New Ways of Doing Sums

The Mathematical Life of George Gabriel Stokes

Peter Lynch
School of Mathematics & Statistics
University College Dublin

National Library of Ireland
17 October 2019

Outline

Maths Week

George Gabriel Stokes

New Book on Stokes

Navier-Stokes Equations

Stokes the Physicist

Campbell-Stokes Sunshine Recorder

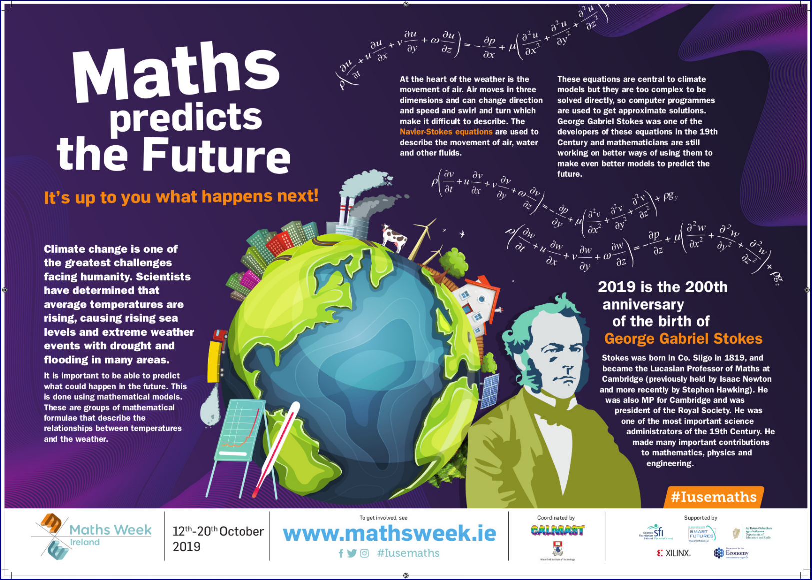
Stokes and the Royal Society

Modelling Weather and Climate

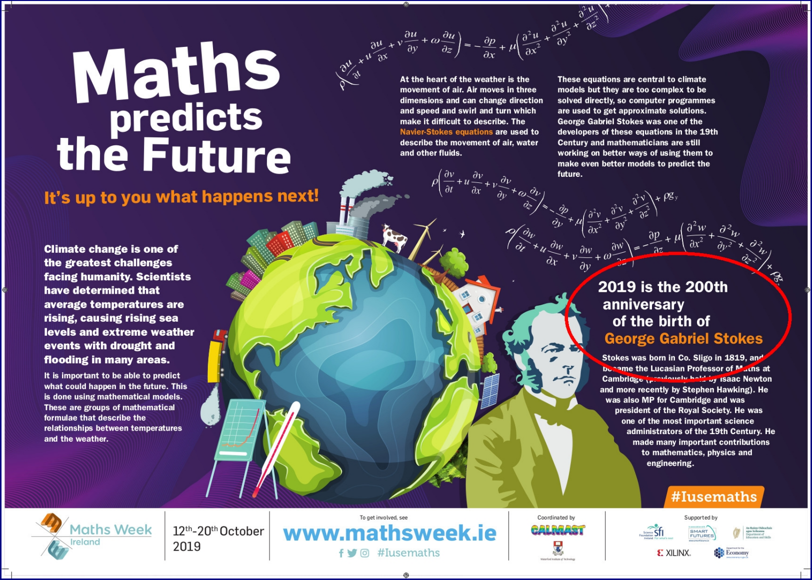
Ocean Waves

Outline

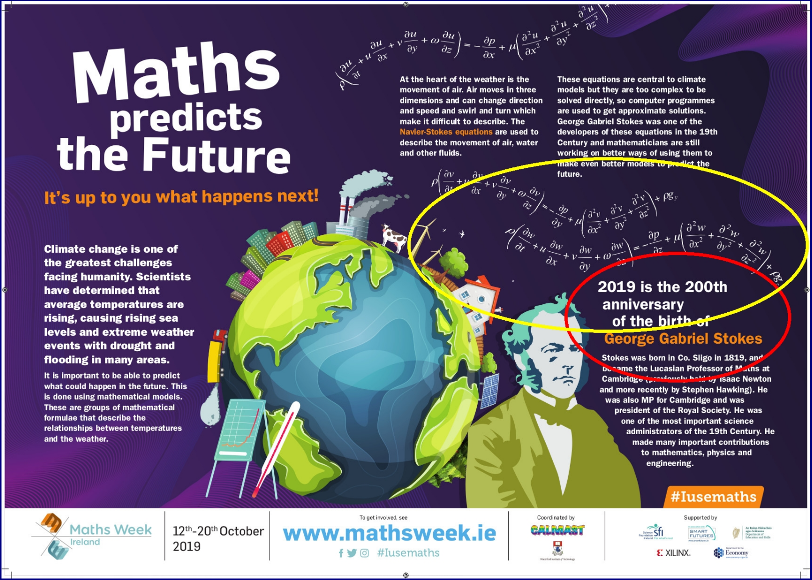
Maths Week 2019



Maths Week 2019



Maths Week 2019



Why is Stokes Important?

