The Fractal Boundary for the Power Tower Function

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Outline

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The Power Tower Function

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or approximately

$$0.066 < x < 1.445$$

We call this function the *power tower function.*

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Thus, the tower is constructed ***downwards.***

It should really be denoted as

$$y(x) = {}^{\cdot^{\cdot^{\cdot}}} x^{x^x}$$

as each new x is adjoined to the ***bottom*** of the tower.

Up and Down Values for $x = 3$

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$$\left(3^3\right)^3 = 27^3 = 19{,}683 \qquad 3^{\left(3^3\right)} = 3^{27} = 7.6256 \times 10^{12}$$

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IT IS ESSENTIAL TO EVALUATE DOWNWARDS

MNEMONIC: Think of e^{x^2}

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Sample Values

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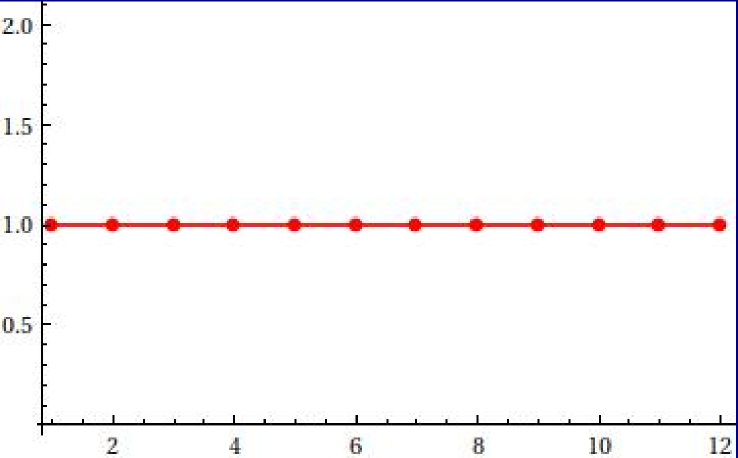
We will see that we may get

- Convergence to a finite value.
- Divergence to infinity.
- Oscillation between two or more values.
- More irregular (chaotic) behaviour (?).

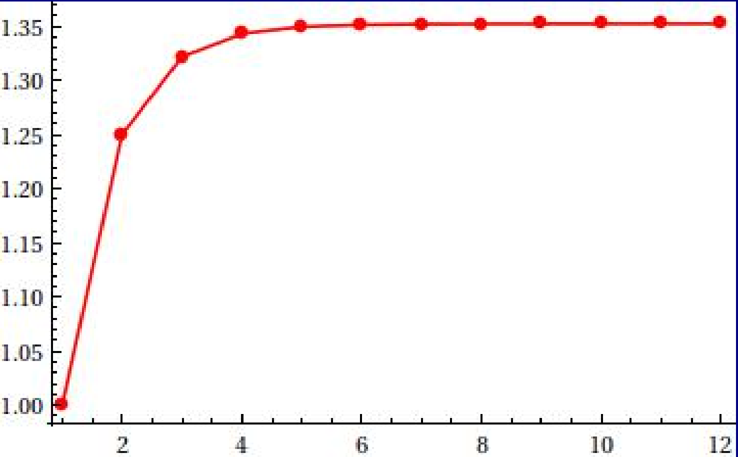
Sample Values: $x = 1$

For $x = 1$, every term in the sequence is equal to 1.



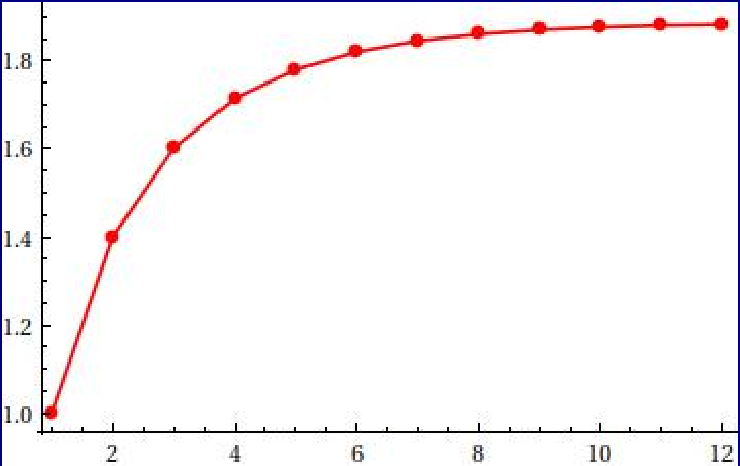
Sample Values: $x = 1\frac{1}{4}$

For $x = 1.25$, the values in the sequence grow:



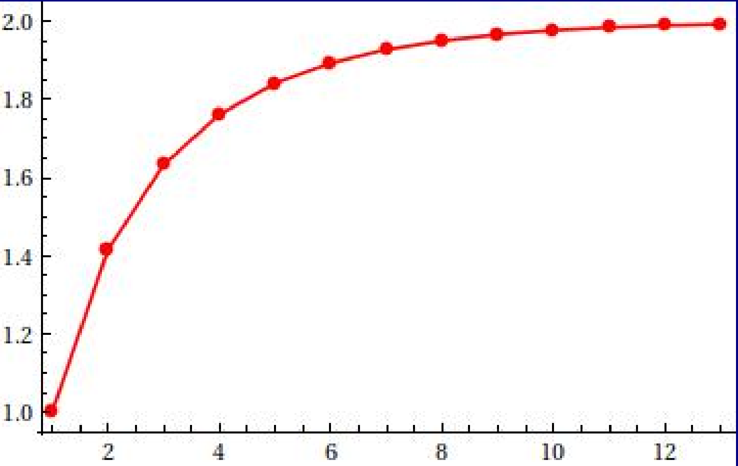
Sample Values: $x = 1\frac{2}{5}$

For $x = 1.40$, the values grow to a larger value:



