A Painless Overview of the Riemann Hypothesis

[Proof Omitted]

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Outline

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Tossing a Coin

Everyone knows that if we toss a fair coin, the chances are equal for Heads and Tails.

Let us give scores to Heads and Tails:

$$\kappa = \begin{cases} +1 & \textbf{for Heads} \\ -1 & \textbf{for Tails} \end{cases}$$

Now if we toss many times, the sum $S_N = \sum_1^N \kappa_n$ is *a random walk* in one dimension.

The expected distance from zero after N tosses is

$$E(|S_N|) = \sqrt{\frac{2N}{\pi}} \sim N^{\frac{1}{2}}$$

Prime Factors

Every natural number *n* can be uniquely factored into a product of primes.

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$$

Let us count multiplicities, so that 15 has two prime factors, while 18 has three:

$$15 = \underbrace{3 \times 5}_{2} \qquad 18 = \underbrace{2 \times 3 \times 3}_{3}$$

You might expect equal chances for a number to have an even or odd number of prime factors.

You might be right!

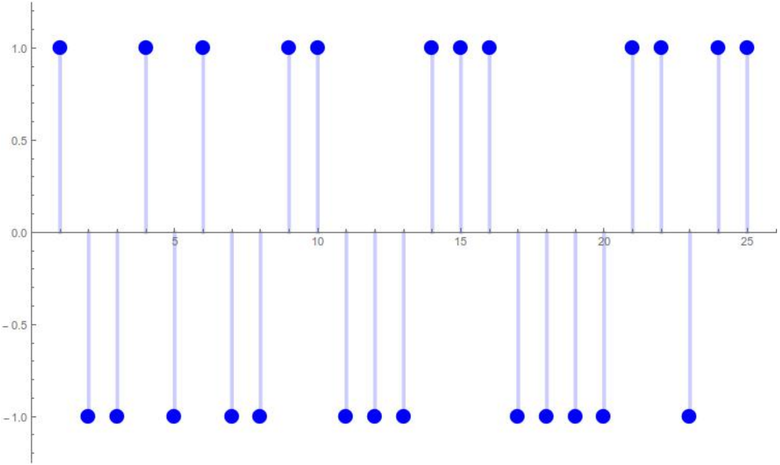
Like Tossing a Coin?

- **Write all the natural numbers.**
- **Write all their prime factors.**
- **Assign $\lambda = +1$ to numbers with even $\#$ factors.**
- **Assign $\lambda = -1$ to numbers with odd $\#$ factors.**

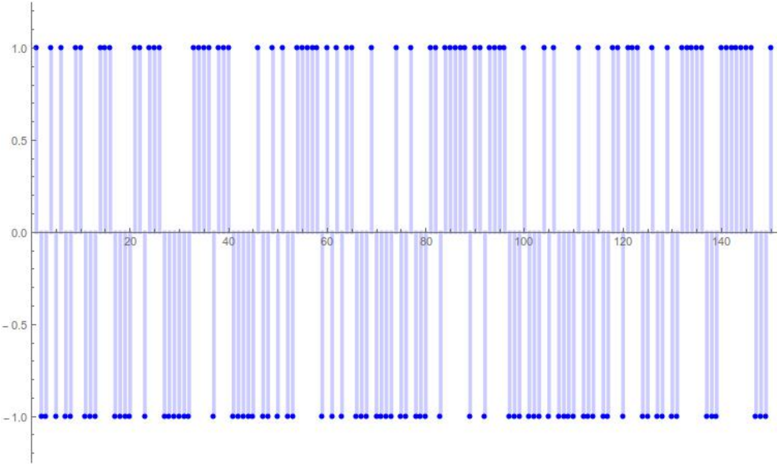
n	2	3	4	5	6	7	8	9	10	11	12
	2	3	$2 \cdot 2$	5	$2 \cdot 3$	7	$2 \cdot 2 \cdot 2$	$3 \cdot 3$	$2 \cdot 5$	11	$2 \cdot 2 \cdot 3$
$\lambda(n)$	-1	-1	$+1$	-1	$+1$	-1	-1	$+1$	$+1$	-1	-1

The function $\lambda(n)$ is called the Liouville Function.

Liouville Function $\lambda(n)$ for $n \leq 25$



Liouville Function $\lambda(n)$ for $n \leq 150$



Like Tossing a Coin?

The function $\lambda(n)$ appears to fluctuate randomly. There is much cancellation for the function

$$\Lambda(N) = \sum_{n=1}^{N} \lambda(n)$$

If $\lambda(n)$ were truly random, we would expect

$$\Lambda(N) = O\left(N^{\frac{1}{2}}\right)$$

If you can show that, for all $\epsilon > 0$,

$$\Lambda(N) = O\left(N^{\frac{1}{2}} + \epsilon\right)$$

you should pick up a cool \$1 million, because this is equivalent to the Riemann Hypothesis.

The Riemann Hypothesis (RH)

The Riemann hypothesis (RH) is widely regarded as the most celebrated problem in modern mathematics.

The hypothesis connects objects in two apparently unrelated mathematical contexts:

- **Prime numbers [fundamentally discrete].**
- **Analytic functions [essentially continuous].**

$$\pi(x) \longleftrightarrow \zeta(x)$$

RH can be formulated in diverse and seemingly unrelated ways. This is one of its attractions.

The Riemann Hypothesis (RH)

The Riemann zeta function is defined by

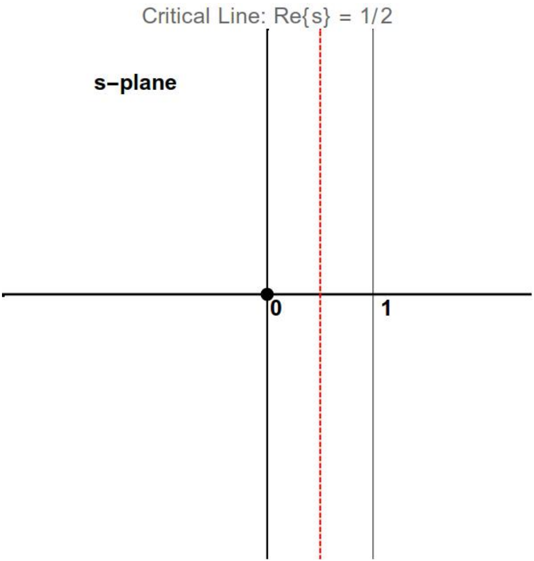
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1$$

The usual statement of the hypothesis is:

> "The complex zeros of the Riemann zeta function all lie on the critical line $\Re(s) = \frac{1}{2}$."

Since the series does not converge on this line, analytic continuation is needed.

The Complex *s*-plane



True or False?

There is powerful heuristic evidence for RH:

The first ten trillion zeros of $\zeta(s)$ are on the line.

To a non-mathematician, this amounts to proof.

But there are examples of hypotheses supported by computational evidence but known to be false.

