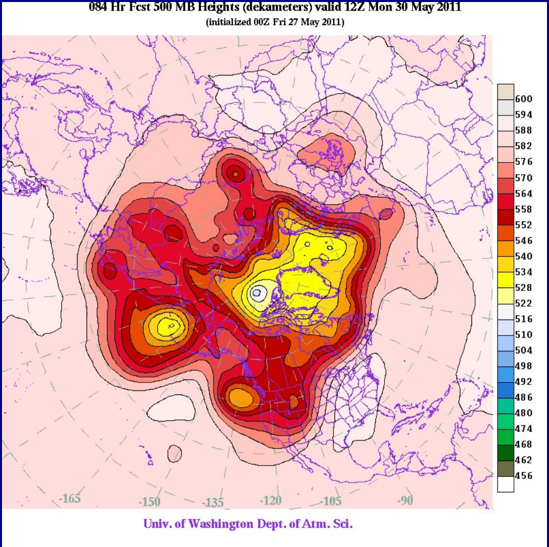
The first few vibrational energy levels of the CO2 molecule

$$2 \times 667 = 1334 \approx 1388$$

Stretching frequency ≈ Twice bending frequency.

Waves in the Atmosphere

500 hPa forecast for midday today



Potential Vorticity Conservation

From the *Shallow Water Equations*, we derive the principle of conservation of potential vorticity:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

where ζ is the relative vorticity, f is the planetary vorticity and h is the fluid depth.

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where ζ is the relative vorticity, f is the planetary vorticity and h is the fluid depth.

Under the assumptions of quasi-geostrophic theory, the dynamics reduce to an equation for ψ alone:

$$\frac{\partial}{\partial t}[\nabla^2\psi - F\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0$$

This is the barotropic quasi-geostrophic potential vorticity equation, used to model weather systems.

Rossby Waves

Wave-like solution of the vorticity equation:

$$\psi = A\cos(kx + \ell y - \sigma t)$$

satisfies the equation provided

$$\sigma = -\frac{k\beta}{k^2 + \ell^2 + F}\,.$$

This is the celebrated Rossby wave formula

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With more than one wave, the components *interact with each other* through the nonlinear terms.

Resonant Rossby Wave Triads

Case of special interest: Two wave components produce a third such that its interaction with each generates the other.

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By a multiple time-scale analysis, we derive the *modulation equations* for the wave amplitudes:

$$
\begin{aligned}
i\dot{A} &= B^*C \\
i\dot{B} &= CA^* \\
i\dot{C} &= AB
\end{aligned}
$$

[Canonical form of the *three-wave equations*].

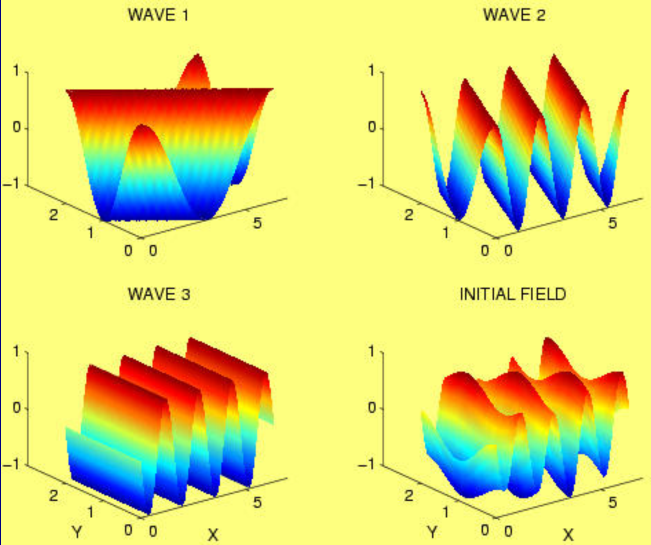
The Spring Equations

and the

Triad Equations are

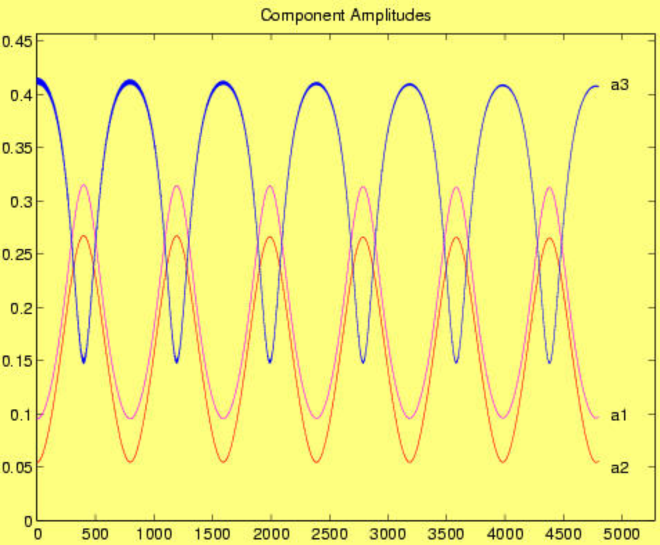
are

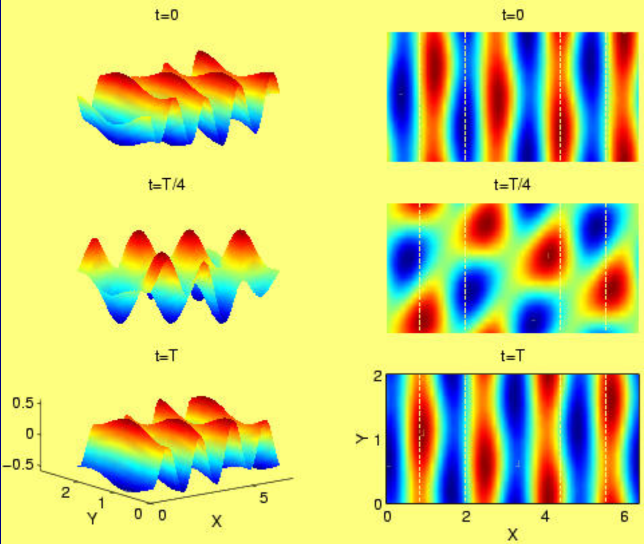
Mathematically Identical!