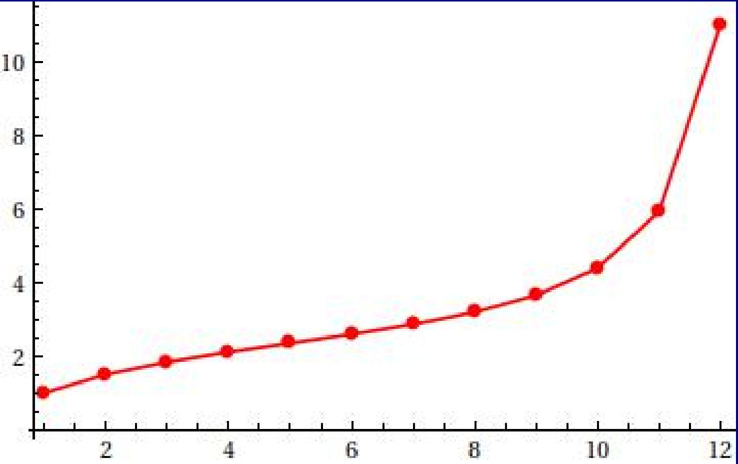
Sample Values: $x = \sqrt{2}$

For $x = \sqrt{2}$, the values grow to $y = 2$.



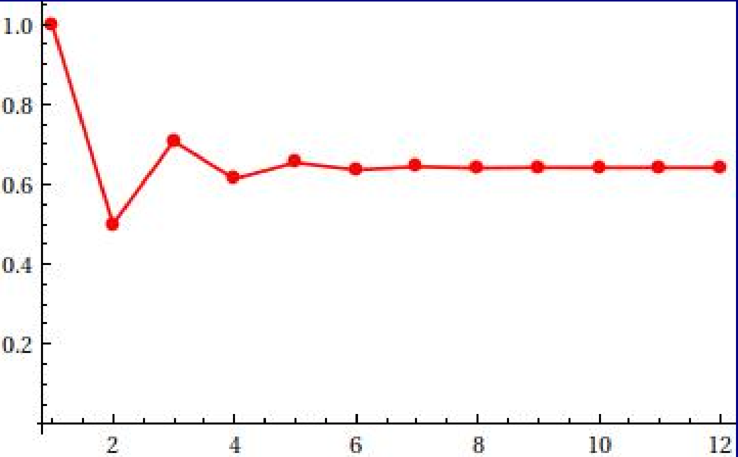
Sample Values: $x = 1\frac{1}{2}$

For $x = 1.5$, the terms appear to grow without limit.



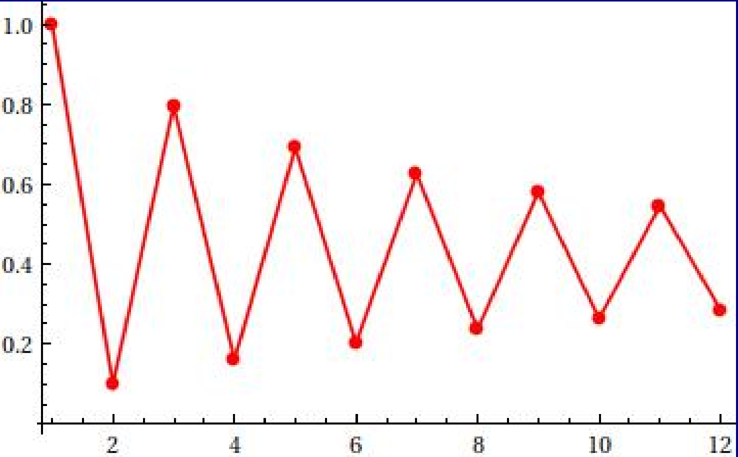
Sample Values: $x = \frac{1}{2}$

For $x = 0.5$, we see oscillating behaviour, converging.



