# The Curious Behaviour of the Rock'n'roller

## Part II

Peter Lynch & Miguel Bustamante
School of Mathematical Sciences

ISSEC — Irish Mechanics Society
Joint Meeting, May 2010

# A Bowling-ball from Stillorgan

Thanks to Brian O'Connor (School of Physics) for slicing the top off

# Recession I

# The Physical System

Consider a spherical rigid body with an asymmetric mass distribution.

Specifically, we consider a loaded sphere.

The dynamics are essentially the same as for the tippe-top, which has been studied extensively.

# The Physical System

Consider a spherical rigid body with an asymmetric mass distribution.

Specifically, we consider a loaded sphere.

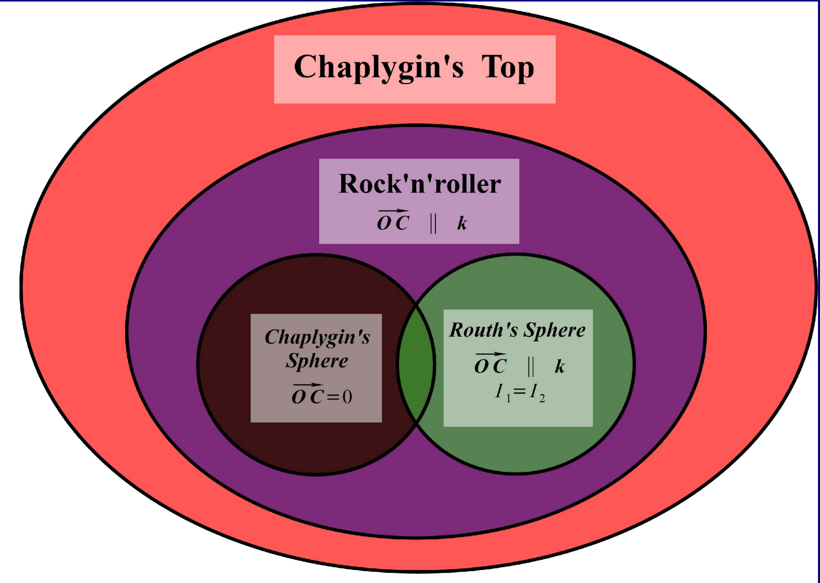
The dynamics are essentially the same as for the **tippe-top**, which has been studied extensively.

Unit radius and unit mass.

Centre of mass off-set a distance $a$ from the centre.

Moments of inertia $I_1$, $I_2$ and $I_3$, with $I_1 \approx I_2 < I_3$.

# The Hierarchy of Models



# Recap on 2008 Talk

**The Routh Sphere does not recess.**

**Recession needs a perturbation, or friction.**

# Recap on 2008 Talk

The Routh Sphere does not recess.

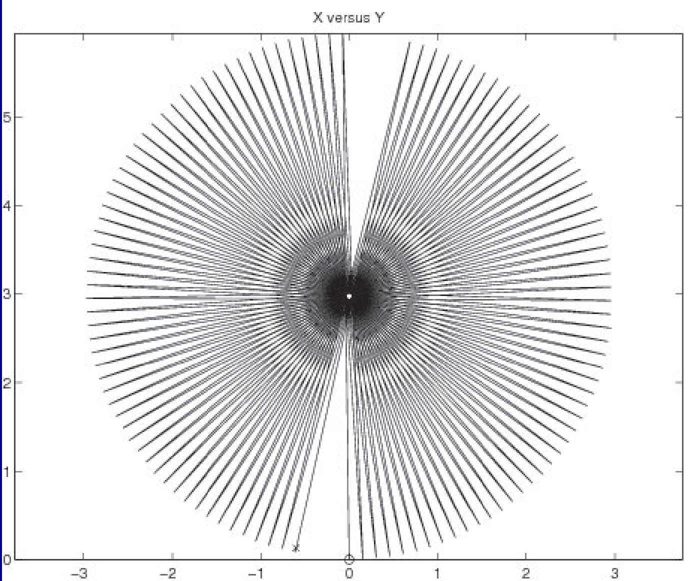
Recession needs a perturbation, or friction.

It was thought likely that appropriate friction forces could explain recession.