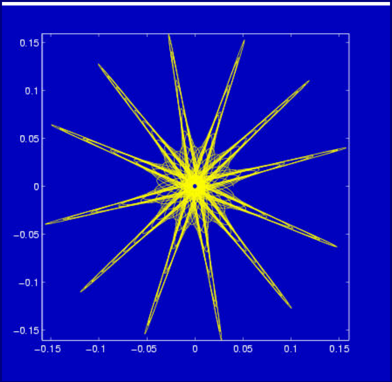
Components of a resonant Rossby wave triad
All fields are scaled to have unit amplitude.

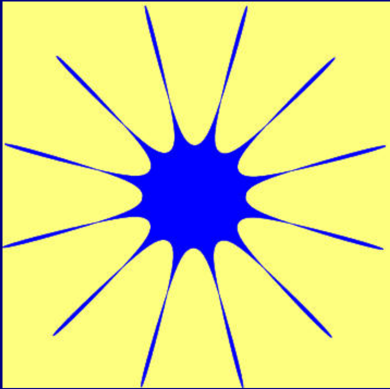
Variation with time of the amplitudes of three components of the stream function.





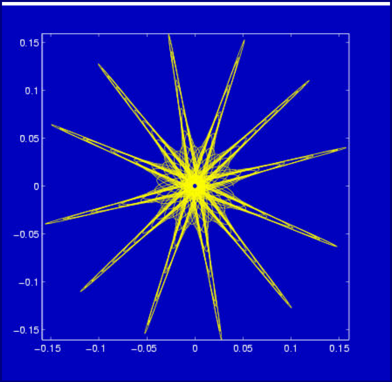
Stream function at three times during an integration of duration $T = 4800$ days.

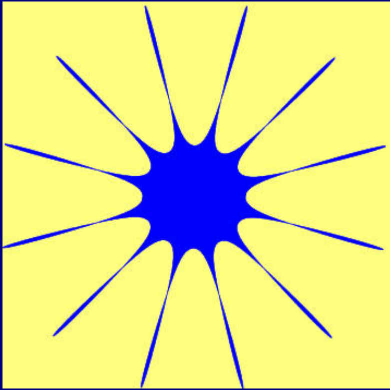




Left: Horizontal projection of spring solution, y vs. x.

Right: Polar plot of A_{maj} versus θ for resonant triad.





Left: Horizontal projection of spring solution, y vs. x.

Right: Polar plot of A_{maj} versus θ for resonant triad.

Take another peek at the Applet!

Review

I hope I have convinced you that:

> **This simple system looks like a toy at best, but its behaviour is astonishingly complex, with many facets of more than academic lustre . . .** (*Breitenberger and Mueller, 1981*)

. . . and that the Swinging Spring is a valuable model of some important aspects of atmospheric dynamics.

Banknotes with Mathematicians

[Applied Mathematicians and Physicists]

Galileo Galilei

Isaac Newton

Christiaan Huygens

Leonard Euler

Carl Friedrich Gauss

Blaise Pascal

Rene Descartes

Benjamin Franklin

Olivia Newton John

Erwin Schrödinger

Albert Einstein

ENIAC and PHONIAC

Charney, et al., *Tellus*, 1950.

$$\begin{bmatrix}\text{Absolute}\\ \text{Vorticity}\end{bmatrix} = \begin{bmatrix}\text{Relative}\\ \text{Vorticity}\end{bmatrix} + \begin{bmatrix}\text{Planetary}\\ \text{Vorticity}\end{bmatrix} \qquad \eta = \zeta + f\,.$$

- The atmosphere is treated as a single layer.
- The flow is assumed to be nondivergent.
- Absolute vorticity is conserved.

$$\frac{d(\zeta + f)}{dt} = 0.$$

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- The flow is assumed to be nondivergent.
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$$\frac{d(\zeta + f)}{dt} = 0.$$

This equation looks simple. But it is **nonlinear:**

$$\frac{\partial}{\partial t}[\nabla^2\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0\,,$$

Recreating the ENIAC Forecasts

The ENIAC integrations have been recreated using:

- A MATLAB program to solve the BVE
- Data from the NCEP/NCAR reanalysis

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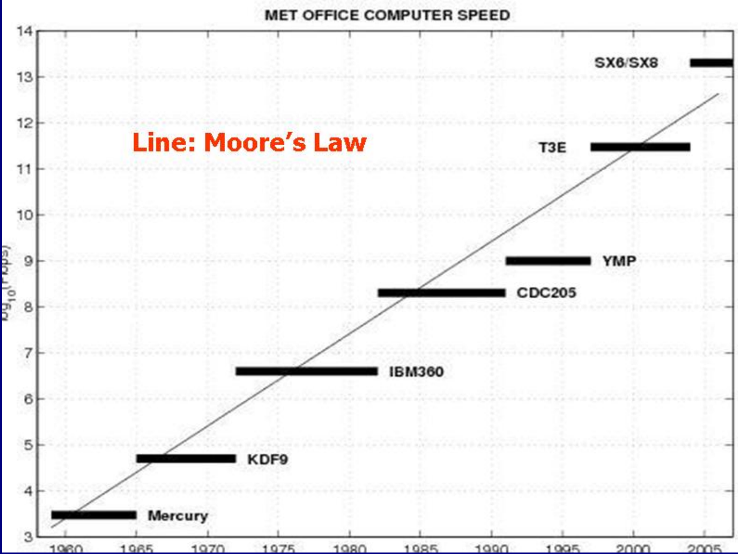
- Data from the NCEP/NCAR reanalysis

The matlab code is available on the author's website
http://maths.ucd.ie/∼plynch/eniac



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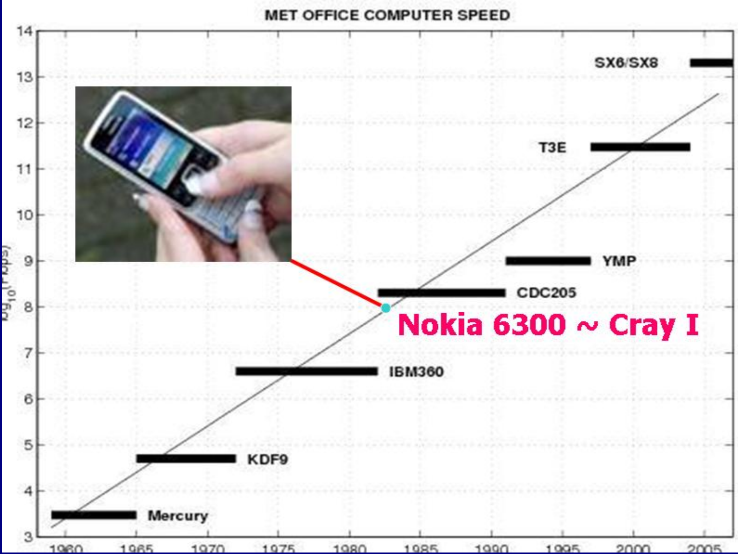
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CDC205
Nokia 6300 ~ Cray I
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Forecasts by PHONIAC

Peter Lynch & Owen Lynch

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We therefore decided to repeat the ENIAC integrations using a programmable mobile phone.