In 1912 J. E. Littlewood proved that the function

$$\mathrm{Li}\,(n) - \pi(n)$$

becomes negative for some finite n. *But this does not happen in the computational range.*

No. 8 on Hilbert's List of 23 Problems

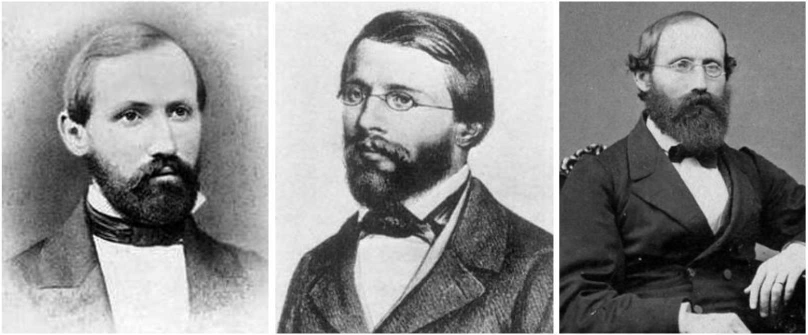
The Riemann Hypothesis was highlighted by David Hilbert at the International Congress of Mathematicians in Paris in 1900.

The following words are attributed to Hilbert:

"If I were to awaken after sleeping for a thousand years, my first question would be: Has the Riemann Hypothesis been proven?"

Outline

Bernhard Riemann (1826–1856)



Bernhard Riemann (1826-66)

Bernhard Riemann (1826–1866)

- Born in Breselenz in Hanover.
- Son of a Lutheran pastor.
- Timid and reserved by nature.
- School in Lüneburg: teacher noticed his talents.
- Mastered Legendre's *Theory of Numbers.*

Bernhard Riemann (1826–1866)

In 1846 Riemann began his studies at Göttingen (Gauss still working but near the end of his career).

Moved to Berlin, where Dirichlet worked.

1851: Riemann awarded a doctorate (in Göttingen).

Thesis on the foundations of complex variable theory.

Bernhard Riemann (1826–1866)

1853: Riemann presented work on trigonometric series for his Habilitation. Constructed what we now call the Riemann integral.

Presentation: *The Foundations of Geometry.*

Riemann's vision of geometry was profound in its sweeping generality.

Riemannian Geometry was the framework for Einstein's General Theory of Relativity.

Bernhard Riemann (1826–1866)

1859: Riemann appointed Professor at Göttingen.

Elected member of Berlin Academy (also in 1859).

Presented his *single contribution to number theory,* on the distribution of prime numbers:

It contained his conjecture (Riemann's Hypothesis).

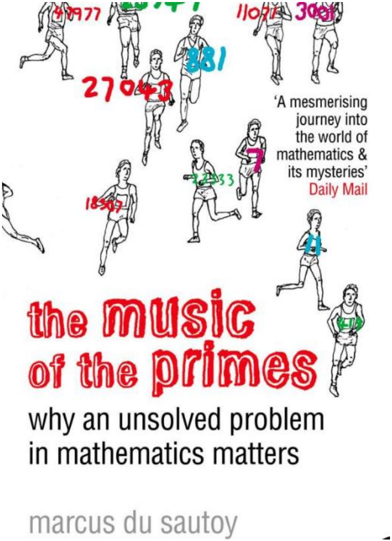
Some of Riemann's Mathematics

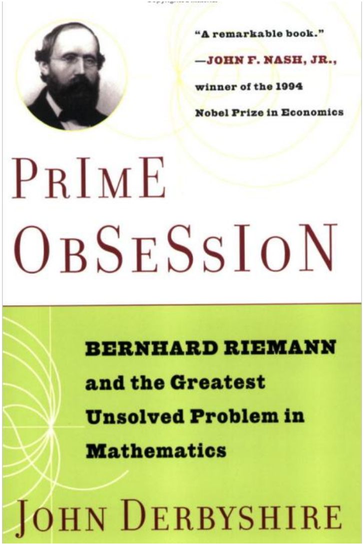
- Complex variable theory
- Number theory
- Geometry
- Integration
- Calculus of variations
- Theory of electricity.

- Riemann integral
- Riemann mapping theorem
- Riemann sphere
- Riemann sheets
- Riemann curvature tensor
- Cauchy-Riemann equations
- Riemannian manifolds.

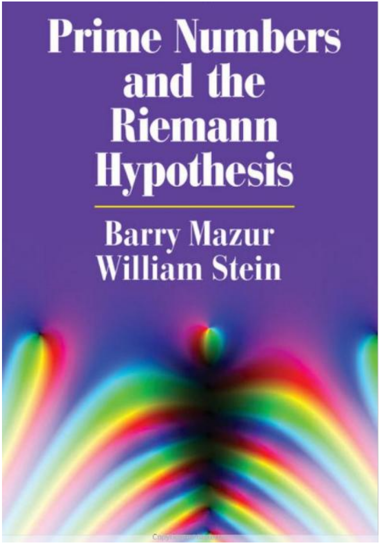
Outline

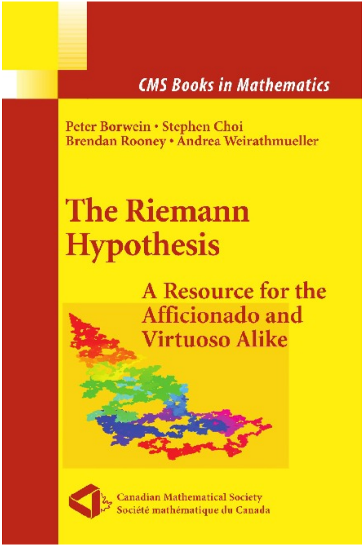
Books about the Riemann Hypothesis





Books about the Riemann Hypothesis





Outline

Prime Numbers

Primes are *the atoms of the number system* . . .
and have fascinated mathematicians for millennia.

Euclid gave a remarkably simple proof
that that there is an infinity of primes.

Many mysteries about primes remain:

God may not play dice with the Universe, but something strange is going on with the prime numbers. (Paul Erdős, after Albert Einstein)

The Prime Counting Function $\pi(n)$

We define the prime counting function $\pi(n)$.
It is the number of primes $\leq n$:

$$
\begin{aligned}
\pi(1) &= 0 \\
\pi(2) &= 1 \\
\pi(3) = \pi(4) &= 2 \\
\pi(5) = \pi(6) &= 3 \\
\pi(7) = \pi(8) = \pi(9) = \pi(10) &= 4 \\
\pi(11) = \pi(12) &= 5 \\
&\cdots \\
\pi(100) &= 25
\end{aligned}
$$

Prime Staircase Graph

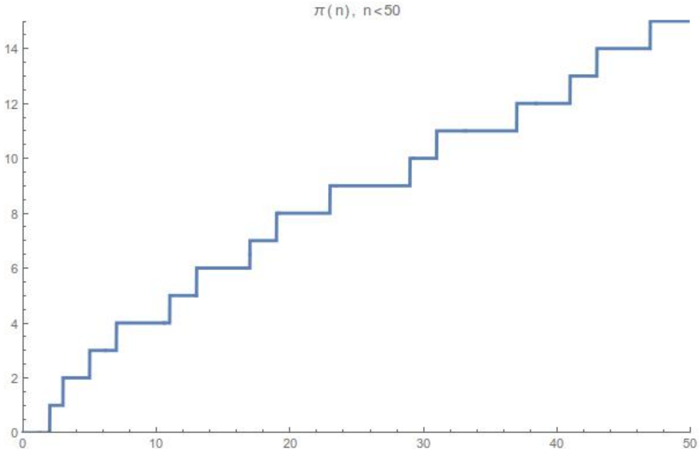


Figure : The prime counting function $\pi(x)$ for $0 \leq x \leq 50$.