We need the Navier-Stokes equations for:

- Designing aircraft
- Predicting the weather
- Forecasting climate change
- Modelling blood flow in the body
- Studying propulsion and lubrication
- The dynamics of swimming
- Designing wind turbines
- Etc., etc., etc.

Outline

Surf off the Sligo Coast

George Gabriel Stokes, 1819–1903

George Gabriel Stokes,
born in Skreen, Sligo on 13 August 1819.


Sligo
Skreen
Collooney

Childhood and Education

George Gabriel Stokes was born in Skreen, Co. Sligo on 13 August 1819 [200 years ago].

He was the youngest of seven children of Rev. Gabriel Stokes, Rector of the Church of Ireland.

From an early age, Stokes showed signs of brilliance:

His school-teacher wrote that

> "Master George was working out
> new ways of doing sums,
> better than those in the book."

The 4 R's

The ‘Old’ Rectory at Skreen (c. 1900)

Descendants of Gabriel Stokes (1682–1768), Great-grandfather of GGS

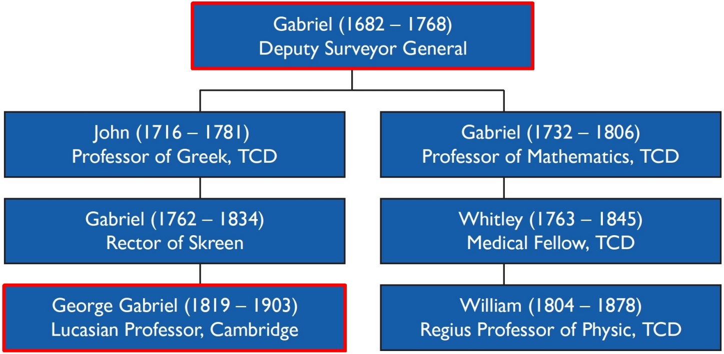


Fig. 1.2 The academic descendants of Gabriel Stokes (1682–1768).

Descendents of Stokes's great-grandfather Gabriel.

Childhood and Education

- Educated in Skreen, Dublin and Bristol.
- 1837: Pembroke College in Cambridge.
- 1841: Graduated as Senior Wrangler.

First place in the Mathematical Tripos.
Winner of the prestigious Smith's Prize.

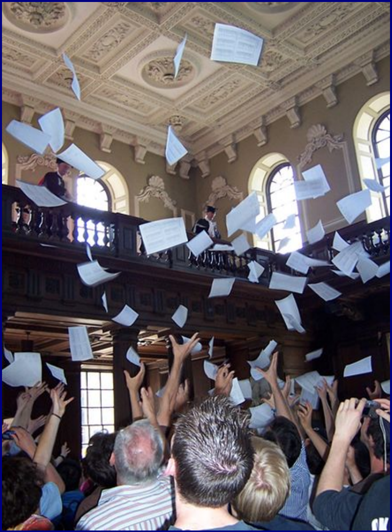
Senior Wrangler

Success in the Tripos was a passport to a great career.

A relative wrote that Stokes had only

"...to decide whether he would be Prime Minister, Lord Chancellor or Archbishop of Canterbury."

Wrangling and the Tripos (modern style)

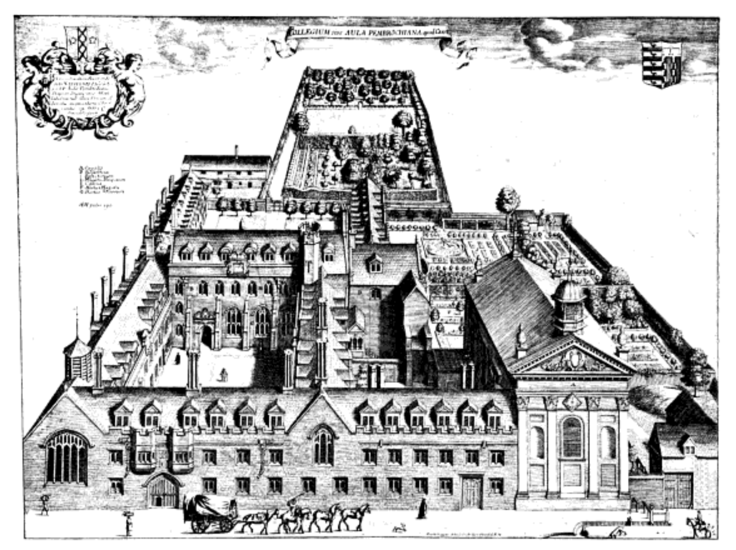


Giving out the results of the Mathematical Tripos.

Stokes as Senior Wrangler (1841)

Elected a Fellow of
Pembroke College
on the basis of
his results in the
Mathematical Tripos.