Forecasts by PHONIAC

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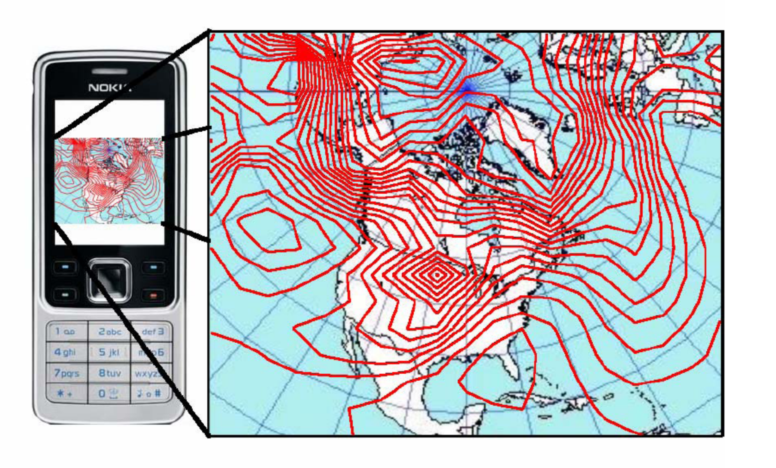
A modern hand-held mobile phone has far greater power than the ENIAC had.

We therefore decided to repeat the ENIAC integrations using a programmable mobile phone.

We converted the program ENIAC.M to PHONIAC.JAR, a J2ME application, and implemented it on a mobile phone.

This technology has great potential for generation and delivery of operational weather forecast products.

PHONIAC: Portable Hand Operated Numerical Integrator and Computer



Weather, November 2008

Forecasts by PHONIAC

Weather – November 2008, Vol. 63, No. 11

Peter Lynch[1] and Owen Lynch[2]

[1]*University College Dublin, Meteorology and Climate Centre, Dublin*
[2]*Dublin Software Laboratory, IBM Ireland*

The first computer weather forecasts were made in 1950, using the ENIAC (Electronic Numerical Integrator and Computer). The ENIAC forecasts led to operational numerical weather prediction within five years, and paved the way for the remarkable advances in weather prediction and climate modelling that have been made over the past half century. The basis for the forecasts was the barotropic vorticity equation (BVE). In the present study, we describe the solution of the BVE on a mobile phone (cell-phone), and repeat one of the ENIAC forecasts. We speculate on the possible applications of mobile phones for micro-scale numerical weather prediction.

The ENIAC Integrations

and John von Neumann (1950; cited below as CFvN). The story of this work was recounted by George Platzman in his Victor P. Starr Memorial Lecture (Platzman, 1979). The atmosphere was treated as a single layer, represented by conditions at the 500 hPa level, modelled by the BVE. This equation, expressing the conservation of absolute vorticity following the flow, gives the rate of change of the Laplacian of height in terms of the advection. The tendency of the height field is obtained by solving a Poisson equation with homogeneous boundary conditions. The height field may then be advanced to the next time level. With a one hour time-step, this cycle is repeated 24 times for a one-day forecast.

The initial data for the forecasts were prepared manually from standard operational 500 hPa analysis charts of the U.S. Weather Bureau, discretised to a grid of 19 by 16 points, with grid interval of 736 km. Centred spatial finite differences and a leapfrog time-scheme were used. The boundary conditions for height were held constant throughout each 24-hour integration. The forecast starting at 0300 UTC, January 5, 1949 is shown in

vorticity. The forecast height and vorticity are shown in the right panel. The feature of primary interest was an intense depression over the United States. This deepened, moving NE to the 90 °W meridian in 24 hours. A discussion of this forecast, which underestimated the development of the depression, may be found in CFvN and in Lynch (2008).

Dramatic growth in computing power

The oft-cited paper in *Tellus* (CFvN) gives a complete account of the computational algorithm and discusses four forecast cases. The ENIAC, which had been completed in 1945, was the first programmable electronic digital computer ever built. It was a gigantic machine, with 18,000 thermionic valves, filling a large room and consuming 140 kW of power. Input and output was by means of punch-cards. McCartney (1999) provides an absorbing account of the origins, design, development and destiny of ENIAC.

Advances in computer technology over the past half-century have been spectacular. The increase in computing power is encap-

A Challenge to you all . . .

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Run an NWP model on a Smart Phone

A Challenge to you all . . .

Run an NWP model on a Smart Phone

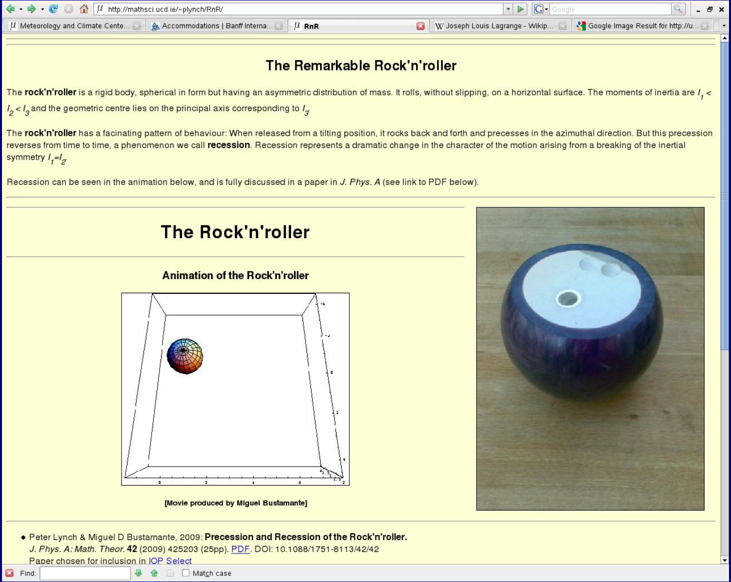
There are many more possibilities for these devices.