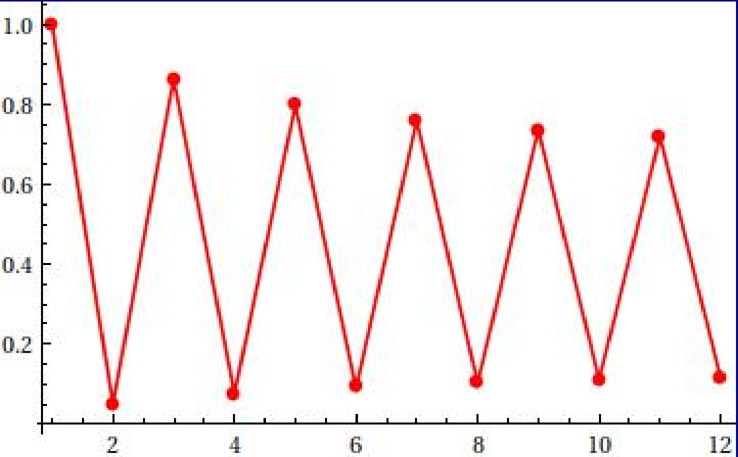
Sample Values: $x = \frac{1}{10}$

For $x = 0.1$, we again see oscillating behaviour



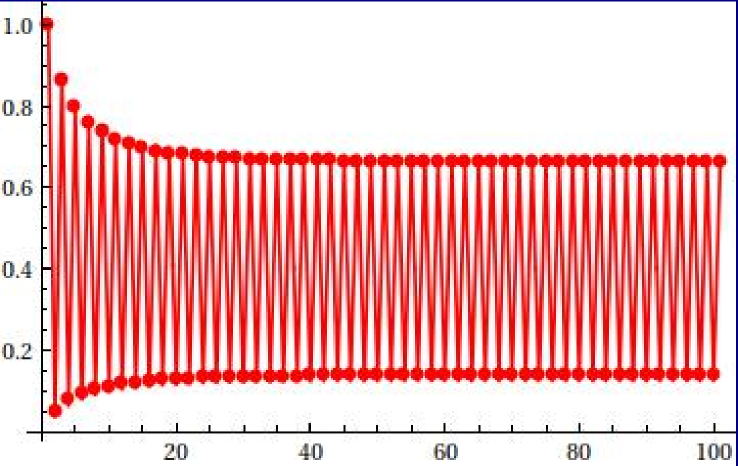
Sample Values: $x = \frac{1}{20}$

For $x = 0.05$, convergence is less obvious.



Sample Values: $x = \frac{1}{20}$

In fact, there is oscillation, no convergence.



Behaviour for Large and Small x

It is clear that

$$\lim_{x\to\infty} x^x = \infty$$

So for large x the power tower function diverges.

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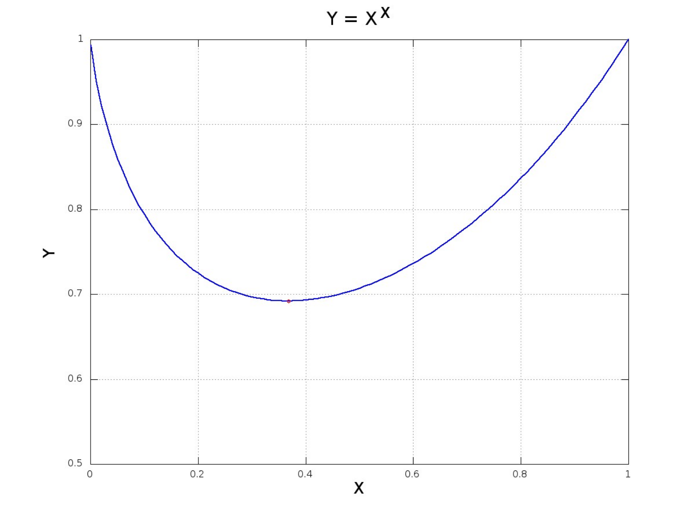
This accounts for the counter-intuitive behaviour of the power tower for very small x.

For small x, alternate terms are close to 0 and to 1, so the sequence oscillates and does not converge.

Behaviour for Small x



$y = x^x$ **for** $x \in [0, 1]$**. Minimum at** $x = 1/e \approx 0.368$

Outline

Iterative Process

If the power tower function is to have any meaning, we need to show that it has well-defined values.

We consider the iterative process

$$y_1 = x \qquad y_{n+1} = x^{y_n} .$$

This generates the infinite sequence

$$\{y_1, y_2, y_3, \dots\} = \{x, x^x, x^{x^x}, \dots\}$$

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$$\{y_1, y_2, y_3, \dots\} = \{x, x^x, x^{x^x}, \dots\}$$

If the sequence converges to $y = y(x)$, it follows that

$$y = x^y$$

But $y = x^y$ leads to an explicit expression for x:

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Taking the derivative of this function we get

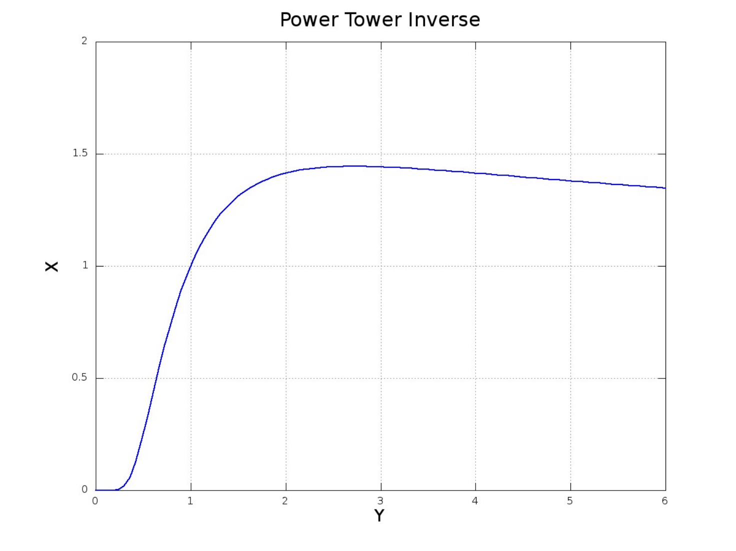
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{1 - \log y}{y^2}\right) x$$

which vanishes when $\log y = 1$ or $y = e$.
At this point, $x = \exp(1/e)$.