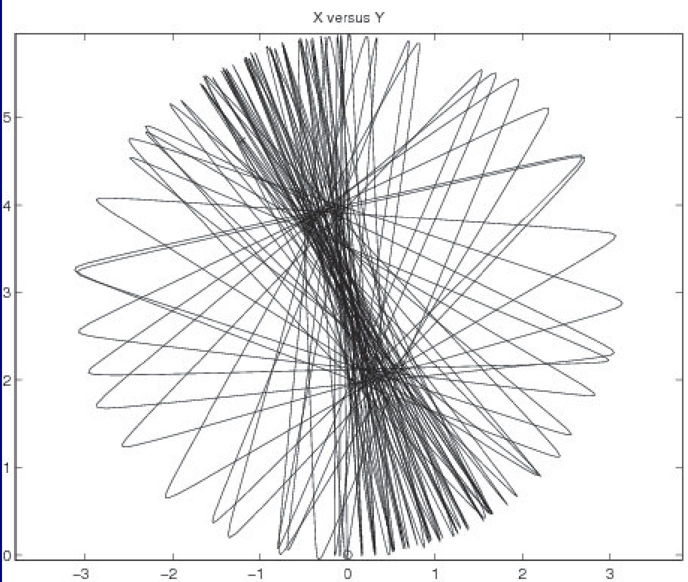
- Rolling friction
- Sliding friction
- Spinning friction
- Air resistance

*Perhaps I can tell you by Philippe's 65th!*

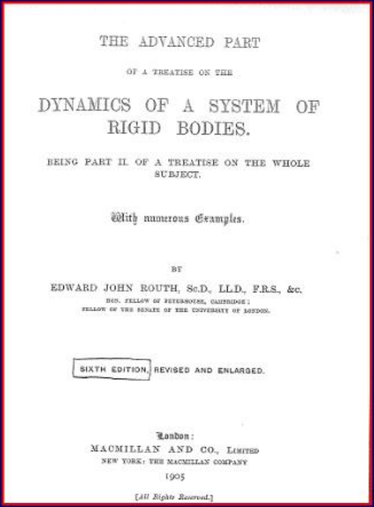
# Symmetric Case: Routh Sphere ($I_1 = I_2$)



# Asymmetric Case: Rock'n'roller ($I_1 < I_2$)



# The Routh Sphere: $I_1 = I_2$


THE ADVANCED PART

OF A TREATISE ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

BEING PART II. OF A TREATISE ON THE WHOLE SUBJECT.

With numerous Examples.

BY

EDWARD JOHN ROUTH, Sc.D., LL.D., F.R.S., &c.

HON. FELLOW OF PETERHOUSE, CAMBRIDGE;
FELLOW OF THE SENATE OF THE UNIVERSITY OF LONDON.

SIXTH EDITION, REVISED AND ENLARGED.

London:
MACMILLAN AND CO., LIMITED
NEW YORK: THE MACMILLAN COMPANY
1905

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**Cover of Routh's *Dynamics* Part II**

**In the Cambridge Mathematical Tripos Examination of 1854, James Clark Maxwell came second.**

**Edward John Routh came first (senior wrangler).**

# The Routh Sphere: $I_1 = I_2$

## In an inertial frame

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} \qquad\qquad \frac{d\mathbf{L}}{dt} = \mathbf{G}$$

**Euler angles $(\theta, \phi, \psi)$ related to angular velocity**

$$\omega_1 = \dot{\theta}\,, \qquad \omega_2 = s\dot{\phi}\,, \qquad \omega_3 = c\dot{\phi} + \dot{\psi}\,.$$

**where $s = \sin\theta$ and $c = \cos\theta$**

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**Rotating frame of reference: angular velocity is**

$$\omega = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$$

**Rotating frame of reference: angular momentum is**

$$\mathbf{L} = \mathbf{I_1}\omega_1\mathbf{i} + \mathbf{I_1}\omega_2\mathbf{j} + \mathbf{I_3}\omega_3\mathbf{k}\,.$$

## In the rotating (body) frame, the equations become

$$\frac{d\mathbf{v}}{dt} + \Omega \times \mathbf{v} = \mathbf{F}$$

and

$$\frac{d\mathbf{L}}{dt} + \Omega \times \mathbf{L} = \mathbf{G}$$

$$\begin{aligned}
\dot{v}_1 + \Omega_2 v_3 - \Omega_3 v_2 &= F_1 \\
\dot{v}_2 + \Omega_3 v_1 - \Omega_1 v_3 &= F_2 \\
\dot{v}_3 + \Omega_1 v_2 - \Omega_2 v_1 &= F_3
\end{aligned}$$

$$\begin{aligned}
\mathbf{I_1}\dot{\omega}_1 + \mathbf{I_3}\Omega_2\omega_3 - \mathbf{I_1}\Omega_3\omega_2 &= G_1 \\
\mathbf{I_1}\dot{\omega}_2 + \mathbf{I_1}\Omega_3\omega_1 - \mathbf{I_3}\Omega_1\omega_3 &= G_2 \\
\mathbf{I_3}\dot{\omega}_3 &= G_3
\end{aligned}$$