The Rock'n'roller

A Bowling-ball from Stillorgan

Thanks to Brian O'Connor (School of Physics) for slicing the top off

Recession I: see website



The Physical System

Consider a spherical rigid body with an asymmetric mass distribution.

Specifically, we consider a loaded sphere.

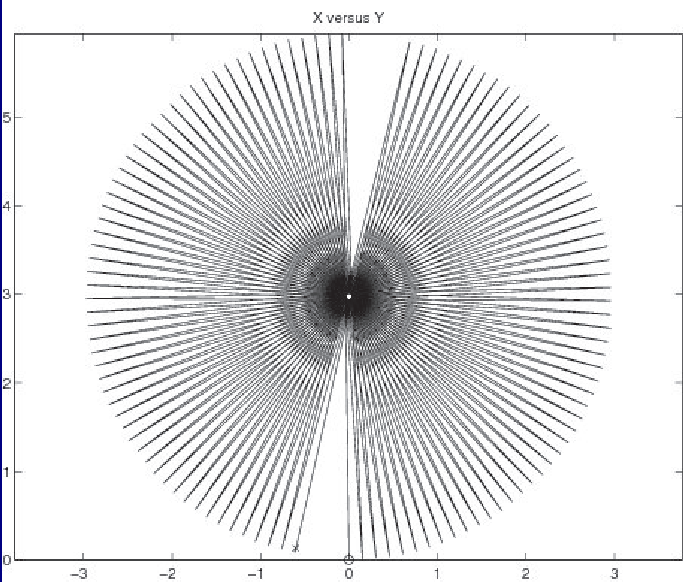
The dynamics are essentially the same as for the tippe-top, which has been studied extensively.

Unit radius and unit mass.

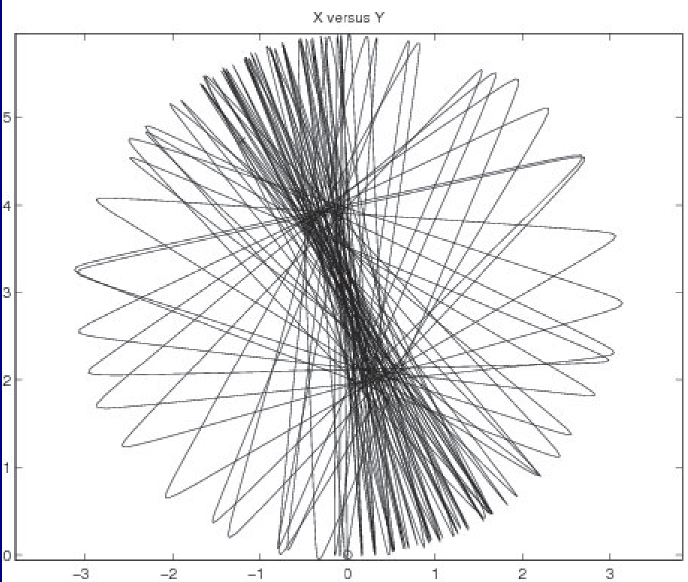
Centre of mass off-set a distance *a* from the centre.

Moments of inertia I_1, I_2 and I_3, with $I_1 \approx I_2 < I_3$.

Symmetric Case: Routh Sphere ($I_1 = I_2$)



Asymmetric Case: Rock'n'roller ($I_1 < I_2$)



The Lagrangian

The Lagrangian of the system is easily written down:

$$L = \tfrac{1}{2}(\mathbf{I_1}\omega_1^2 + \mathbf{I_2}\omega_2^2 + \mathbf{I_3}\omega_3^2) + \tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - ga(1 - \cos\theta)$$

The equations may then be written (in vector form):

$$\mathbf{\Sigma}\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}\,, \qquad \mathbf{K}\dot{\boldsymbol{\omega}} = \mathbf{P}_\omega$$

where the matrices $\mathbf{\Sigma}$ and K are known and

$$\mathbf{P}_\omega = \begin{pmatrix} -(g + \omega_1^2 + \omega_2^2)as\chi + (\mathbf{I_2} - \mathbf{I_3} - af)\omega_2\omega_3 \\ (g + \omega_1^2 + \omega_2^2)as\sigma + (\mathbf{I_3} - \mathbf{I_1} + af)\omega_1\omega_3 \\ (\mathbf{I_1} - \mathbf{I_2})\omega_1\omega_2 + as(-\chi\omega_1 + \sigma\omega_2)\omega_3 \end{pmatrix}$$

Note that neither K nor $\mathbf{P}_\omega$ depends explicitly on ϕ.

Nonholonomic Constraints

Assume nonholonomic constraints

$$g_k(q_\rho, \dot{q}_\rho) = 0 \, .$$

When the constraints are linear in the velocities, we can write the equations as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_k \mu_k \frac{\partial g_k}{\partial \dot{q}_i} = 0 \, .$$

For the Rock'n'roller, we have one holonomic constraint and two nonholonomic constraints.

The enigma of nonholonomic constraints

M. R. Flannery[a)]
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

(Received 16 February 2004; accepted 8 October 2004)

The problems associated with the modification of Hamilton's principle to cover nonholonomic constraints by the application of the multiplier theorem of variational calculus are discussed. The reason for the problems is subtle and is discussed, together with the reason why the proper account of nonholonomic constraints is outside the scope of Hamilton's variational principle. However, linear velocity constraints remain within the scope of D'Alembert's principle. A careful and comprehensive analysis facilitates the resolution of the puzzling features of nonholonomic constraints.
[DOI: 10.1119/1.1830501]

***Am. J. Phys.*, Vol 73, 265-272 (2005)**

Constants of Motion for Routh Sphere

The total energy is conserved:

$$K = \tfrac{1}{2}[u^2 + v^2 + w^2] + \tfrac{1}{2}[\mathbf{I_1}\omega_1^2 + \mathbf{I_2}\omega_2^2 + \mathbf{I_3}\omega_3^2] + mga(1 - \cos\theta)\,.$$

Jellett's constant is the scalar product:

$$C_J = \mathbf{L}\cdot\mathbf{r} = \mathbf{I_1} s(\sigma\omega_1 + \chi\omega_2) + \mathbf{I_3} f\,\omega_3 = \text{constant}\,.$$

where $f = \cos\theta - a$, $\sigma = \sin\psi$ **and** $\chi = \cos\psi$.