Percentage of Primes Less than N

Table : Percentage of Primes less than N

N	$\pi(N)$	Percent
1,000	168	16.8%
1,000,000	78,498	7.8%
1,000,000,000	50,847,534	5.1%
1,000,000,000,000	37,607,912,018	3.8%

We can see that the percentage of primes is falling off with increasing size . . .

. . . but the rate of decrease is very slow.

The Function $\zeta(s)$ for $s \in \mathbb{N}$

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It is easy to show that this converges for $\Re(s) > 1$.

For $s = 1$ we get the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

For $s = 2$ it is the "Basel series", summed by Euler:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

Euler's Great Contribution

Euler studied the series for $k \in \mathbb{N}$:

$$\zeta(k) = \sum \frac{1}{n^k}$$

Euler's product formula was a major contribution:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} .$$

This links the zeta function to the prime numbers.

Gauss and Legendre

In 1792 Carl Friedrich Gauss, only 15 years old, found that the proportion of primes less that n decreased approximately as $1/\log n$.

So the number of primes less than n is

$$\pi(n) \sim \frac{n}{\log n}$$

Around 1795 Legendre noticed a similar pattern of the primes, but it was to take another century before a proof emerged.

“The Most Remarkable Result . . . ”

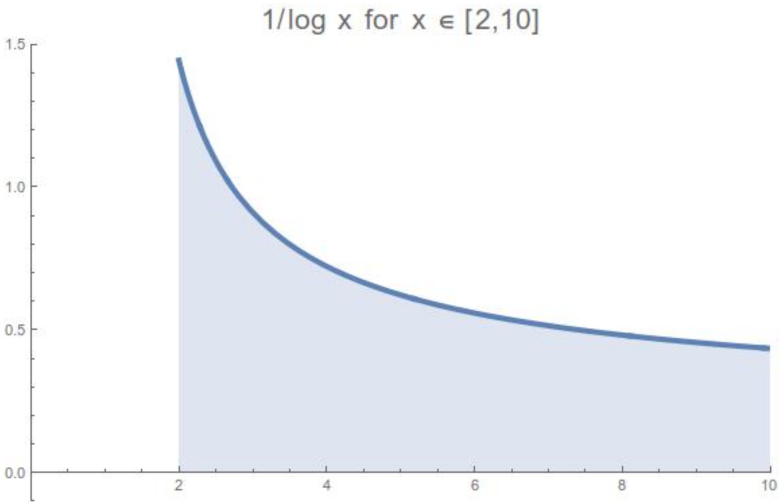
In a letter dated 1823, Niels Henrik Abel described the distribution of primes as “the most remarkable result in all of mathematics.”

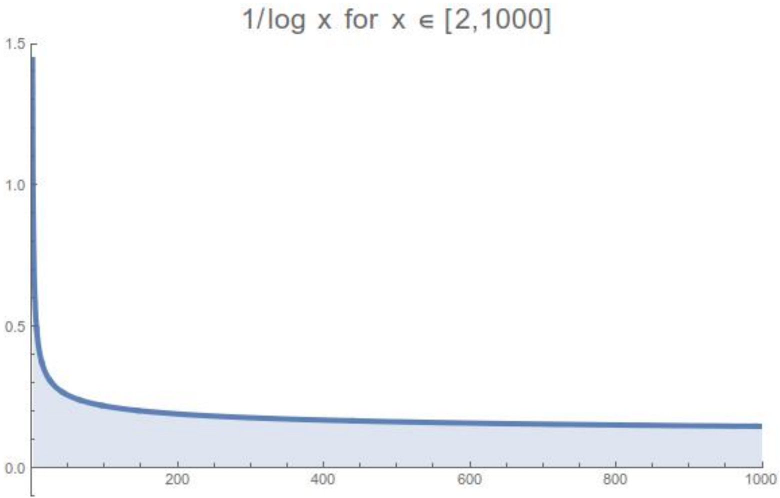
In 1838, Dirichlet discovered an approximation to $\pi(n)$ using the logarithmic integral

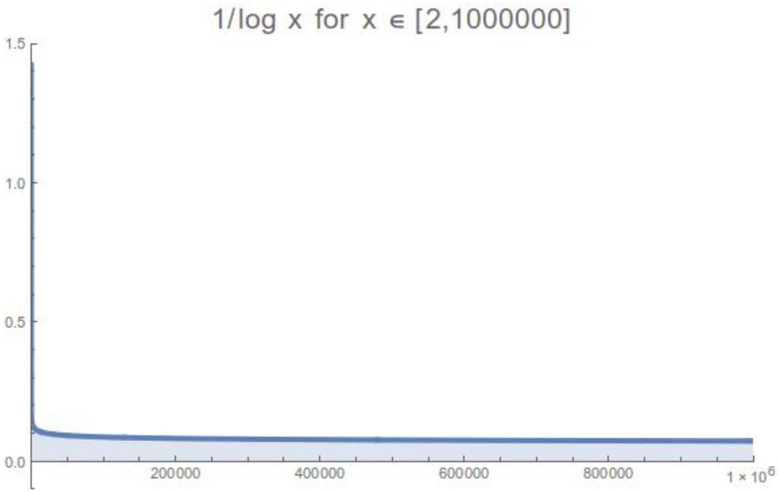
$$\pi(n) \approx \mathrm{Li}(n) = \int_2^n \frac{\mathrm{d}x}{\log x}$$

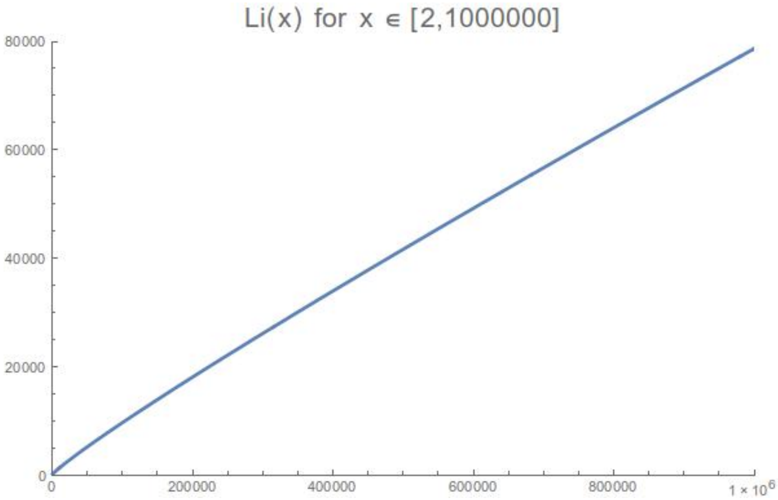
This gives a significantly better estimate of $\pi(n)$ than the simple ratio $n/\log n$.

You’ve guessed it: Gauss had already discovered (but not uncovered) this result.