# The Lagrangian

**The Lagrangian of the system is easily written down:**

$$L = \frac{1}{2}(\mathbf{I_1}\omega_1^2 + \mathbf{I_2}\omega_2^2 + \mathbf{I_3}\omega_3^2) + \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - ga(1 - \cos\theta)$$

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**The equations may then be written (in vector form):**

$$\mathbf{\Sigma}\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}\,, \qquad \mathbf{K}\dot{\boldsymbol{\omega}} = \mathbf{P}_\omega$$

**where the matrices $\mathbf{\Sigma}$ and $\mathbf{K}$ are known and**

$$\mathbf{P}_\omega = \begin{pmatrix} -(g + \omega_1^2 + \omega_2^2)as\chi + (\mathbf{I_2} - \mathbf{I_3} - af)\omega_2\omega_3 \\ (g + \omega_1^2 + \omega_2^2)as\sigma + (\mathbf{I_3} - \mathbf{I_1} + af)\omega_1\omega_3 \\ (\mathbf{I_1} - \mathbf{I_2})\omega_1\omega_2 + as(-\chi\omega_1 + \sigma\omega_2)\omega_3 \end{pmatrix}$$

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$$\mathbf{\Sigma}\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}\,, \qquad \mathbf{K}\dot{\boldsymbol{\omega}} = \mathbf{P}_\omega$$

**where the matrices $\mathbf{\Sigma}$ and K are known and**

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**Note that neither K nor $\mathbf{P}_\omega$ depends explicitly on $\phi$.**

# Nonholonomic Constraints

**We assume perfectly rough contact (rolling motion).**

**Holonomic constraints** $f_k(q_\rho) = 0$ **can be handled by modifying the Lagrangian:**

$$L \longrightarrow L + \sum \lambda_k f_k$$

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$$L \longrightarrow L + \sum \lambda_k f_k$$

For non-holonomic constraints this doesn't work.

Misunderstandings on non-holonomy abound:

- Whittaker and Landau & Lifshitz get it right!
- Goldstein *et al.* (2002) get it wrong!
- See Flannery (2005) for a review.

# The enigma of nonholonomic constraints

M. R. Flannery[a)]
*School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332*

(Received 16 February 2004; accepted 8 October 2004)

The problems associated with the modification of Hamilton's principle to cover nonholonomic constraints by the application of the multiplier theorem of variational calculus are discussed. The reason for the problems is subtle and is discussed, together with the reason why the proper account of nonholonomic constraints is outside the scope of Hamilton's variational principle. However, linear velocity constraints remain within the scope of D'Alembert's principle. A careful and comprehensive analysis facilitates the resolution of the puzzling features of nonholonomic constraints. 
[DOI: 10.1119/1.1830501]

***Am. J. Phys.*, Vol 73, 265-272 (2005)**

# Nonholonomic Constraints

**Assume nonholonomic constraints**

$$g_k(q_\rho, \dot{q}_\rho) = 0\,.$$

**When the constraints are linear in the velocities, we can write the equations as:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_k \mu_k \frac{\partial g_k}{\partial \dot{q}_i} = 0\,.$$

**For the Rock'n'roller, we have one holonomic constraint and two nonholonomic constraints.**

# Constants of Motion for Routh Sphere

**There are three degrees of freedom and three constants of integration.**

# Constants of Motion for Routh Sphere

There are three degrees of freedom and three constants of integration.
The kinetic energy is

$$K = \tfrac{1}{2}[u^2 + v^2 + w^2] + \tfrac{1}{2}[\mathbf{I_1}\omega_1^2 + \mathbf{I_2}\omega_2^2 + \mathbf{I_3}\omega_3^2]$$

The potential energy is

$$V = mga(1 - \cos\theta)\,.$$

Since there is no dissipation,

$$E = K + V = \mathrm{constant}\,.$$

# Constants of Motion for Routh Sphere

**Jellett's constant is the scalar product:**

$$C_J = \mathbf{L} \cdot \mathbf{r} = \mathbf{I_1} s(\sigma\omega_1 + \chi\omega_2) + \mathbf{I_3} f\,\omega_3 = \text{constant}\,.$$