Pembroke College, 1690



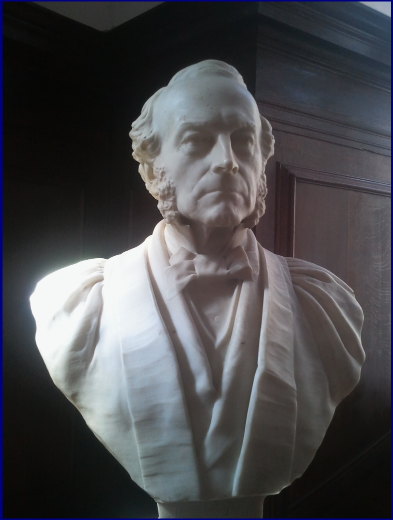
Pembroke College, Victorian Era

Pembroke College Today

Stokes's Progress

In 1849, Stokes was appointed **Lucasian Professor** of Mathematics.

He held this chair for over fifty years.



The Lucasian Chair of Mathematics

Table of Lucasian Professors of Mathematics

Isaac Barrow	1663–9	Charles Babbage	1828–39
Isaac Newton	1669–1702	Joshua King	1839–49
William Whiston	1702–10	**George Gabriel Stokes**	**1849–1903**
Nicholas Saunderson	1711–39	Joseph Larmor	1903–32
John Colson	1739–60	Paul Dirac	1932–69
Edward Waring	1760–98	James Lighthill	1969–80
Isaac Milner	1798–1820	Stephen Hawking	1980–2009
Robert Woodhouse	1820–2	Brian Green	2009–15
Thomas Turton	1822–6	Michael Cates	2015–
George Airy	1826–8		

The Lucasian Chair of Mathematics

Table of Lucasian Professors of Mathematics

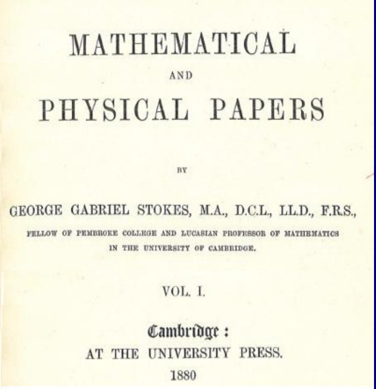
Isaac Barrow	1663–9	Charles Babbage	1828–39
Isaac Newton	1669–1702	Joshua King	1839–49
William Whiston	1702–10	**George Gabriel Stokes**	**1849–1903**
Nicholas Saunderson	1711–39	Joseph Larmor	1903–32
John Colson	1739–60	Paul Dirac	1932–69
Edward Waring	1760–98	James Lighthill	1969–80
Isaac Milner	1798–1820	Stephen Hawking	1980–2009
Robert Woodhouse	1820–2	Brian Green	2009–15
Thomas Turton	1822–6	Michael Cates	2015–
George Airy	1826–8		

Stokes's Wife Mary

In 1859 Stokes married
Mary Susannah,
daughter of
Thomas Romney
Robinson,
astronomer at
Armagh Observatory.

George and Mary
had five children.

Mathematical & Physical Papers



MATHEMATICAL

AND

PHYSICAL PAPERS

BY

GEORGE GABRIEL STOKES, M.A., D.C.L., LL.D., F.R.S.,

FELLOW OF PEMBROKE COLLEGE AND LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. I.

Cambridge:
AT THE UNIVERSITY PRESS.
1880

Stokes's Collected Works, in 5 volumes, include some 140 papers.

Mathematical & Physical Papers

ERRATA.

P. 103, l. 14, *for* their *read* there.
P. 198, l. 3, *for* p^{r-1} *read* p_{r-1}.

ERRATUM.

P. 318, Equations (17) and (18). *For* − *read* + before the terms multiplied by sin 3φ and cos 3φ.

UNIV. OF CALIFORNIA

MATHEMATICAL AND PHYSICAL PAPERS.

[From the *Transactions of the Cambridge Philosophical Society*, Vol. VII. p. 439.]

ON THE STEADY MOTION OF INCOMPRESSIBLE FLUIDS.

[Read April 25, 1842.]

IN this paper I shall consider chiefly the steady motion of fluids in two dimensions. As however in the more general case of motion in three dimensions, as well as in this, the calculation is simplified when $udx + vdy + wdz$ is an exact differential, I shall first consider a class of cases where this is true. I need not explain the notation, except where it may be new, or liable to be mistaken.

To prove that $udx + vdy + wdz$ is an exact differential, in the case of steady motion, when the lines of motion are open curves, and when the fluid in motion has come from an expanse of fluid of indefinite extent, and where, at an indefinite distance, the velocity is indefinitely small, and the pressure indefinitely near to what it would be if there were no motion.

By integrating along a line of motion, it is well known that we get the equation

$$\frac{p}{\rho} = V - \tfrac{1}{2}(u^2 + v^2 + w^2) + C \quad \text{.................(1)},$$

where $dV = Xdx + Ydy + Zdz$, which I suppose an exact differential. Now from the way in which this equation is obtained,

Mathematical & Physical Papers

it appears that C need only be constant for the same line of motion, and therefore in general will be a function of the parameter of a line of motion. I shall first shew that in the case considered C is absolutely constant, and then that whenever it is, $udx + vdy + wdz$ is an exact differential*.