Moreover, it is easily shown that

$$\lim_{y \to 0} x = 0 \qquad \text{and} \qquad \lim_{y \to \infty} x = 1$$

Plot of $x = y^{1/y}$

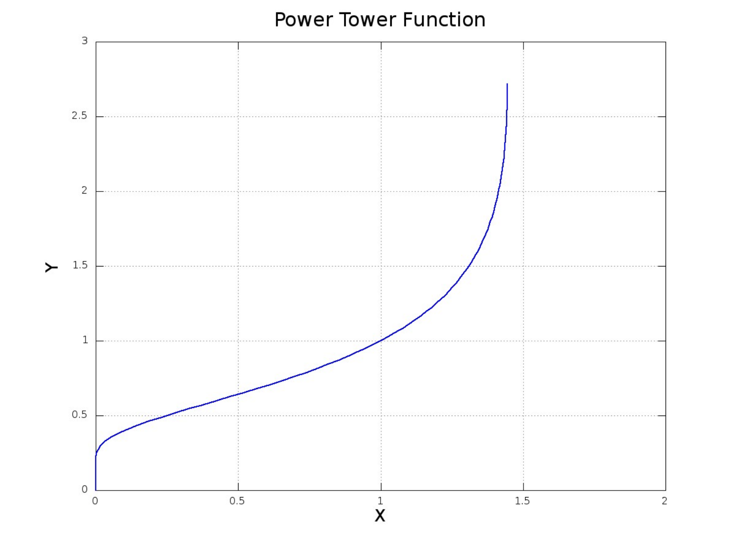


Plot of $x = y^{1/y}$

We plotted the function $x = y^{1/y}$ above.

- **It is defined for all positive y.**
- **Its derivative vanishes at $y = e$ where it takes its maximum value $\exp(1/e)$.**
- **It is monotone increasing on the interval $(0, e)$ and has an inverse function on this interval.**
- **This inverse is the power tower function:**

Power tower function for $x < \exp(1/e)$.



Iterative Solution

The logarithm of $y = x^y$ **gives** $\log y = y \log x$**.**

That is

$$y = \exp(y \log x) \qquad \textbf{or} \qquad y = \exp(\xi y)$$

where $\xi = \log x$**.**

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where $\xi = \log x$**.**

This is suited for iterative solution:
given a value of x **(or** ξ**), we seek a value** y **such that the graph of** $\exp(\xi y)$ **intersects the diagonal line** $y = y$**.**

Starting from some value $y_{(0)}$ we iterate:

$$y_{(1)} = \exp(\xi y_{(0)})\,, \quad \ldots \quad y_{(n+1)} = \exp(\xi y_{(n)})$$

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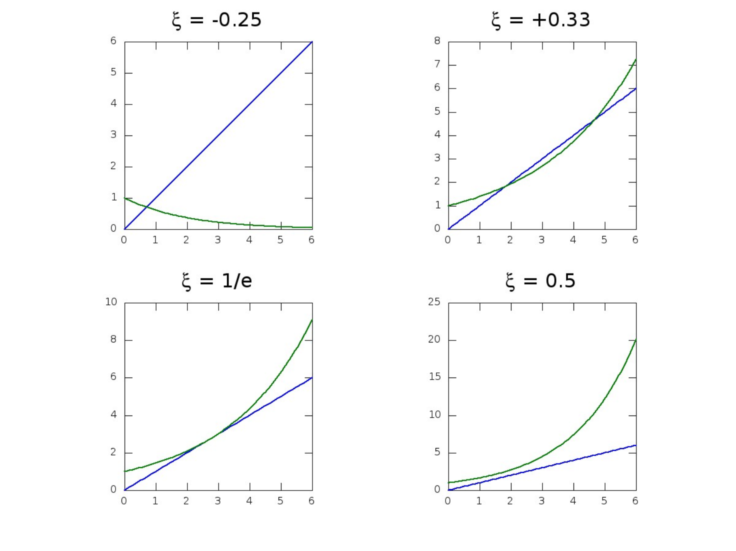
We graph $\exp(\xi y)$ **for selected of values of** ξ**.**

$$x \in [e^{-e}, e^{1/e}] \Longrightarrow \xi \in [-e, 1/e]$$

- **For $\xi < 0$, corresponding to $x < 1$, there is a single root (top left panel).**
- **For $0 < \xi < 1/e$ (that is, for $1 < x < e^{1/e}$), there are two roots (top right panel).**
- **For $\xi = 1/e$ ($x = e^{1/e}$), there is one double root (bottom left panel).**
- **Finally, for $\xi > 1/e$ ($x > e^{1/e}$), there are no roots (bottom right panel).**

Graphs of y & $\exp(\xi y)$ for some values ξ



Graphs of y & $\exp(\xi y)$ for some values ξ

We compute iterations of:

$$y_{(n+1)} = \exp(\xi y_{(n)})$$

The iterative method converges only if the derivative

$$\frac{\mathrm{d}}{\mathrm{d}y} \exp(\xi y) = \xi y$$

of the right side has modulus less than unity.

This criterion is satisfied for $-e < \xi < 0$, and also for the smaller of the two roots when $0 < \xi < 1/e$.

We therefore expect to obtain a single solution for $-e < \xi < 1/e$ or $\exp(-e) < x < \exp(1/e)$.

Outline

Swiss mathematician Johann Heinrich Lambert (1728–1777) introduced a function that is of wide value and importance.

The Lambert W-function is the inverse of $z = w\exp(w)$:

$$w = W(z) \qquad \Longleftrightarrow \qquad z = w\exp(w)\,.$$

A plot of $w = W(z)$ is presented below.

We confine attention to real values of $W(z)$, which means that $z \geq -1/e$.