Stephen O'Brien & John L Synge first gave this interpretation

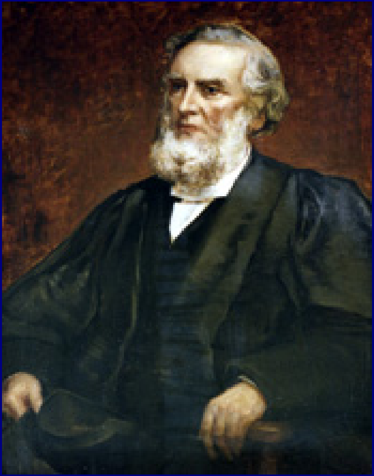
Routh's constant (difficult to interpret physically):

$$C_R = \left[\sqrt{\mathbf{I_3} + s^2 + (\mathbf{I_3}/\mathbf{I_1})f^2}\right]\omega_3 = \text{constant}\,.$$

Edward J Routh

1831–1907

John H Jellett



1817–1888

Journal of Physics A: Mathematical and Theoretical

Precession and recession of the rock'n'roller

IOPSELECT

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Journal Journal of Physics A: Mathematical and Theoretical Create an alert RSS this journal

Issue Volume 42, Number 42

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doi: 10.1088/1751-8113/42/42/425203

Article **References**

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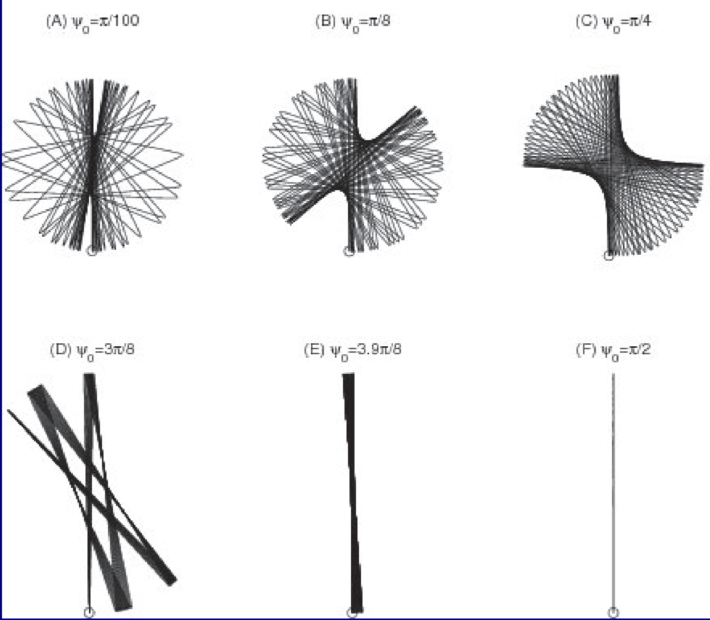
Abstract We study the dynamics of a spherical rigid body that rocks and rolls on a plane under the effect of gra
distribution of mass is non-uniform and the centre of mass does not coincide with the geometric centre
symmetric case, with moments of inertia $I_1 = I_2 < I_3$, is integrable and the motion is completely regular.



(A) $\psi_0=\pi/100$
(B) $\psi_0=\pi/8$
(C) $\psi_0=\pi/4$
(D) $\psi_0=3\pi/8$
(E) $\psi_0=3.9\pi/8$
(F) $\psi_0=\pi/2$

Orbit of stars in a Globular Cluster

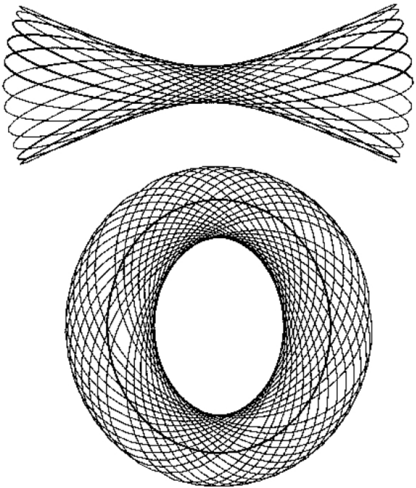
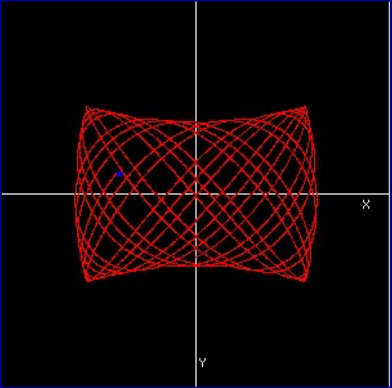
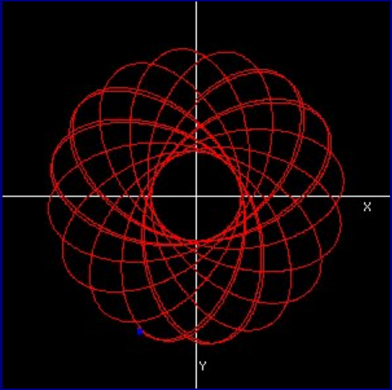


Figure 3.8 Two orbits of a common energy in the potential Φ_L of equation (3.103) when $v_0 = 1$, $q = 0.9$ and $R_c = 0.14$: top, a box orbit; bottom, a loop orbit. The closed parent of the loop orbit is also shown. The energy, $E = -0.337$, is that of the isopotential surface that cuts the long axis at $x = 5R_c$.

A Globular Cluster (m22)





Box orbit (left) and loop orbit (right)

Quaternionic Formulation

The Euler angles have a singularity when $\theta = 0$
The angles ϕ and ψ are not uniquely defined there.