To determine the value of C for any particular line of motion, it is sufficient to know the values of p, and of the whole velocity, at any point along that line. Now if there were no motion we should have

$$\frac{p_1}{\rho} = V + C_1 \text{ ....(2)},$$

p_1 being the pressure in that case. But considering a point in this line at an indefinite distance in the expanse, the value of p at that point will be indefinitely nearly equal to p_1, and the velocity will be indefinitely small. Consequently C is more nearly equal to C_1 than any assignable quantity: therefore C is equal to C_1; and this whatever be the line of motion considered; therefore C is constant.

In ordinary cases of steady motion, when the fluid flows in open curves, it does come from such an expanse of fluid. It is conceivable that there should be only a canal of fluid in this expanse in motion, the rest being at rest, in which case the velocity at an infinite distance might not be indefinitely small. But experiment shews that this is not the case, but that the fluid flows in from all sides. Consequently at an indefinite distance the velocity is indefinitely small, and it seems evident that in that case the pressure must be indefinitely near to what it would be if there were no motion.

Differentiating therefore (1) with respect to x, we get

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{dv}{dx} - w\frac{dw}{dx};$$

but

$$\frac{1}{\rho}\frac{dp}{dx} = X - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz};$$

whence

$$v\left(\frac{dv}{dx} - \frac{du}{dy}\right) + w\left(\frac{dw}{dx} - \frac{du}{dz}\right) = 0.$$

[* See note, page 3.]

Similarly,

$$w\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + u\left(\frac{du}{dy} - \frac{dv}{dx}\right) = 0,$$

$$u\left(\frac{du}{dz} - \frac{dw}{dx}\right) + v\left(\frac{dv}{dz} - \frac{dw}{dy}\right) = 0;$$

whence*

$$\frac{dv}{dx} = \frac{du}{dy}, \quad \frac{dw}{dy} = \frac{dv}{dz}, \quad \frac{du}{dz} = \frac{dw}{dx},$$

and therefore $udx + vdy + wdz$ is an exact differential.

When $udx + vdy + wdz$ is an exact differential, equation (1) may be deduced in another way †, from which it appears that C is constant. Consequently, in any case, $udx + vdy + wdz$ is, or is not, an exact differential, according as C is, or is not, constant.

Steady Motion in Two Dimensions.

I shall first consider the more simple case, where $udx + vdy$ is an exact differential. In this case u and v are given by the equations

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \text{ ....(3)},$$

$$\frac{du}{dy} - \frac{dv}{dx} = 0 \text{ ....(4)};$$

and p is given by the equation

$$\frac{p}{\rho} = V - \tfrac{1}{2}(u^2 + v^2) + C.$$

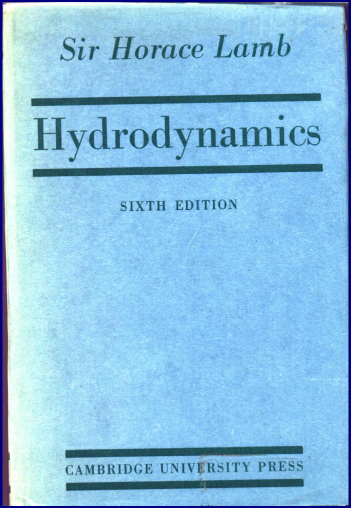
The differential equation to a line of motion is

$$\frac{dy}{dx} = \frac{v}{u}.$$

* [This conclusion involves an oversight (see *Transactions*, p. 465) since the three preceding equations are not independent, as may readily be seen. I have not thought it necessary to re-write this portion of the paper, since in the two classes of steady motion to which the paper relates, namely those of motion in two dimensions, and of motion symmetrical about an axis, the three analogous equations are reduced to one, and the proposition is true. None of the succeeding results are affected by this error, excepting that the second paragraph of p. 11 must be restricted to the two cases above mentioned.]

† See Poisson, *Traité de Mécanique*.

A Crude but indicative Metric



In his book *Hydrodynamics*, (6th edition), Horace Lamb has more than 50 page references to Stokes.

Some Contributions of Stokes

- Stokes Drag
- Stokes Drift
- Stokes's Law
- Stokes Waves
- Stokes's Theorem
- Stokes Parameters
- Stokes Phenomenon
- The Navier-Stokes Equations
- Campbell-Stokes Sunshine Recorder

Stokes Drag and Stokes's Law

A Child's Query:

Child: **Daddy, why don't clouds fall down?**

Daddy: **Clouds do fall, but very slowly!**

Stokes Drag and Stokes's Law

A Child's Query:

Child: **Daddy, why don't clouds fall down?**

Daddy: **Clouds do fall, but very slowly!**

Stokes formulated the drag law for small particles:

$$F = 6\pi\mu r v$$

This leads an expression for the terminal velocity:

$$v_s = \frac{2r^2\rho g}{9\mu}$$

A droplet of radius 5 microns falls with a terminal speed of about 3 mm/s (about four days for 1 km).