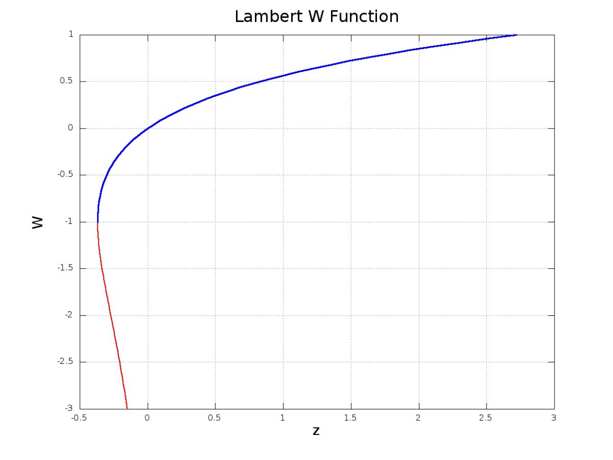


Figure : Lambert W-function $w = W(z)$. The inverse of $z = w\exp(w)$.

Applications of the W-Function

MATHEMATICS

- Transcendental equations.
- Solving differential equations.
- In combinatorics.
- Delay differential equations.
- Iterated exponentials.
- Asymptotics.

PHYSICS

- Analysis of algorithms.
- Water waves.
- Combustion problems.
- Population growth.
- Eigenstates of H_2 molecule.
- Quantum gravity.

Power Tower Function and W

For the Power Tower Function, x in terms of y is:

$$x = y^{1/y}$$

This is well defined for all positive y.

Its inverse has a branch point at $(x, y) = (e^{1/e}, e)$.

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Its inverse has a branch point at $(x, y) = (e^{1/e}, e)$.

If $\xi = \log x$ we have $y = \exp(\xi y)$. We can write

$$(-\xi y)\exp(-\xi y) = (-\xi)$$

We now define $z = -\xi$ and $w = -\xi y$ and have $z = w\exp(w)$. By the definition of the Lambert W-function, this is

$$w = W(z)$$

Returning to variables x and y, we conclude that

$$y = \frac{W(-\log x)}{-\log x}$$

which is the expression for the power tower function in terms of the Lambert W-function.

This enables analytical continuation of the power tower function to the complex plane.

The relationship between the power tower function and the Lambert W-function allows us to extend the power tower function to the complex plane.

The function has a logarithmic branch point at $x = 0$.

The behaviour of the different branches of the W-function are described in [Corless96].

Outline

We now examine the PTF for complex z.

Specifically, we look at the case $z = i$:

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Assuming the sequence $\{q_n\}$ converges to Q,

$$Q = i^Q$$

Again,

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Writing $Q = \varrho \exp(i\vartheta)$ it follows that

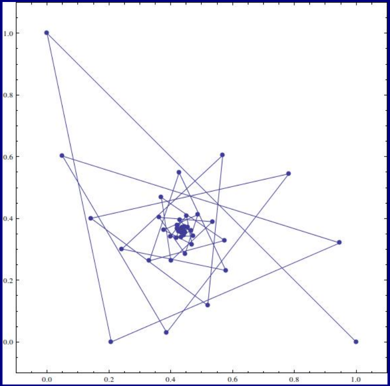
$$\vartheta \tan\vartheta = \log\left[\frac{\pi}{2}\frac{\cos\vartheta}{\vartheta}\right] \qquad \textbf{and} \qquad \varrho = \frac{2}{\pi}\frac{\vartheta}{\cos\vartheta}$$

This is easily solved to give

$$Q = (0.438283, 0.360592)$$

Here we show the sequence $\{q_n\}$.

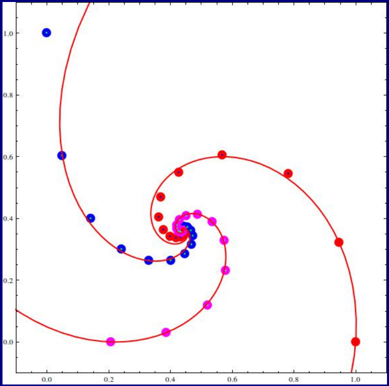
The points spiral around the limit point Q, converging towards it.



The points q_n fall into three distinct sets.

Three logarithmic spirals are superimposed on the plot.

Is this pattern accidental?



Outline

Asymptotic Behaviour

We fitted a logarithmic spiral to the sequence $\{z_n(i)\}$.

The points of the sequence were close to such a curve but did not lie exactly upon it.

Therefore, we looked at the asymptotic behaviour of the sequence for large n.

We consider the specific case $z = i$ and suppose that $z_n = (1+\epsilon)Z$ where ϵ is small.

Then we find that $z_{n+1} = Z^\epsilon \cdot Z$ so that

$$\left(\frac{z_{n+1} - Z}{z_n - Z}\right) = \left(\frac{Z^\epsilon - 1}{\epsilon}\right) .$$

By L'Hôpital's rule, the limit of the right-hand side as $\epsilon \to 0$ is $\log Z$. Thus for small ϵ (large n) we have

$$(z_{n+1} - Z) \approx \log Z \cdot (z_n - Z)$$

and the sequence of differences $\{z_{n+k} - Z\}$ lies approximately on a logarithmic spiral

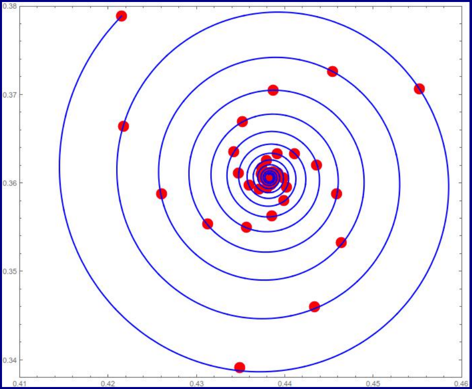
$$z_{n+k} \approx Z + (\log Z)^k \cdot (z_n - Z) .$$

Logarithmic Spiral

$\{z_n(i)\}$ **for** $n \geq 30$**.**

Points $z_n(i)$ **spiral around the limit point** $(0.438283, 0.360592)$

The logarithmic spiral gives an excellent fit.