**where** $f = \cos\theta - a$, $\sigma = \sin\psi$ **and** $\chi = \cos\psi$.
**S O'Brien & J L Synge first gave this interpretation.**

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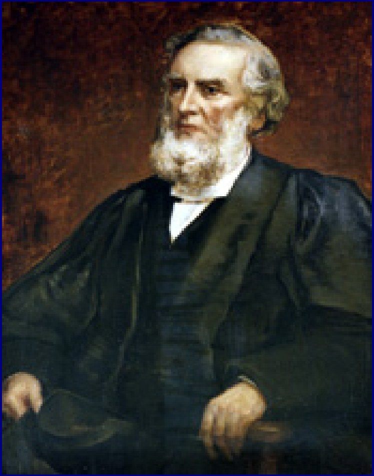
**Constant $C_R$ implies conservation of *sign* of $\omega_3$ ...**
**... but this does not automatically preclude recession!**

## Edward J Routh

**1831–1907**

## John H Jellett



**1817–1888**

# Integrability of Routh Sphere

**Using Routh's constant, we have $\omega_3 = \omega_3(\theta)$.**

**Then, using Jellett's constant, we have $\omega_2 = \omega_2(\theta)$.**

**Using the energy equation, we can now write:**

$$\dot{\theta}^2 = f(\theta)\,.$$

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**Using the energy equation, we can now write:**

$$\dot{\theta}^2 = f(\theta) \, .$$

**For a given $\theta$, both $\omega_2$ and $\omega_3$ are fixed:**
**This confirms that recession is impossible.**

# Integrability of the Rock'n'roller

**The only known constant of motion is total energy $E$.**

**There remains a symmetry: the system is unchanged under the transformation**

$$\phi \longrightarrow \phi + \delta\phi$$

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There remains a symmetry: the system is unchanged under the transformation

$$\phi \longrightarrow \phi + \delta\phi$$

The spirit of **Noether's Theorem** would indicate another constant associated with this symmetry;

So far, we have not found a "missing constant".

# Rock'n'roller

## The Jellett and Routh quantities

$$Q_J = \mathbf{L} \cdot \mathbf{r} = \mathbf{I_1} s(\sigma\omega_1 + \chi\omega_2) + \mathbf{I_3} f\,\omega_3$$

$$Q_R = \left[\sqrt{\mathbf{I_3} + s^2 + (\mathbf{I_3}/\mathbf{I_1})f^2}\right]\omega_3$$

**are no longer conserved for the Rock'n'roller.**

# Rock'n'roller

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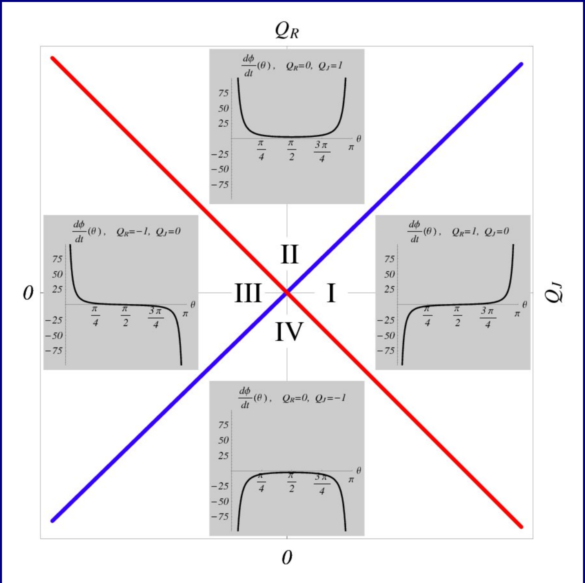
**are no longer conserved for the Rock'n'roller.**

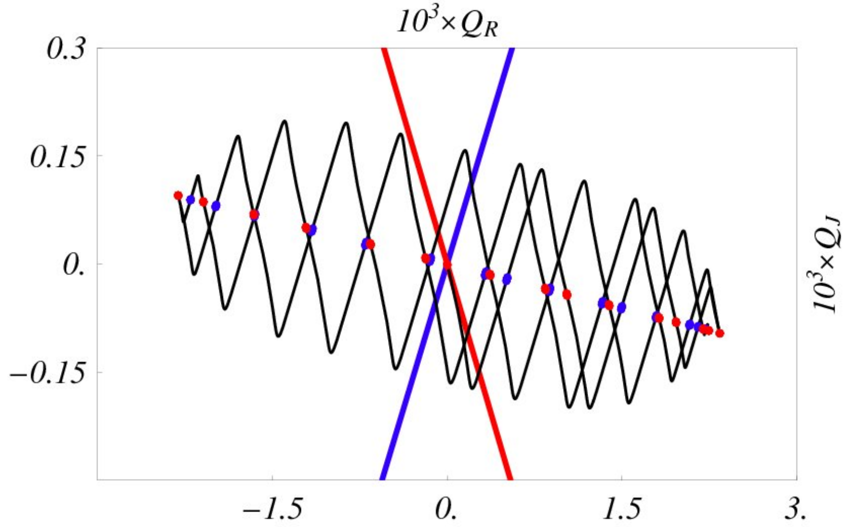
**We have found, analytically, that recession occurs when critical values of these quantities are crossed:**