1/log x for x ∈ [2,10]
1.5
1.0
0.5
0.0
2
4
6
8
10


1/log x for x ∈ [2,1000]
1.5
1.0
0.5
0.0
200
400
600
800
1000

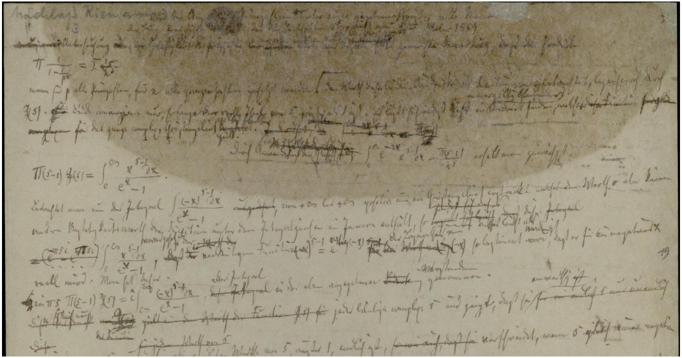

1/log x for x ∈ [2,1000000]
1.5
1.0
0.5
0.0
200000
400000
600000
800000
1 × 10^6


Li(x) for x ∈ [2,1000000]
80000
60000
40000
20000
0
0
200000
400000
600000
800000
1 × 10^6

Outline

Über die Anzahl der Primzahlen . . .

In 1859, Bernhard Riemann published his paper on the distribution of the prime numbers.



Like all his mathematical contributions, it was a work of astonishing novelty of ideas.

Über die Anzahl der Primzahlen . . .

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Transcribed by D. R. Wilkins

Preliminary Version: December 1998

A selection of papers on the Riemann hypothesis has been assembled by David Wilkins of TCD
`http://www.maths.tcd.ie/~dwilkins/`

www.maths.tcd.ie/~dwilkins/

The Mathematical Papers of Georg Friedrich Bernhard Riemann (1826-1866)

The following papers of Bernhard Riemann are available here:

Papers published by Riemann

Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse
(Inauguraldissertation, Göttingen, 1851)

Ueber die Gesetze der Vertheilung von Spannungselectricität in ponderabeln Körpern, wenn diese nicht als vollkommene Leiter oder Nichtleiter, sondern als dem Enthalten von Spannungselectricität mit endlicher Kraft widerstrebend betrachtet werden
(Amtlicher Bericht über die 31. Versammlung deutscher Naturforscher und Aerzte zu Göttingen im September 1854)

Zur Theorie der Nobili'schen Farbenringe
(Annalen der Physik und Chemie, 95 (1855))

Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen
(Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 7 (1857))

Theorie der Abel'schen Functionen
(Journal für die reine und angewandte Mathematik, 54 (1857))

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse
(Monatsberichte der Berliner Akademie, November 1859)

Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite
(Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 8 (1860))

Ein Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoides
(Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 9 (1860))

Ueber das Verschwinden der Theta-Functionen
(Journal für die reine und angewandte Mathematik, 65 (1866))

Riemann's H-index could not have exceeded 9.

The Golden Key

Euler had discovered a connection between the zeta function and the prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

John Derbyshire calls this formula *The Golden Key.*

Riemann considered $\zeta(s)$ for complex values of s.
***He used the Golden Key* to derive a relationship between $\zeta(s)$ and the prime counting function:**

$$\zeta(s) \quad \Longleftrightarrow \quad \pi(n)$$

Zeros $\Longleftrightarrow$ Primes

Relating $\pi(x)$ to the zeros of $\zeta(s)$

Riemann defined a *prime power counting function*

$$J(x) = \pi(x) + \frac{1}{2}\pi(\sqrt[2]{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \frac{1}{4}\pi(\sqrt[4]{x}) + \cdots$$

This is actually a finite sum.

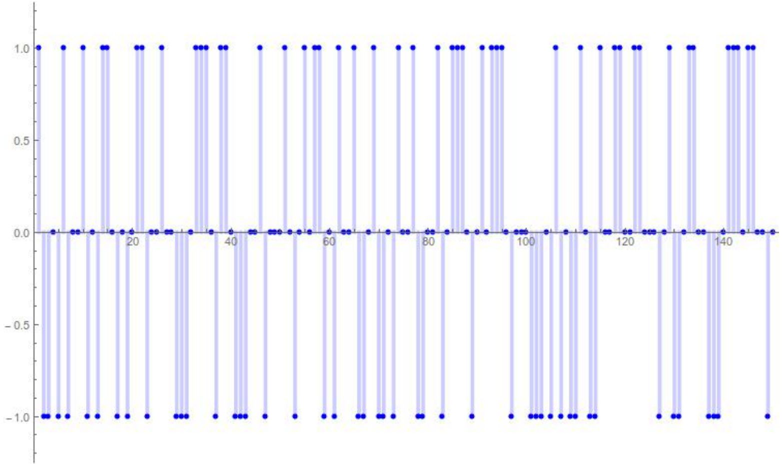
He used Möbius inversion to get the (finite) sum

$$\pi(x) = \sum_n \frac{\mu(n)}{n} J(\sqrt[n]{x})$$

where $\mu(n)$ is the Möbius function.

He now had $\pi(x)$ in terms of $J(x)$.

Möbius Function $\mu(n)$ for $n \leq 150$



Riemann's Main — Dazzling — Result

Riemann then expressed $\zeta(s)$ as a function of $J(x)$:

$$\log \zeta(s) = s \int_1^\infty J(x) x^{-s-1}\,\mathrm{d}x$$

He inverted this to get $J(x)$ in terms of $\zeta(s)$:

$$J(x) = \underbrace{\mathrm{Li}\,(x)}_{\textit{MAIN TERM}} - \underbrace{\sum_\rho \mathrm{Li}\,(x^\rho)}_{\textit{CORRECTION}} + \underbrace{\int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\log t} - \log 2}_{\textit{SMALL TERMS}}$$

where ρ runs over the complex zeros of $\zeta(s)$.

He now had $\pi(x)$ in terms of the zeros of $\zeta(s)$.

How? By Extending $\zeta(s)$ to $\mathbb{C}\backslash\{1\}$

Riemann showed that $\zeta(s)$ can be extended to the entire s-plane (except for $s = 1$).

He found a *functional equation*

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s)$$

The r.h.s. equals the l.h.s. with s replaced by $1 - s$.

There is a symmetry about the critical line $\Re(s) = \frac{1}{2}$.

Trivial & Nontrivial Zeros

Since $\Gamma(s)$ has simple poles for $s \in \{0, -1, -2, \ldots\}$ the zeta function has simple zeros at $s = -2, -4, \ldots$.

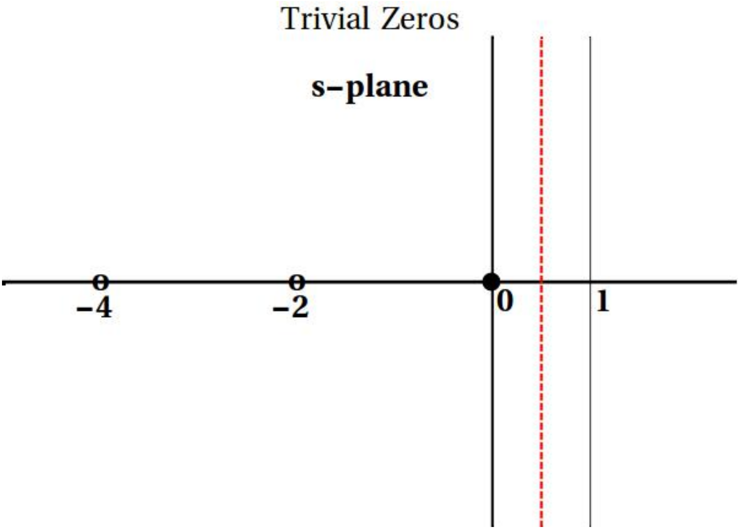
These are the *trivial zeros* of $\zeta(s)$.