We can obviate this problem by using Euler's symmetric parameters

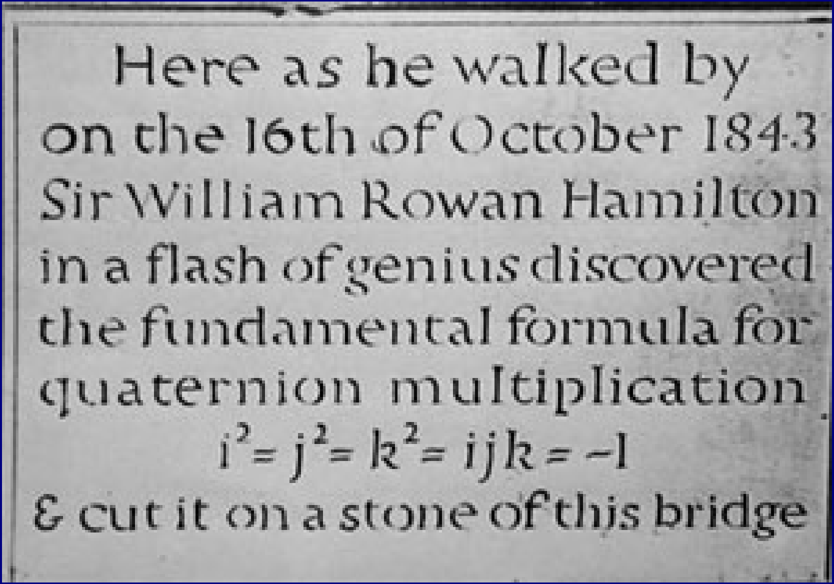
$$\gamma = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi) \qquad \xi = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi)$$

$$\zeta = \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \qquad \eta = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)$$

There are the components of a unit quaternion

$$\mathbf{q} = \gamma + \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$$

$$\gamma^2 + \xi^2 + \eta^2 + \zeta^2 = 1$$


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Expressions for the angular rates of change:

$$\dot{\theta} = \frac{(\xi\dot{\xi} + \eta\dot{\eta}) - (\gamma\dot{\gamma} + \zeta\dot{\zeta})}{\sqrt{(\xi^2 + \eta^2)(\gamma^2 + \zeta^2)}}$$

$$\dot{\phi} = \left(\frac{\gamma\dot{\zeta} - \zeta\dot{\gamma}}{\gamma^2 + \zeta^2}\right) + \left(\frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2}\right)$$

$$\dot{\phi} = \left(\frac{\gamma\dot{\zeta} - \zeta\dot{\gamma}}{\gamma^2 + \zeta^2}\right) - \left(\frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2}\right)$$

The components of angular velocity are

$$\begin{aligned} \omega_1 &= 2[\gamma\dot{\xi} - \xi\dot{\gamma} + \zeta\dot{\eta} - \eta\dot{\zeta}] \\ \omega_2 &= 2[\gamma\dot{\eta} - \eta\dot{\gamma} + \xi\dot{\zeta} - \zeta\dot{\xi}] \\ \omega_3 &= 2[\gamma\dot{\zeta} - \zeta\dot{\gamma} + \eta\dot{\xi} - \xi\dot{\eta}] \end{aligned}$$

The first-order (small θ) equations may be written

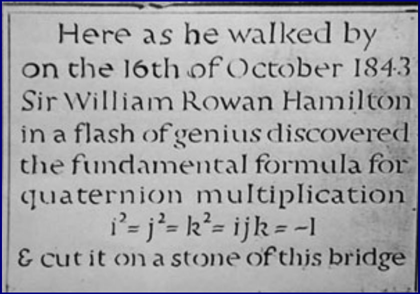
$$\begin{aligned}
\ddot{\gamma} + \left(\frac{\omega_3}{2}\right)^2 \gamma &= 0 \\
\ddot{\zeta} + \left(\frac{\omega_3}{2}\right)^2 \zeta &= 0 \\
\ddot{\xi} + \kappa_{21}\omega_3\dot{\eta} + \Omega_1^2\xi &+ \epsilon'\zeta\left\{(1-\kappa)\omega_3(\gamma\dot{\xi} + \zeta\dot{\eta}) + \Omega_{11}^2(\gamma\eta - \zeta\xi)\right\} = 0 \\
\ddot{\eta} - \kappa_{21}\omega_3\dot{\xi} + \Omega_1^2\eta &- \epsilon'\gamma\left\{(1-\kappa)\omega_3(\gamma\dot{\xi} + \zeta\dot{\eta}) + \Omega_{11}^2(\gamma\eta - \zeta\xi)\right\} = 0
\end{aligned}$$

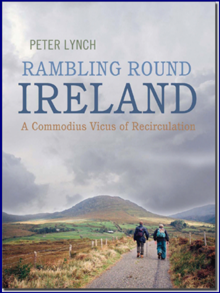
where ϵ' is related to the asymmetry $(I_2 - I_1)/I_1$.

By a simple rotation of coordinates, they can be transformed to a system with constant coefficients.

Thus, the complete solution can be obtained.

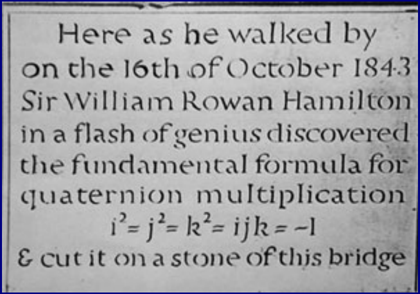
Competition

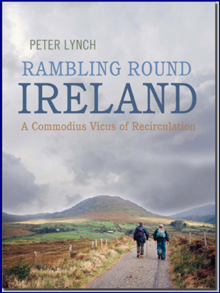




$$i^2 = j^2 = k^2 = ijk = -1$$

Competition





$$i^2 = j^2 = k^2 = ijk = -1$$

(1) Find [i j - j i]. **(2) Find 1 / [i j - j i].** **(3) Find i / j.**

You have two minutes !

$$i^2 = j^2 = k^2 = ijk = -1$$

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$$i(ijk) = (ii)jk = -jk = -i\,, \quad \textbf{So} \quad jk = i$$

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Similarly

$$ji = -k \qquad kj = -i \qquad ik = -j$$

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$$ij = k \qquad jk = i \qquad ki = j$$