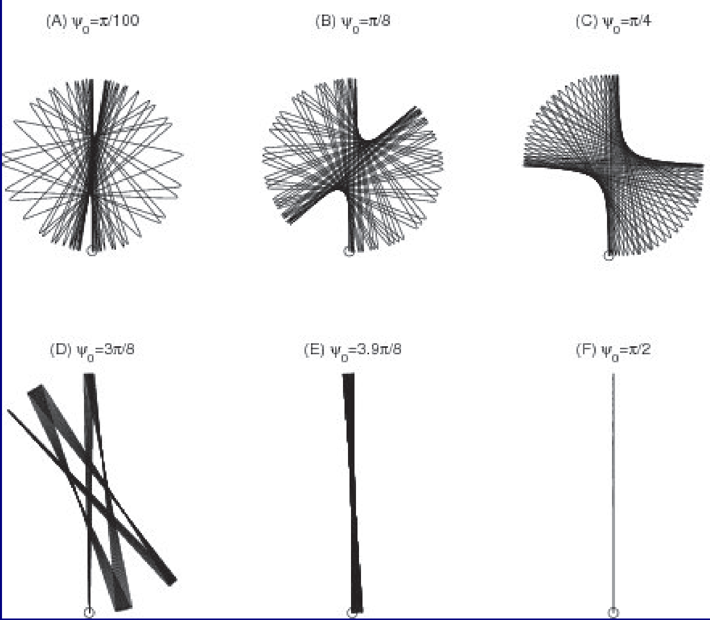
$$Q_J = Q_{J,0}^{\mathrm{crit}} \qquad \text{and} \qquad Q_J = Q_{J,\pi}^{\mathrm{crit}}$$

**These are shown on the figure below.**


$Q_R$
$Q_J$
0
0
I
II
III
IV
$\frac{d\phi}{dt}(\theta)$, $Q_R=0$, $Q_J=1$
$\frac{d\phi}{dt}(\theta)$, $Q_R=-1$, $Q_J=0$
$\frac{d\phi}{dt}(\theta)$, $Q_R=1$, $Q_J=0$
$\frac{d\phi}{dt}(\theta)$, $Q_R=0$, $Q_J=-1$

**$Q_J$ versus $Q_R$**


$10^3 \times Q_R$
0.3
0.15
0.
−0.15
−1.5
0.
1.5
3.
$10^3 \times Q_J$



(A) $\psi_0=\pi/100$
(B) $\psi_0=\pi/8$
(C) $\psi_0=\pi/4$
(D) $\psi_0=3\pi/8$
(E) $\psi_0=3.9\pi/8$
(F) $\psi_0=\pi/2$

# Orbit of stars in a Globular Cluster

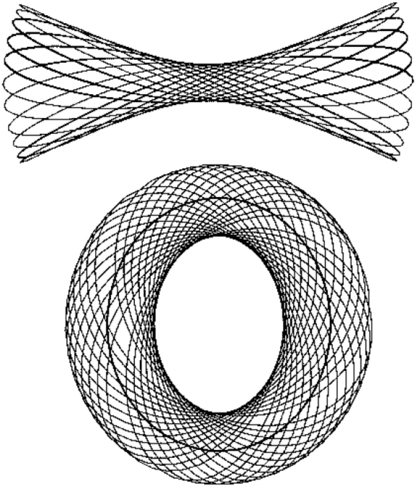


**Figure 3.8** Two orbits of a common energy in the potential $\Phi_L$ of equation (3.103) when $v_0 = 1$, $q = 0.9$ and $R_c = 0.14$: top, a box orbit; bottom, a loop orbit. The closed parent of the loop orbit is also shown. The energy, $E = -0.337$, is that of the isopotential surface that cuts the long axis at $x = 5R_c$.