The zeros of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$ are called the *nontrivial zeros of* $\zeta(s)$.

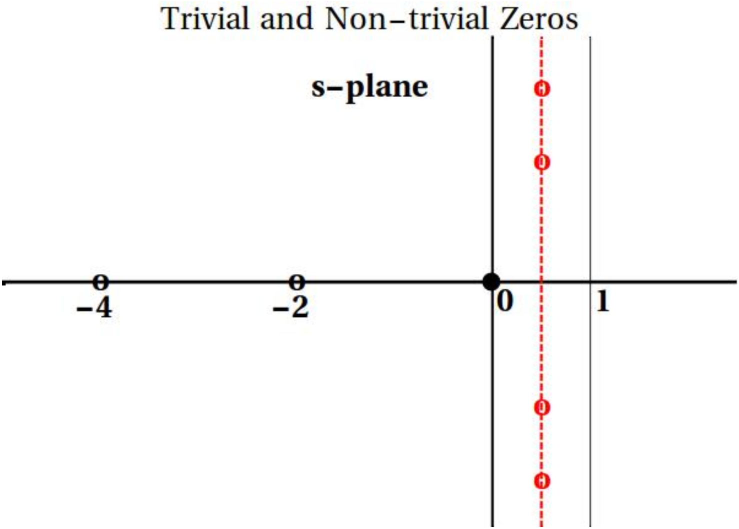
Riemann's Hypothesis:

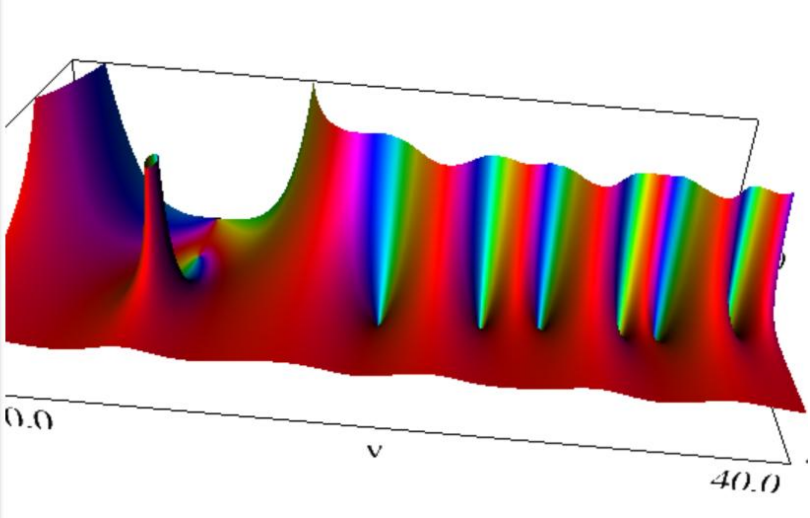
All the nontrivial zeros are on the critical line $\Re(s) = \frac{1}{2}$.

Trivial Zeros of $\zeta(s)$



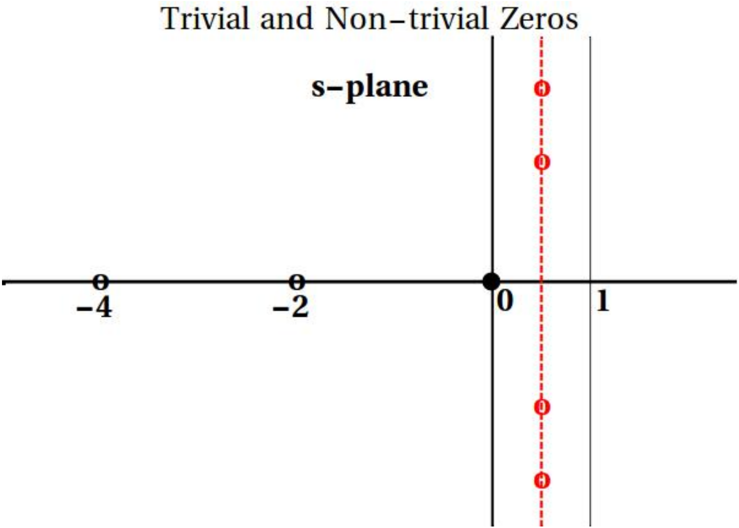
Trivial & Nontrivial Zeros of $\zeta(s)$

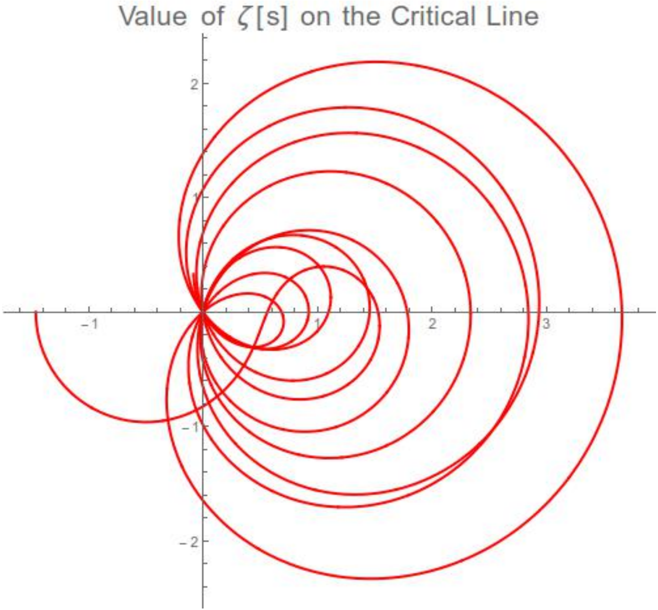




(Plot from `http://dlmf.nist.gov/`)

Trivial & Nontrivial Zeros of $\zeta(s)$




Value of ζ[s] on the Critical Line
2
1
−1
−2
−1
1
2
3

How Many Zeros Are There?

Consider the critical strip up to height T.

Riemann conjectured that the number of zeros in this strip is

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$$

There are infinitely many nontrivial zeros. The zeros get closer for larger T.

This conjecture was proved in 1905 by Hans von Mangoldt (1854–1925).

Musical Interlude

John Derbyshire sings Tom Apostol's song about the Riemann Hypothesis.

`https://www.youtube.com/watch?v=zmCHhGT7KkQ`

The Riemann Spectrum

Assuming RH, let us write the zeros of $\zeta(s)$ as

$$\left\{\tfrac{1}{2} \pm i\theta_k,\ k \in \mathbb{N}\right\}$$

Riemann showed that, given these values, the function $\pi(n)$ could be reconstructed.

Below, we plot the Fourier transform

$$-\sum_{j=1}^{1000} \cos\left(\log(s)\theta_j\right)$$

(Plots are from Mazur and Stein, 2016)

Riemann Spectrum ⇒ Prime Numbers

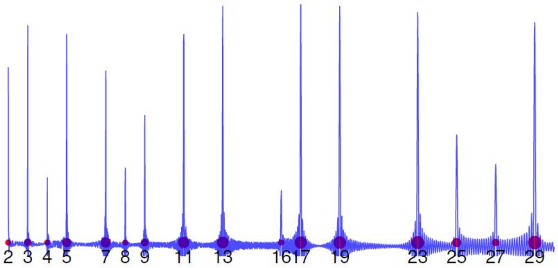


Figure 35.1: Illustration of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$, where $\theta_1 \sim 14.13, \ldots$ are the first 1000 contributions to the Riemann spectrum. The red dots are at the prime powers p^n, whose size is proportional to $\log(p)$.