Similarly

$$ji = -k \qquad kj = -i \qquad ik = -j$$

$$(1) \qquad [ij - ji] = k - (-k) = 2k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$i(ijk) = (ii)jk = -jk = -i\,, \quad \textbf{So} \quad jk = i$$

$$ij = k \qquad jk = i \qquad ki = j$$

Similarly

$$ji = -k \qquad kj = -i \qquad ik = -j$$

$$(1) \qquad [ij - ji] = k - (-k) = 2k$$

$$(2) \qquad \frac{1}{2k} = \frac{k}{2kk} = \frac{k}{-2} = -\tfrac{1}{2}k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$i(ijk) = (ii)jk = -jk = -i\,, \quad \textbf{So} \quad jk = i$$

$$ij = k \qquad jk = i \qquad ki = j$$

Similarly

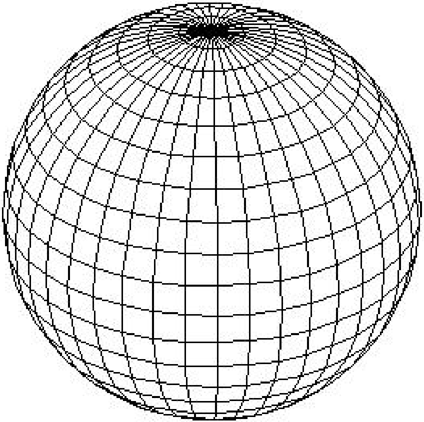
$$ji = -k \qquad kj = -i \qquad ik = -j$$

(1) $$[ij - ji] = k - (-k) = 2k$$

(2) $$\frac{1}{2k} = \frac{k}{2kk} = \frac{k}{-2} = -\tfrac{1}{2}k$$

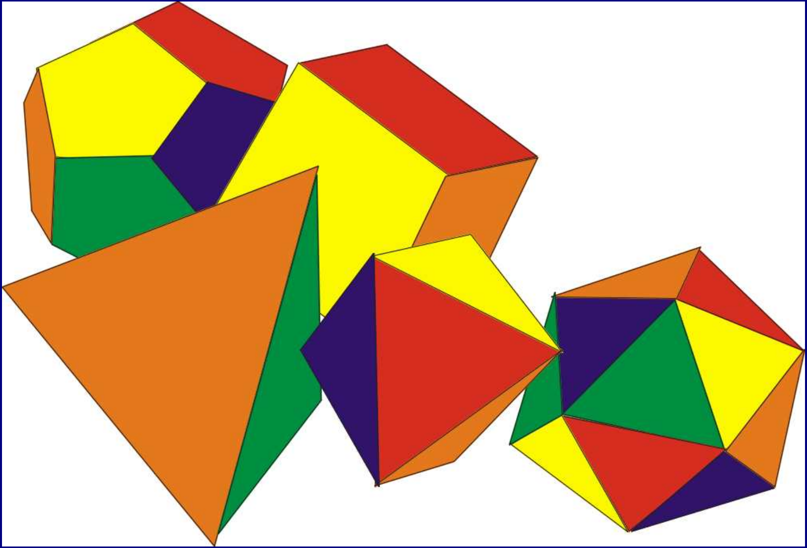
(3) $$i\left(\frac{1}{j}\right) = i\,(-j) = -k \qquad \left(\frac{1}{j}\right)\,i = (-j)\,i = +k$$

Discretizing the Sphere

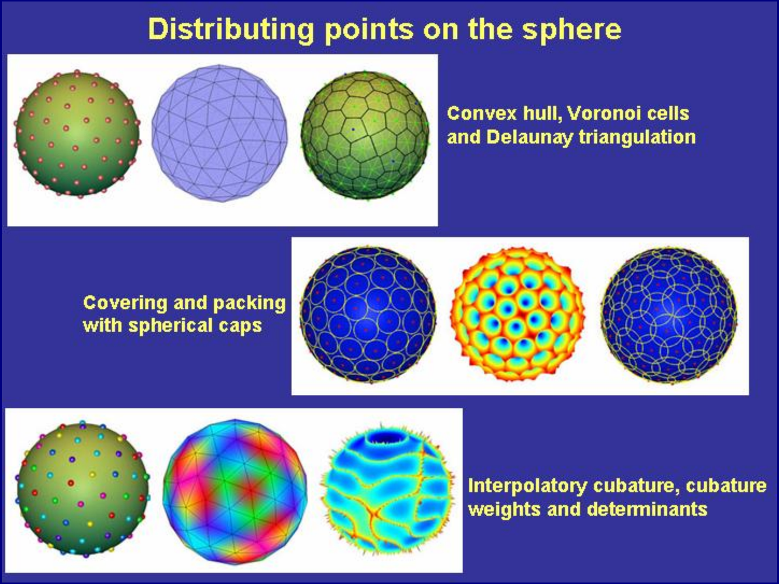


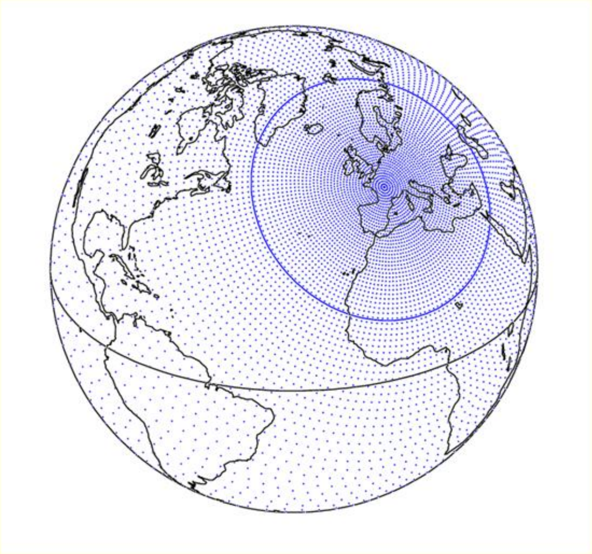
Regular Latitude-Longitude Grid

Challenge: Find a uniform distribution of points — thousands of them — on a sphere.