Some Paradoxes in Hydrodynamics

- D'Alembert's Paradox
- The Reversibility Paradox
- Paradoxes of Airfoil Theory
- The Rayleigh Paradox
- Von Neumann's Paradox
- Kopal's Paradox
- The Eiffel Paradox
- The Rising Bubble Paradox
- The Magnus Effect Paradox
- Stokes's Paradox

Euler's Fluid Equations

Leonhard Euler, born on 15 April, 1707 in Basel. Died on 18 September, 1783 in St Petersburg.

Euler formulated the equations for incompressible, inviscid fluid flow:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla p = \mathbf{g} \,.$$

$$\nabla \cdot \mathbf{V} = 0$$

Jean Le Rond d'Alembert

A body moving at constant speed through a gas or a fluid does not experience any resistance (d'Al. 1752).

Jean Le Rond d'Alembert

D'Alembert expressed his concerns thus:

"I do not see how one can satisfactorily explain, by theory, the resistance of fluids."

He remarked that the theory leads to "a singular paradox which I leave to future geometers for elucidation."

Resolution of d'Alembert's Paradox

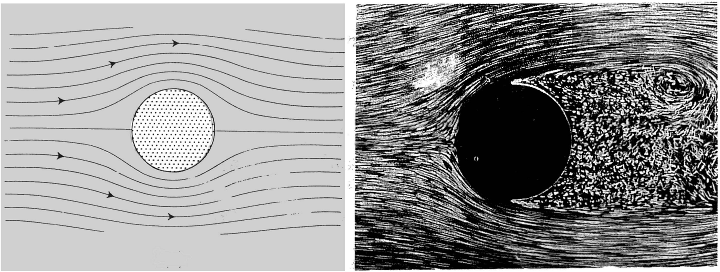


Fig. 9.1 *Flow past a circular cylinder for (a) a hypothetical fluid with zero viscosity, (b) a real fluid with very small viscosity* μ *(from van Dyke 1982).*

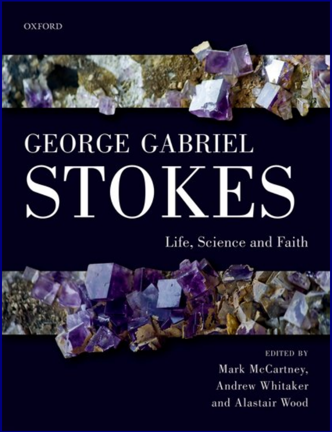
The minutest amount of viscosity has a profound qualitative impact on the character of the solution.

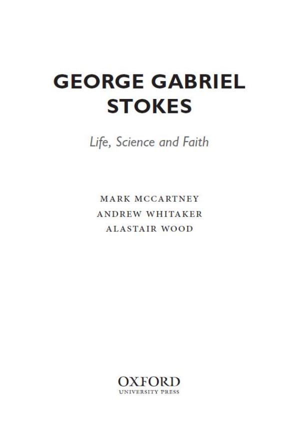
The N-S equations incorporate the effect of viscosity.

Outline

- Maths Week
- George Gabriel Stokes
- **New Book on Stokes**
- Navier-Stokes Equations
- Stokes the Physicist
- Campbell-Stokes Sunshine Recorder
- Stokes and the Royal Society
- Modelling Weather and Climate
- Ocean Waves

Stokes: Life, Science and Faith




GEORGE GABRIEL
STOKES

Life, Science and Faith

MARK MCCARTNEY
ANDREW WHITAKER
ALASTAIR WOOD

OXFORD
UNIVERSITY PRESS

George Gabriel Stokes: Life, Science and Faith.
Eds. Mark McCartney, Andrew Whitaker, and Alastair Wood,
Oxford University Press (2019). ISBN: 978-0-1988-2286-8

Stokes: Life, Science and Faith

Table of Contents

Outline

C. L. M. H. Navier, 1785–1836

Claude Louis Marie Henri Navier,
French engineer and physicist
who specialized in mechanics.

Basic Publications and Review

Navier, C. L. M. H., 1822:
Mémoire sur les lois du mouvement des fluides.
Mém. Acad. Sci. Inst. France, **Vol. 6, 389–440.**

Stokes, G. G., 1845:
On the theories of the internal friction of fluids in motion.
Trans. Cambridge Philos. Soc., **Vol. 8.**

⋆ ⋆ ⋆

Basic Publications and Review

Navier, C. L. M. H., 1822:
Mémoire sur les lois du mouvement des fluides.
Mém. Acad. Sci. Inst. France, **Vol. 6, 389–440.**

Stokes, G. G., 1845:
On the theories of the internal friction of fluids in motion.
Trans. Cambridge Philos. Soc., **Vol. 8.**

$\star \quad \star \quad \star$

Between Hydrodynamics and Elasticity Theory:
The First Five Births of the Navier-Stokes Equation.
Olivier Darrigol, 2002:
***Arch. Hist. Exact Sci.*, Vol. 56, (2), 95–150.**

The Navier-Stokes Equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{V} \,.$$

The Navier-Stokes Equations describe how the change of velocity (the acceleration) is determined by the pressure gradient force and frictional force.