Riemann Spectrum $\Rightarrow$ Prime Numbers

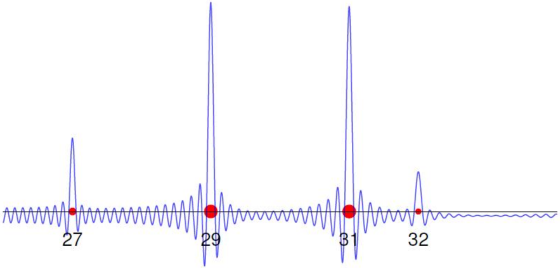


Figure 35.2: Illustration of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$ in the neighborhood of a twin prime. Notice how the two primes 29 and 31 are separated out by the Fourier series, and how the prime powers 3^3 and 2^5 also appear.

Riemann Spectrum ⇒ Prime Numbers

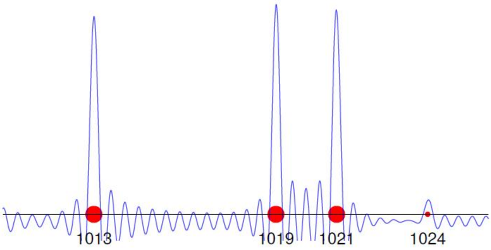


Figure 35.3: Fourier series from 1,000 to 1,030 using 15,000 of the numbers θ_i. Note the twin primes 1,019 and 1,021 and that $1{,}024 = 2^{10}$.

Reconstructing the Staircase Function

From the Riemann Spectrum

$$\left\{\tfrac{1}{2} \pm i\theta_k,\ k \in \mathbb{N}\right\}$$

we can reconstruct the prime counting function $\pi(n)$.

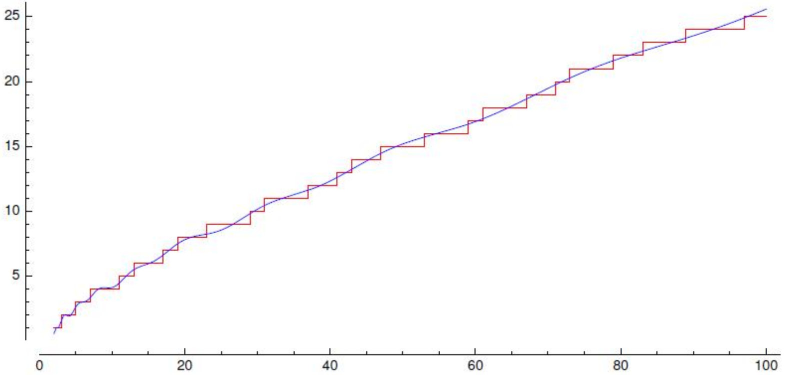
We use the *main result* of Riemann's paper:

$$J(x) = \mathrm{Li}\,(x) - \sum_{\rho} \mathrm{Li}\,(x^{\rho}) + \textbf{Smaller Terms}$$

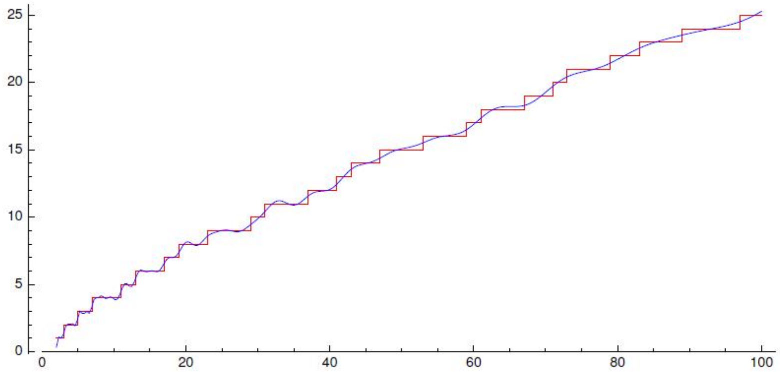
where ρ runs over the complex zeros of $\zeta(s)$.

(Plots below are from Mazur and Stein, 2016)

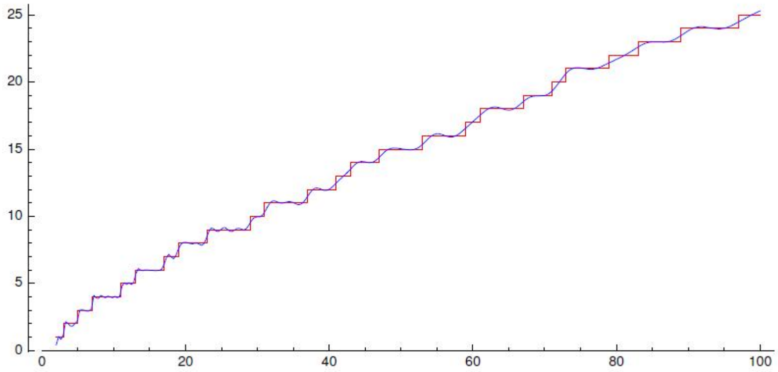
Staircase and R_1



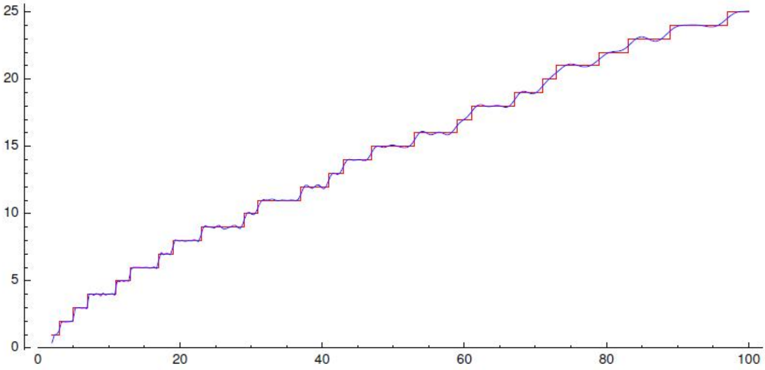
Staircase and R_{10}



Staircase and R_{25}



Staircase and R_{100}



Outline

The Prime Number Theorem (PNT)

The Prime Number Theorem describes the asymptotic distribution of the prime numbers.

The proportion of primes less than n is:

$$\pi(n)/n.$$

The PNT states that this is asymptotic to $1/\log n$

$$\pi(n) \sim \frac{n}{\log n}$$

PNT implies that the n-th prime number p_n is given approximately by $p_n \approx n \log n$.