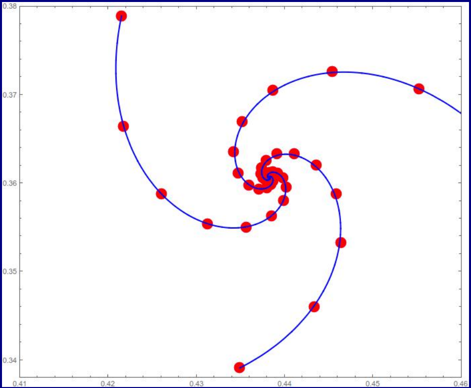


Supernumerary Spirals

Same sequence of points.

Points z_n fall into three sets.

Three logarithmic spirals superimposed.



The Asymptotic Spiral

The three “supernumerary spirals” are no accident.

Such spirals are familiar in many contexts.

In the seeds of a sun-flower, clockwise and anti-clockwise spirals are evident.

By changing the parameter z it is possible to tune the limit $Z(z)$ to have spirals of a particular shape.

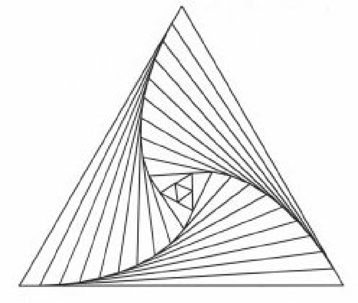
Patterns like this also found in pursuit problems.

A Pursuit Problem

Three ships initially at the vertices of an equilateral triangle.

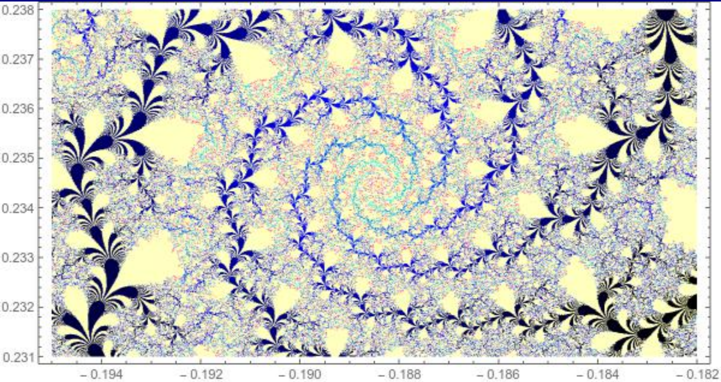
Each bears towards its counter-clockwise neighbour.

Three spiral arms are traced out.



Outline

We can construct a beautiful fractal set using the Power Tower Function with complex arguments.



Repeated exponentiation is called tetration and the fractal is sometimes called the tetration fractal.

We examine the behaviour of the (tetration) function

$$^{\infty}z = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$$

- For some values of z this converges.
- For other values it is periodic.
- For others, it “escapes” to infinity.

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- For others, it “escapes” to infinity.

The boundary of the region for which the function is finite is fractal. Let Π be the set for which ${}^{\infty}z$ is finite.

The “escape set” is the complement of this set.

The boundary of the set Π is exquisitely complex.

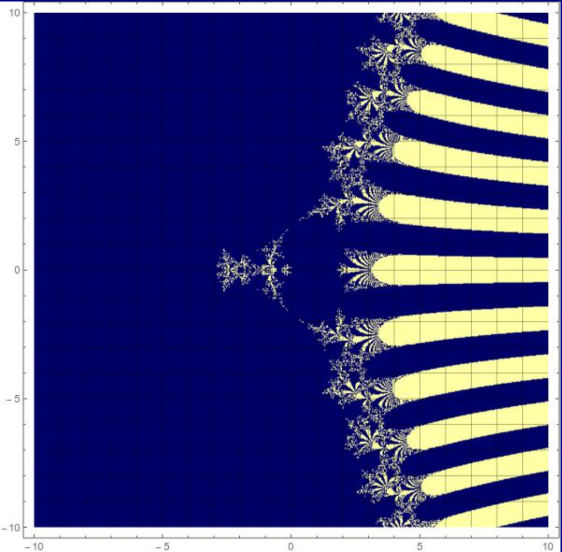

Figure : The power tower fractal for $|x| < 10, |y| < 10$.

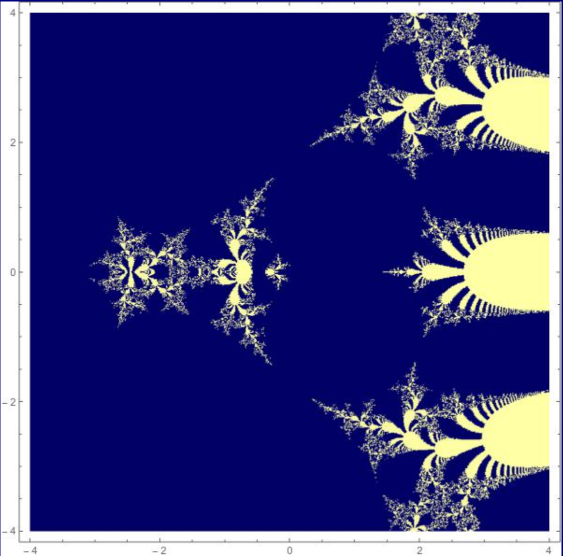

Figure : The power tower fractal for $|x| < 4, |y| < 4$.