The Navier-Stokes Equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{V} \,.$$

The Navier-Stokes Equations describe how the change of velocity (the acceleration) is determined by the pressure gradient force and frictional force.

For motion relative to the rotating earth, we include the Coriolis force and gravity:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + 2\boldsymbol{\Omega} \times \mathbf{V} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{V} + \mathbf{g} \,.$$

Some Applications of the Navier-Stokes Equations

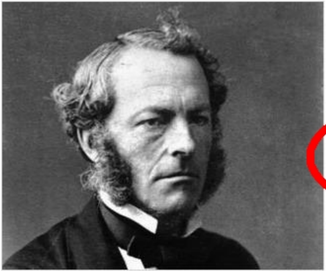
- Designing aircraft
- Modelling blood-flow
- Studying propulsion or lubrication
- Constructing wind turbines
- Forecasting the weather
- Ocean modelling
- Fundamental studies of turbulence.

Clay Maths Institute Millennium Prize

CMI

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS

Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Outline

- Maths Week
- George Gabriel Stokes
- New Book on Stokes
- Navier-Stokes Equations
- **Stokes the Physicist**
- Campbell-Stokes Sunshine Recorder
- Stokes and the Royal Society
- Modelling Weather and Climate
- Ocean Waves

ï»¿

A Talented Experimentalist

Stokes's combined mathematical sophistication with a great experimental facility.

He devised and performed many ingenious experiments in optics.

His work gave evidence supporting the wavelike nature of light.

He carried out a spectral analysis of blood, showing how oxygen is carried by haemoglobin.

Supernumerary Rainbows

Stokes evaluated
Airy's integral

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\,\mathrm{d}t$$

for large values of x.

Stokes developed new methods of exponential asymptotics which still have many important applications.

Fluorescence

Stokes elucidated the strange phenomenon of fluorescence with a radically new theory.

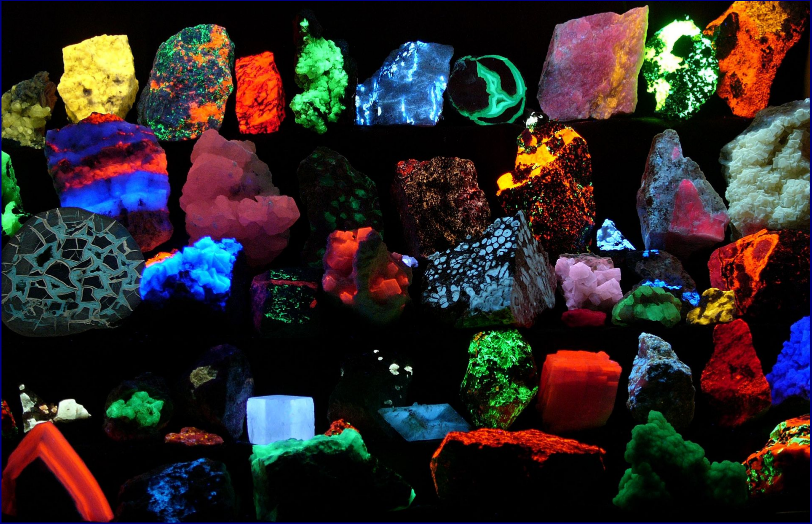
Objects are normally invisible in ultra-violet light, but a fluorescent body emits light at a lower frequency.

Stokes knew that fluorescence is found in many biological systems in the marine environment.

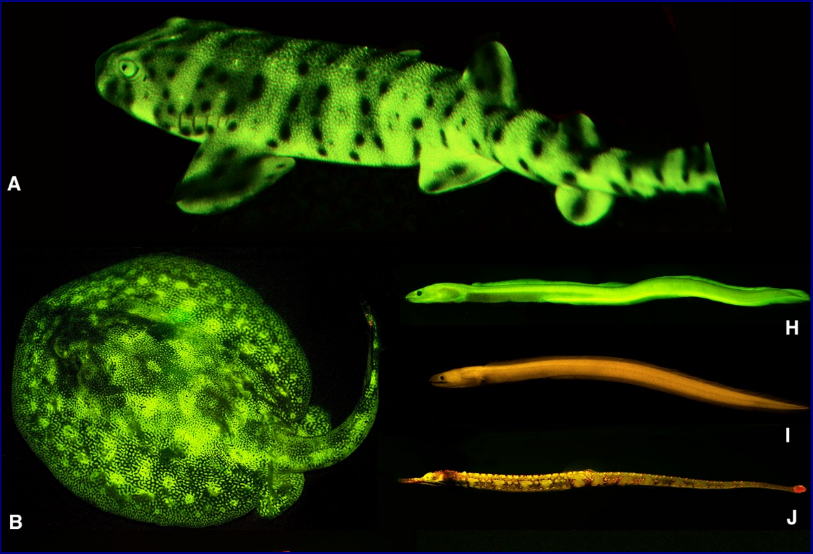
Stokes realised that his work on fluorescence offered a way to measure the UV spectrum of sunlight.