Prime Staircase Graph to 50

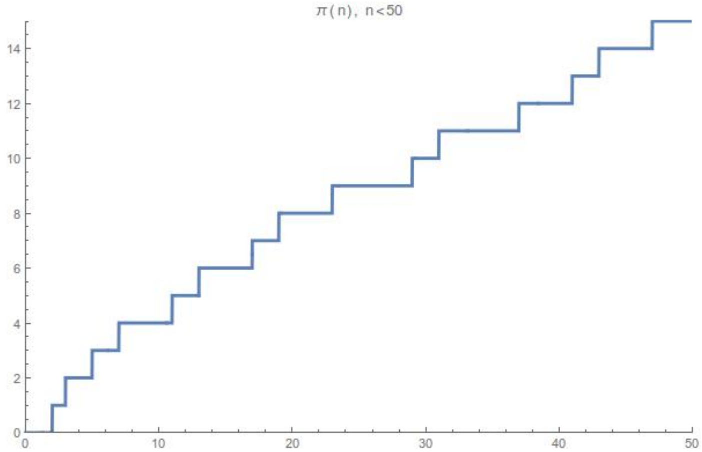


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

Prime Staircase Graph to 500

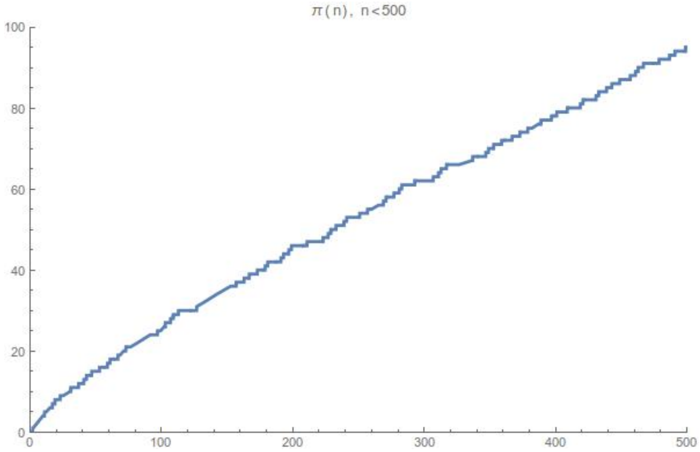


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

Prime Staircase Graph to 5000

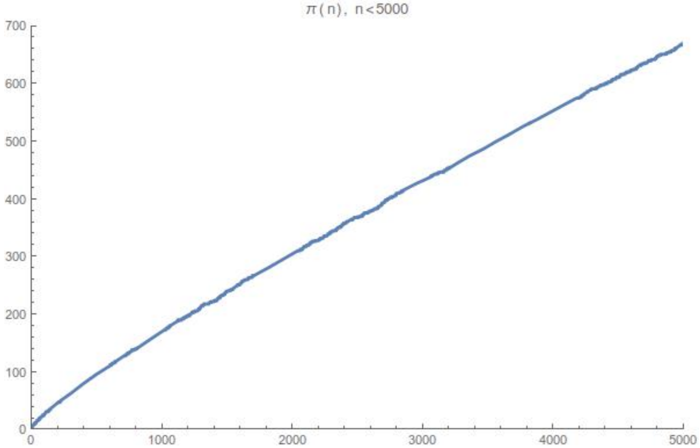


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 5000$.

Prime Staircase Graph to 50000

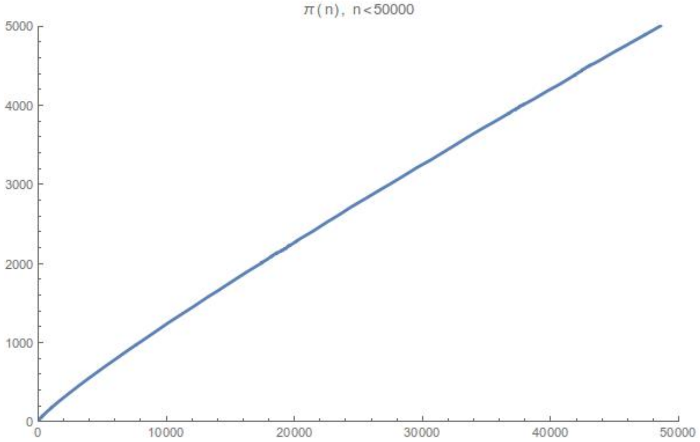
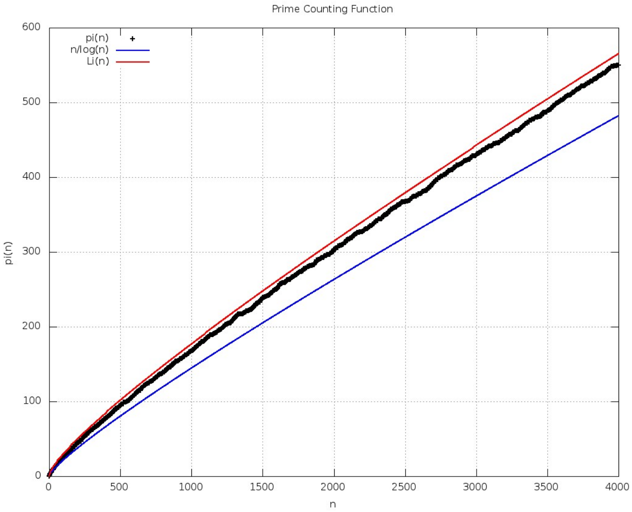


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50000$.

$\pi(x)$ compared to $\mathrm{Li}(x)$ and $x/\log x$



Black: $\pi(x)$. **Red:** $\mathrm{Li}\,x$, **Blue:** $x/\log x$

Prime Number Theorem (PNT)

Riemann’s work inspired two proofs of the PNT, independently by Jacques Hadamard and Charles de la Vallée Poussin, both in 1896.

Charles-Jean Étienne Gustave Nicolas Le Vieux, Baron de la Vallée Poussin (1866–1962)

(died aged 96).

Jacques Salomon Hadamard (1865–1963)

(died aged 98).

Outline

We write the self-evident equation

$$\pi(x) = \underbrace{\mathrm{Li}\,(x)}_{\text{ESTIMATE}} - \underbrace{[\mathrm{Li}\,(x) - \pi(x)]}_{\text{ERROR TERM}}$$

Following the proof of the Prime Number Theorem

$$\pi(x) \sim \mathrm{Li}(x)$$

interest shifted to the error term

$$\mathrm{Li}\,(x) - \pi(x)$$

The goal of proving the Riemann Hypothesis became an obsession for many mathematicians.

Riemann had given a very precise expression for this error, but it remained to be proven.

Prime Number Theorem (PNT)

In 1899, Edmund Landau showed that the Prime Number Theorem is equivalent to proving

$$\lim_{n\to\infty}\left[\frac{\lambda(1)+\lambda(2)+\cdots\lambda(n)}{n}\right]=0\,.$$

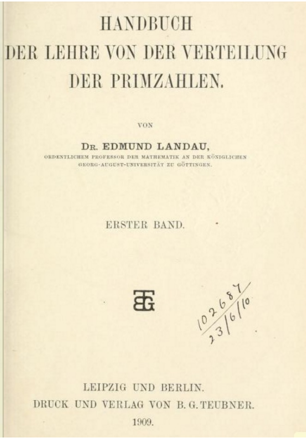
In 1903, Landau gave a simplified proof of the PNT.

His masterpiece (*Handbuch*) in 1909 was the first systematic presentation of analytic number theory.

In this work, Landau introduced the notation

- **Big-Oh or O for asymptotic limits.**
- **$\pi(x)$ for the prime counting function.**

Edmund Landau (1877–1938)



HANDBUCH
DER LEHRE VON DER VERTEILUNG
DER PRIMZAHLEN.

VON

DR. EDMUND LANDAU,
ORDENTLICHEM PROFESSOR DER MATHEMATIK AN DER KÖNIGLICHEN
GEORG-AUGUST-UNIVERSITÄT ZU GÖTTINGEN.

ERSTER BAND.