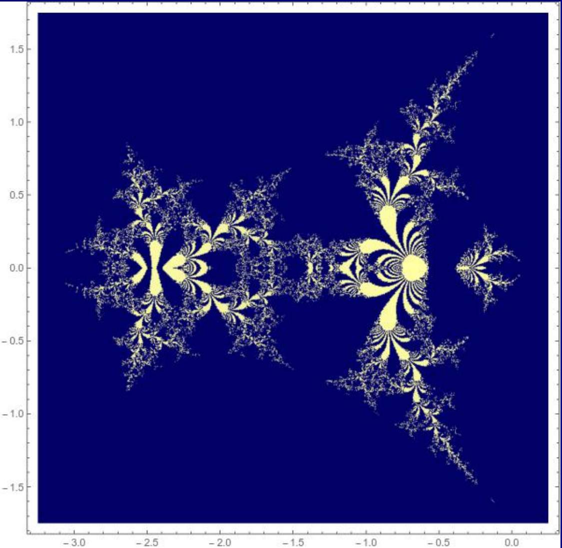

Figure : PTF for $-3.25 \leq x \leq 0.25$, $-1.75 \leq y \leq 1.75$.

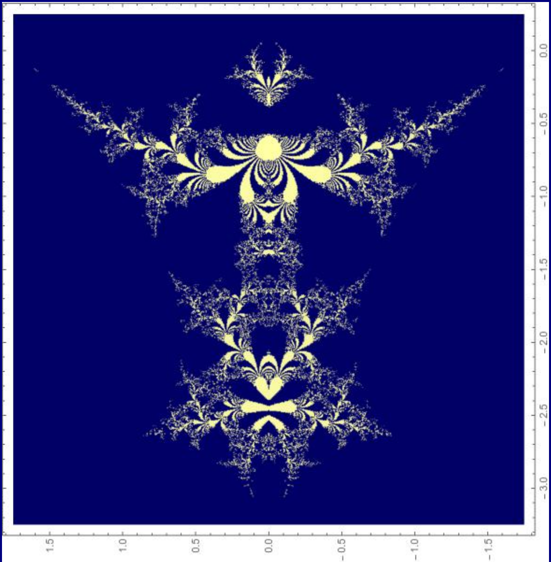

Figure : PTF for $-3.25 \leq x \leq 0.25$, $-1.75 \leq y \leq 1.75$.

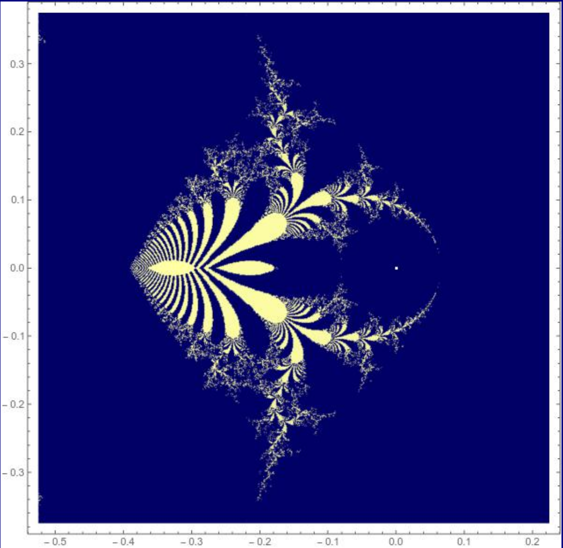

Figure : PTF for $-0.525 \leq x \leq 0.225$, $-0.375 \leq y \leq 0.375$.

A marine creature. Let's call it the lobster.

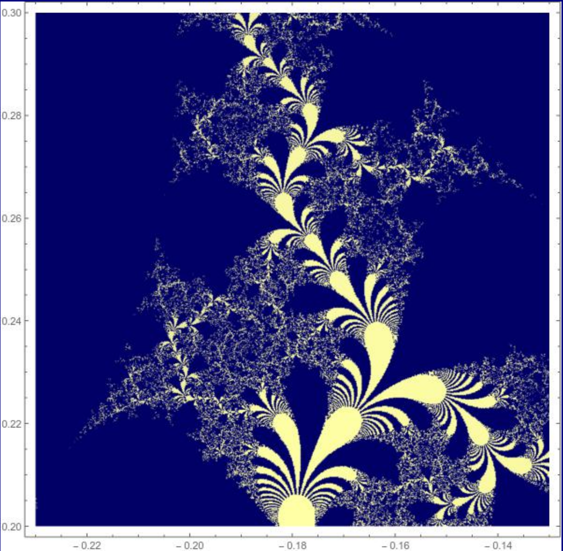

Figure : PTF for $-0.23 \leq x \leq -0.13$, $+0.2 \leq y \leq 0.3$.