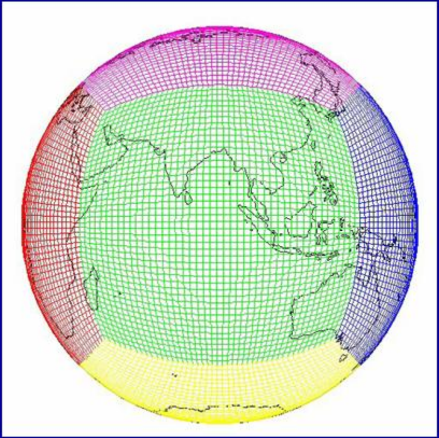
The Five Platonic Solids

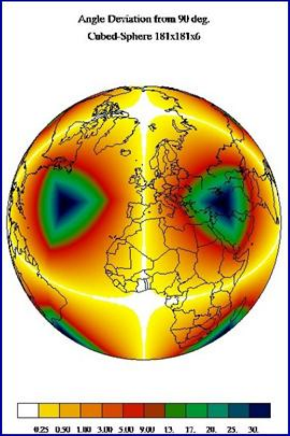

Distributing points on the sphere
Convex hull, Voronoi cells
and Delaunay triangulation
Covering and packing
with spherical caps
Interpolatory cubature, cubature
weights and determinants

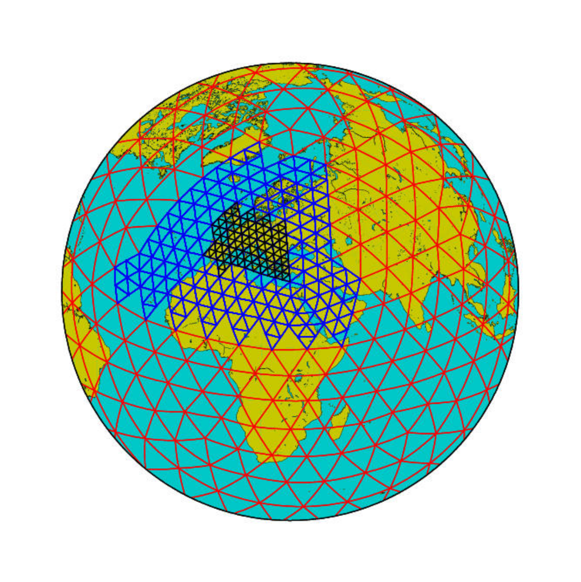


Conformal Stretched Grid

The Cubed Sphere

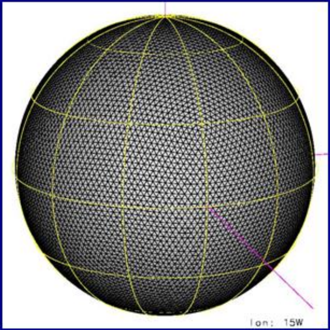


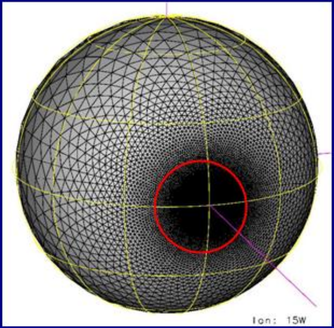




Triangulated Icosahedral Grid

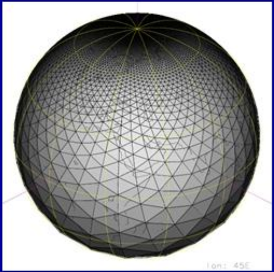
Stretched Icosahedral Grid

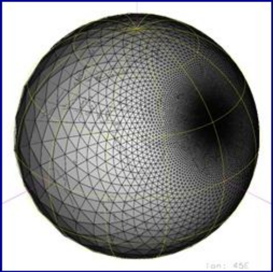




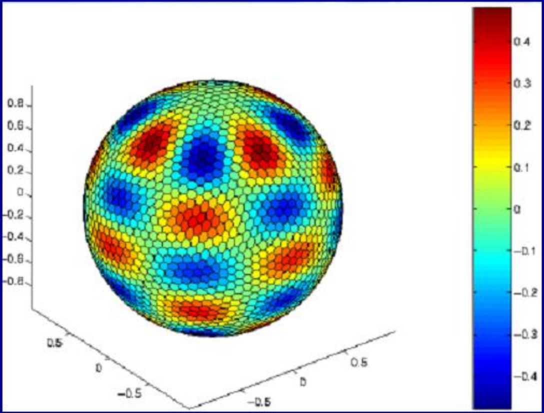
To make a stretched grid

- Gather the grid points in the north pole region (left figure)
- Rotate the grid system to the interested region (right figure)

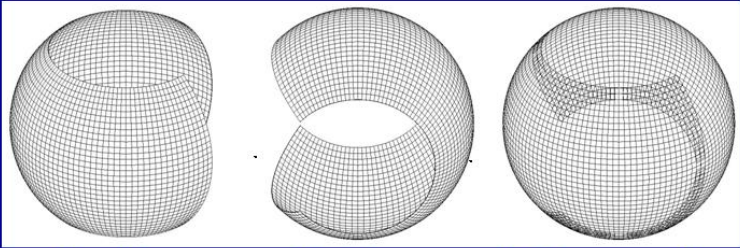




Penta-Hexagonal Grid

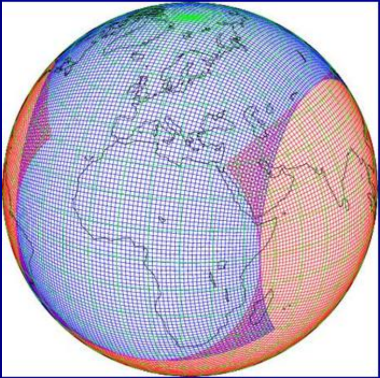


Yin-Yang grid

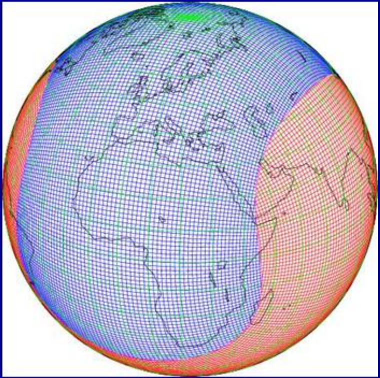


Yang (N) zone　　Yin (E) zone　　Yin-Yang composition

Rectangles, minimal overlap



Overlaps trimmed to median



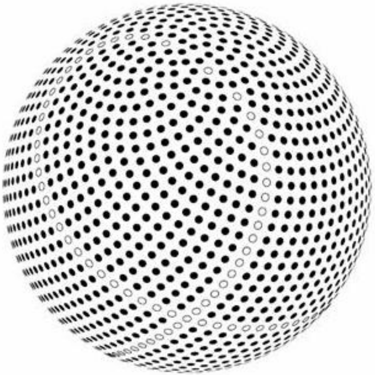


Figure 2. A spherical Fibonacci grid, at resolution $N = 1000$ (2001 grid points). As in Fig. 1, the spiral structure is highlighted by marking every 34th and 55th grid point.

Fibonacci Grid

Inspired by Sun-flowers and Pineapples

The ultimate grid remains elusive.

This is your big chance of fame.