LEIPZIG UND BERLIN.
DRUCK UND VERLAG VON B. G. TEUBNER.
1909.

Theory of the Distribution of the Prime Numbers

Edmund Landau, 1909.

Hardy and Littlewood

G. H. Hardy and J. E. Littlewood.

Hardy and Littlewood

In 1914 Hardy showed that there is an infinity of zeros of $\zeta(s)$ on the critical line.

In the same year, Littlewood showed that the error

$$\mathrm{Li}(x) - \pi(x)$$

changes sign infinitely often.

Skewes' Number Sk is the smallest number for which $\mathrm{Li}\,(x) - \pi(x)$ changes sign.

Estimates. **THEN:** $\mathrm{Sk} \approx 10^{10^{10^{34}}}$ **NOW:** $\mathrm{Sk} \approx 10^{316}$.

It is still beyond the computational range.

Zeros on the Critical Line

1914: ***Hardy*** **showed that $\zeta(s)$ has infinitely many zeros on the critical line.**

1942: ***Selberg*** **showed that a positive proportion of zeros lie on the line.**

1974: ***Levinson*** **showed that at least one third of the zeros are on the line.**

1989: ***Conrey*** **showed that at least 40% of the zeros are on the line.**

But no-one has shown that ALL zeros are on the line.

Outline

RH and Quantum Physics

Are prime numbers sub-atomic physics linked?

The eigenvalues of a Hermitian matrix are real. So . . .

The Hilbert-Pólya Conjecture

The non-trivial zeros of $\zeta(s)$ are the eigenvalues of a Hermitian operator.

- **Is there a Riemann operator?**
- **What dynamical system does it represent?**
- **Will RH be proved by a physicist?**

Of the 9 papers that Riemann published during his lifetime, 4 are on physics !!!

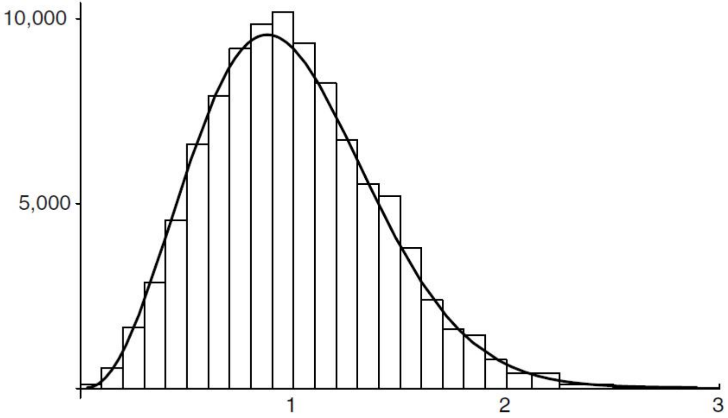
Montgomery-Odlyzko Law

The spectra of random Hermitian matrices are not random: the eigenvalues are spread out as if there is "repulsion" between closely spaced values.

Hugh Montgomery found a similar pattern in the Riemann spectrum. This led to the following “Law”:

The Riemann spectrum is statistically identical to the distribution of eigenvalues of an ensemble of Gaussian Hermitian matrices.

Montgomery-Odlyzko Law



The Montgomery-Odlyzko Law (Distribution of spacings for the 90,001st to 100,000th zeta-function zeros).

Outline

How to Solve It

Approaches to a solution may be through
Algebra, Analysis, Computation, Physics.

- **(Classical) Analytic Number Theory.**
- **Mertens' Function (Liouville Function) approach.**
- **Connes *p*-acid Number approach.**
- **More algebraic approaches.**
- **Quantum Mechanical Operators.**
- **Random Matrix Theory.**
- **Etc., etc., etc.**

Equivalent Hypotheses

There are numerous other conjectures that will sink or swim along with the Riemann Hypothesis.

- **Von Koch:** $\pi(n) = \mathrm{Li}\,(n) + O(x^{\frac{1}{2}} \log x)$.
- **Landau:** $\sum \lambda(n) = O(x^{\frac{1}{2} + \epsilon})$.
- **Mertens:** $M(n) \equiv \sum \mu(n) = O(x^{\frac{1}{2} + \epsilon})$.
- **Etc., etc., etc.**

Borwein *et al.* list 32 equivalent hypothesis *some of which look deceptively simple.*

True or False?

Case for the defence:

- Hardy: infinitude of zeros on critical line.
- RH implies the PNT, known to be true.
- Landau & Bohr: "Most" zeros near critical line.
- Algebraic results: Artin, Weil, Deligne.
- Computation: First 10^{13} zeros are on the line.

Case for the prosecution:

- Riemann had no solid case for his "very likely".
- Littlewood's proof that $\mathrm{Li}\,(x) - \pi(x)$ changes sign.
- Behaviour of $\zeta(\frac{1}{2} + it)$ is "wild" for large t.
- Counter-examples beyond computational range.

Attempts to Proof the RH

A long line of reputable mathematicians have attempted to prove the Riemann Hypothesis.

1885 Thomas Stieltjes announces a proof.
It never appeared.

1945 Hans Rademacher “almost” proves RH false.
Article appears in *Time* Magazine.

2002 Louis de Branges posts a proof on his website.
It has not been accepted by other mathematicians.

THE RIEMANN HYPOTHESIS

LOUIS DE BRANGES*

ABSTRACT. A proof of the Riemann hypothesis is obtained for the zeta functions constructed from a discrete vector space of finite dimension over the skew–field of quaternions with rational numbers as coordinates in hyperbolic analysis on locally compact Abelian groups obtained by completion. Zeta functions are generated by a discrete group of symplectic transformations. The coefficients of a zeta function are eigenfunctions of Hecke operators defined by the group. In the nonsingular case the Riemann hypothesis is a consequence of the maximal accretive property of a Radon transformation defined in Fourier analysis. In the singular case the Riemann hypothesis is a consequence of the maximal accretive property of the restriction of the Radon transformation to a subspace defined by parity. The Riemann hypothesis for the Euler zeta function is a corollary.

Figure : The latest proof by De Branges (12 Dec 2016)

Outline

So What?

Why are we interested in the Riemann Hypothesis?
Because it's there! It is the Everest of mathematics.

Because it has much more to teach us.
We may be only at the beginning of Number Theory.

- **The beauty of the analysis**
- **The sheer joy of discovery**
- **Validation of numerous theorems**
- **Applications (???) (for funding purposes !!!)**
  - **Cryptography (?)**
  - **Riemann spectrum in quantum physics (?)**
  - **Better mousetraps (!)**
  - **Anything else you care to add (!)**

When Will We Know?

**The proof (or refutation) may come tomorrow,
or we may have to wait for a century or more.**

**“I have no idea what the consequences will be,
and I don’t think anyone else has either.
I am certain, though, that they will be tremendous.”**

John Derbyshire.

Wir müssen wissen. Wir werden wissen.

Outline

Books about the Riemann Hypothesis

- **Borwein, Peter, Stephen Choi, Brendan Rooney, Andrea Weirathmueller, 2008: The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike. Can. Math. Soc. Books. Springer.**
- **Darbyshire, John (2004): Prime Obsession. Plume Publishing.**
- **Du Sautoy, Marcus (2004): The Music of the Primes. Harper Perennial.**
- **Mazur, Barry and William Stein, 2016: Prime Numbers and the Riemann Hypothesis. Camb. Univ. Press.**