We benefit from his work through fluorescent lamps, where a phosphor coating fluoresces, emitting light.

Fluorescence in the Mineral World



Fluorescence in the Animal World



Fluorescence in the Social World

Outline

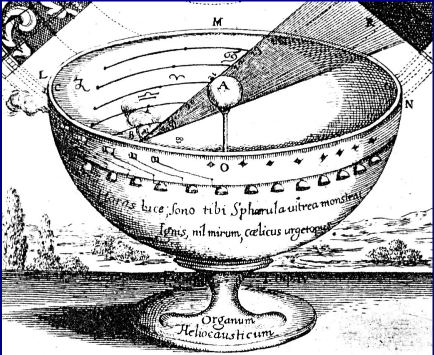
An Early Sunshine Recorder

Athanasius Kircher was Professor of Mathematics and Hebrew at the *Collegio Romano*.

Around 1646 he devised a recording sundial called the Horologium Helio-causticum.

The Horologium Helio-causticum

A Sundial is drawn in the shell, "together with things for burning and making sounds."



"With light and sound the glassy sphere shows the hours; truly, it is the work of the heavenly fire."

Campbell's Sunshine Recorder.

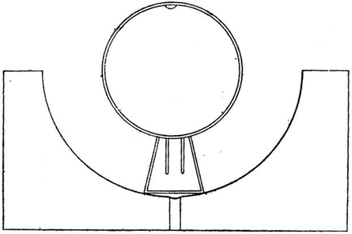


FIG. 1.—Section of Mr. Campbell's original Sunshine Recorder.

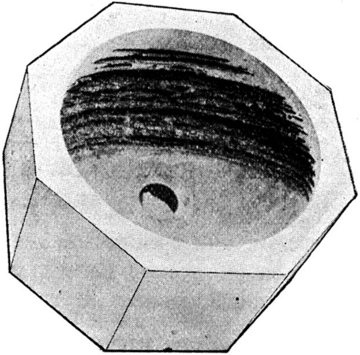
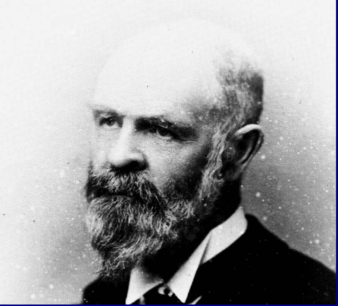


FIG. 2.—Wooden Sunshine Bowl.

The "self-registering sundial" of Francis Campbell (c. 1853).

Robert Henry Scott (1833–1916)



Robert Scott, born in Dublin, was founder of Valentia Observatory and first Director of the British Met Office.

Scott proposed some improvements to Campbell's sunshine recorder.

The detailed design was due to G. G. Stokes.

Stokes' Quarterly Journal Paper

Description of the Card Supporter for Sunshine Recorders adopted at the Meteorological Office

George Gabriel Stokes

Quarterly Journal of the Royal Meteorological Society, **Vol. 6 (1880) 83–94.**

"The method of recording sunshine by the burning of an object placed in the focus of a glass sphere freely exposed to the rays of the sun, which was devised by Mr. Campbell, commends itself by its simplicity, and seems likely to come into pretty general use."

Stokes' Quarterly Journal Paper

Description of the Card Supporter for Sunshine Recorders adopted at the Meteorological Office

George Gabriel Stokes

Quarterly Journal of the Royal Meteorological Society, Vol. 6 (1880) 83–94.

In the discussion following the reading of the paper, a Mr. Mawley remarked:

"The fact of this sunshine-recorder being in all respects an English invention, adds much to its interest."

Stokes Honoured in Ireland?

Surely there must be an
Irish postage stamp
featuring Stokes?

In a quick web-search,
no such stamp was found.

Stokes Honoured in Ireland?

Surely there must be an
Irish postage stamp
featuring Stokes?

In a quick web-search,
no such stamp was found.

Campbell-Stokes Sunshine Recorder

FIGURE 138
Campbell-Stokes sunshine recorder.

No moving parts.

Campbell-Stokes Sunshine Recorder

FIGURE 138
Campbell-Stokes sunshine recorder.

No moving parts.

One moving part!
(In Biblical Coordinates)

Outline

Royal Society

1851: Stokes elected a Fellow of the Royal Society.

(Along with William Thomson, Thomas H. Huxley and John Tyndall.)

1854–1884: Stokes Secretary of the Royal Society

President from 1885 to 1890.

"I am naturally of rather a retiring character,
and should feel not a little out of my element
in being brought so prominently forward."

Stokes to Th. R. Robinson.

T. H. Huxley criticised Stokes for his "ultra-conservative and theological viewpoint."

Stokes as President of Royal Society

President of Royal Society

According to his daughter, "Stokes was apt to look bored when being painted and to draw down the corners of his mouth. Thus, the portrait by Herkomer at the Royal Society is not satisfactory to those who knew him best."