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## Precession and recession of the rock'n'roller

IOPSELECT


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**Journal** Journal of Physics A: Mathematical and Theoretical Create an alert RSS this journal

**Issue** Volume 42, Number 42

**Citation** Peter Lynch and Miguel D Bustamante 2009 *J. Phys. A: Math. Theor.* **42** 425203
doi: 10.1088/1751-8113/42/42/425203


**Article** **References**

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**Abstract** We study the dynamics of a spherical rigid body that rocks and rolls on a plane under the effect of gra
distribution of mass is non-uniform and the centre of mass does not coincide with the geometric centre
symmetric case, with moments of inertia $I_1 = I_2 < I_3$, is integrable and the motion is completely regular.

# Quaternionic Formulation

The Euler angles have a singularity when $\theta = 0$
The angles $\phi$ and $\psi$ are not uniquely defined there.

# Quaternionic Formulation

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**The angles $\phi$ and $\psi$ are not uniquely defined there.**

**We can obviate this problem by using Euler's symmetric parameters**

$$\gamma = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi) \qquad \xi = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi)$$

$$\zeta = \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \qquad \eta = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)$$

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$$\zeta = \cos\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\phi + \psi) \qquad \eta = \sin\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\phi - \psi)$$

**There are the components of a unit quaternion**

$$\mathbf{q} = \gamma + \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$$

$$\gamma^2 + \xi^2 + \eta^2 + \zeta^2 = 1$$


Here as he walked by
on the 16th of October 1843
Sir William Rowan
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

## Expressions for the angular rates of change:

$$
\begin{aligned}
\dot{\theta} &= \frac{(\xi\dot{\xi} + \eta\dot{\eta}) - (\gamma\dot{\gamma} + \zeta\dot{\zeta})}{\sqrt{(\xi^2 + \eta^2)(\gamma^2 + \zeta^2)}} \\
\dot{\phi} &= \left(\frac{\gamma\dot{\zeta} - \zeta\dot{\gamma}}{\gamma^2 + \zeta^2}\right) + \left(\frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2}\right) \\
\dot{\phi} &= \left(\frac{\gamma\dot{\zeta} - \zeta\dot{\gamma}}{\gamma^2 + \zeta^2}\right) - \left(\frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2}\right)
\end{aligned}
$$

## The components of angular velocity are

$$
\begin{aligned}
\omega_1 &= 2[\gamma\dot{\xi} - \xi\dot{\gamma} + \zeta\dot{\eta} - \eta\dot{\zeta}] \\
\omega_2 &= 2[\gamma\dot{\eta} - \eta\dot{\gamma} + \xi\dot{\zeta} - \zeta\dot{\xi}] \\
\omega_3 &= 2[\gamma\dot{\zeta} - \zeta\dot{\gamma} + \eta\dot{\xi} - \xi\dot{\eta}]
\end{aligned}
$$

## The first-order (small $\theta$) equations may be written

$$
\begin{aligned}
\ddot{\gamma} + \left(\frac{\omega_3}{2}\right)^2 \gamma &= 0 \\
\ddot{\zeta} + \left(\frac{\omega_3}{2}\right)^2 \zeta &= 0 \\
\ddot{\xi} + \kappa_{21}\omega_3\dot{\eta} + \Omega_1^2\xi &+ \epsilon'\zeta\left\{(1-\kappa)\omega_3(\gamma\dot{\xi} + \zeta\dot{\eta}) + \Omega_{11}^2(\gamma\eta - \zeta\xi)\right\} = 0 \\
\ddot{\eta} - \kappa_{21}\omega_3\dot{\xi} + \Omega_1^2\eta &- \epsilon'\gamma\left\{(1-\kappa)\omega_3(\gamma\dot{\xi} + \zeta\dot{\eta}) + \Omega_{11}^2(\gamma\eta - \zeta\xi)\right\} = 0
\end{aligned}
$$

**where $\epsilon'$ is related to the asymmetry $(I_2 - I_1)/I_1$.**

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\end{aligned}
$$

**where $\epsilon'$ is related to the asymmetry $(I_2 - I_1)/I_1$.**

**By a simple rotation of coordinates, they can be transformed to a system with constant coefficients.**

**Thus, the complete solution can be obtained.**

# Conclusion

Recession is found in a wide variety of physical contexts.

Through the quaternion analysis, we can explain the phenomenon in simple terms.

Details remain to be worked out.

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Come back for Part III in a few years.

**Thank You**