Prominent Members of the Royal Society

Royal Society

William Thomson followed Stokes as PRS.
Stokes was awarded the Copley Medal in 1893.

Fig. 8.4 The Royal Society Copley Medal awarded to Stokes in 1893. Image courtesy Nick Lefebvre.

Outline

ï»¿

Modelling the Changing Climate

Amongst the most pressing problems facing humanity today.

Enormous uncertainties exist concerning the future climate.

The best means we have for reducing these is by means of computer simulations.

At the heart of every climate model lie the Navier-Stokes equations.

The same models are used regularly for short and medium range weather forecasts.

Equations of the Atmosphere

Gas Law: (Boyle's Law and Charles' Law.)

Conservation of Mass for Air

Conservation of Mass for Water

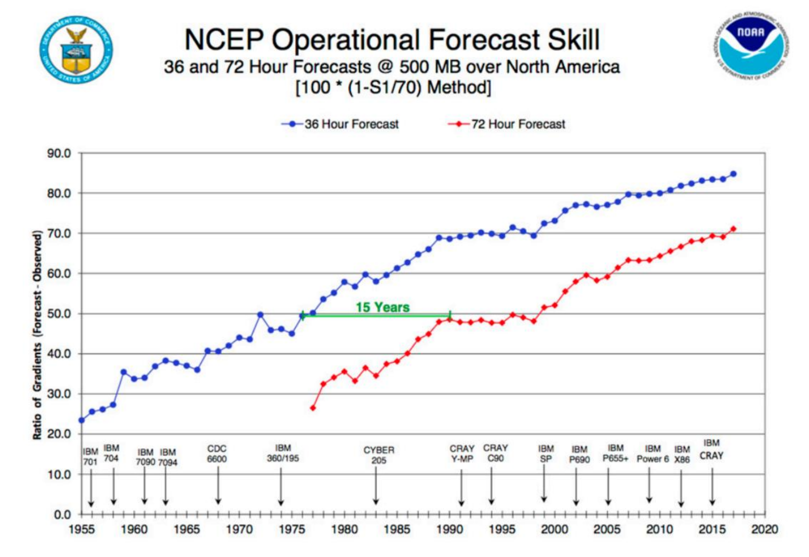
Equations of Motion: Navier-Stokes Equations

Thermodynamic Equation

Weather Forecasting in a Nut-Shell

- The atmosphere is a physical system
- Its behaviour is governed by the laws of physics
- These laws are expressed quantitatively as mathematical equations
- Using observations, we can specify the atmospheric state: "Today's Weather"
- Using the equations, we can calculate how this state changes with time: "Tomorrow's Weather"

Long-term Skill Growth



Forecast of Hurricane Sandy

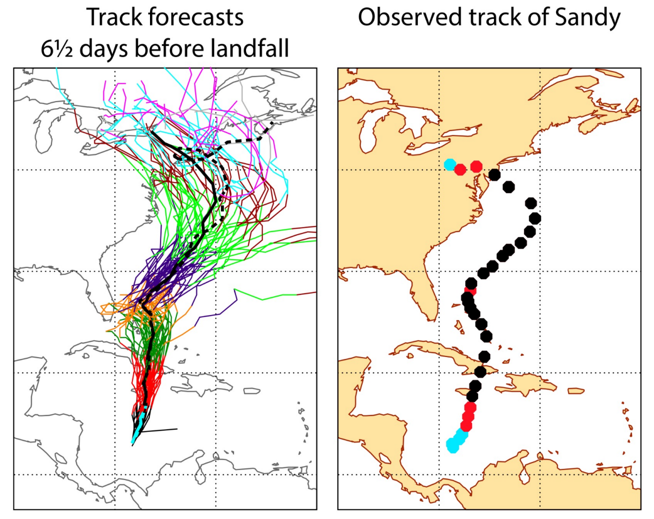


Figure : Landfall, New Jersey, 30 October 2012

Outline

- Maths Week
- George Gabriel Stokes
- New Book on Stokes
- Navier-Stokes Equations
- Stokes the Physicist
- Campbell-Stokes Sunshine Recorder
- Stokes and the Royal Society
- Modelling Weather and Climate
- **Ocean Waves**

Ocean Waves

Stokes, growing up on Ireland's Wild Atlantic Way, was a skilled swimmer and a keen observer of nature.

During holidays in Skreen and elsewhere in Ireland, he made observations of waves and swell.

He considered the "highest possible periodic wave."

He showed that the wave of maximum height has a crest with an angle of 120°.

The Wild Atlantic Way

A Sligo Man

Stokes never forgot his origins in Skreen.

He returned to Sligo and elsewhere in Ireland regularly for summer vacations.

In one of his mathematical papers he wrote of

“the surf that breaks upon the western coasts as a result of storms out in the Atlantic”,

recalling the majestic rollers thundering in as he strolled as a boy along Dunmoran Strand.

Thank you