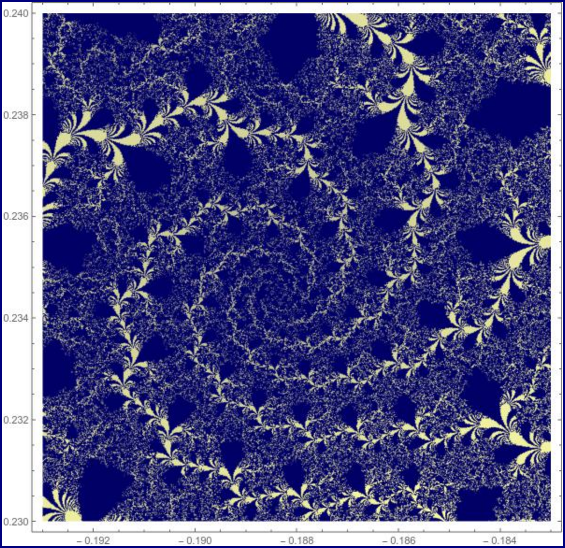

Figure : PTF for $-0.193 \leq x \leq -0.183$, $+0.23 \leq y \leq 0.24$.

Spiral structure in the antenna.

Images from Website of Paul Bourke

http://paulbourke.net/fractals/tetration/

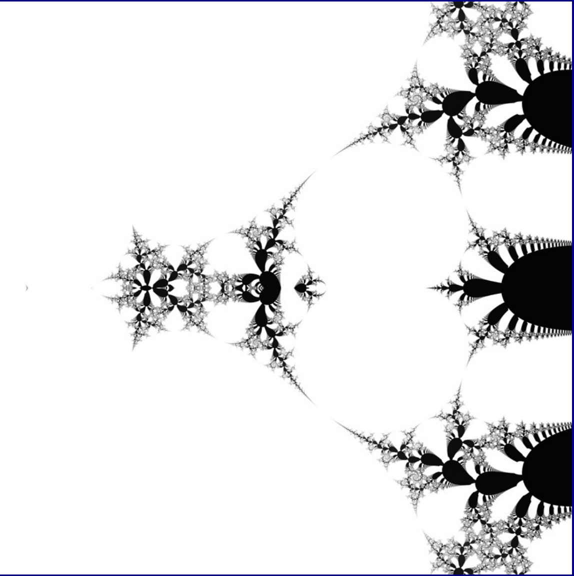

Figure : Center = (-0.5,0.0), range = 9.0

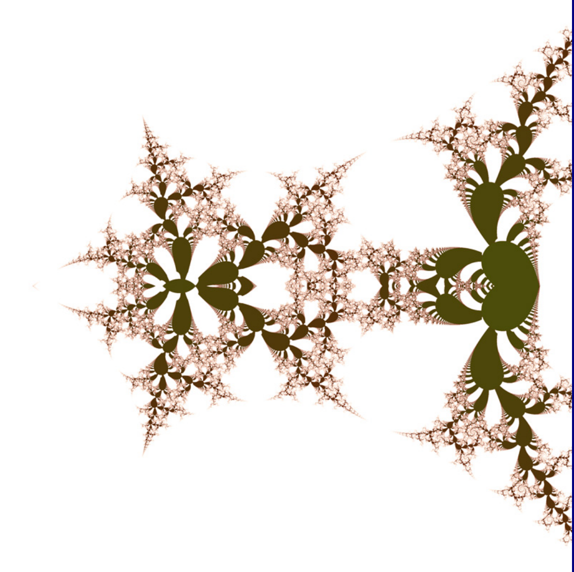

Figure : Center = (-1.9,0.0), range = 3.0

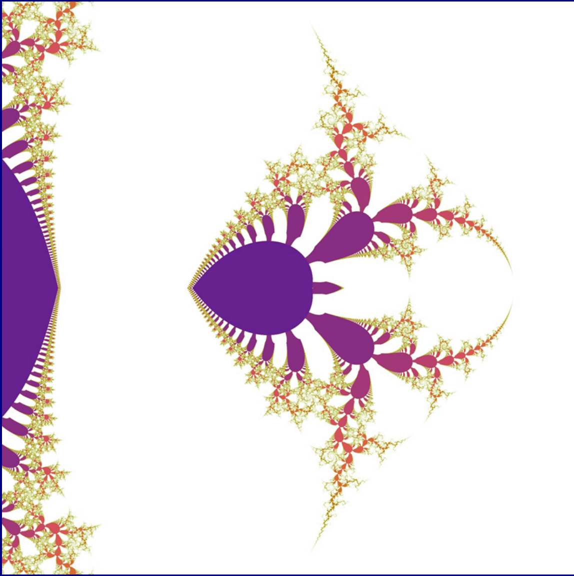

Figure : Center = (-0.25,0.0), range = 0.8

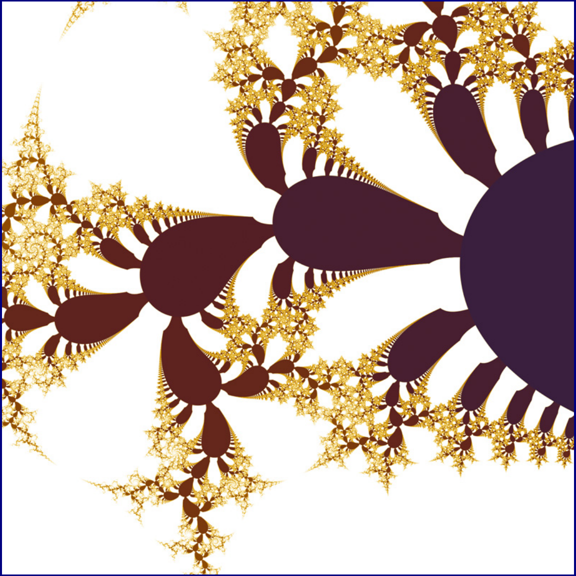

Figure : Center = (2.2,-2.5), range = 2.0

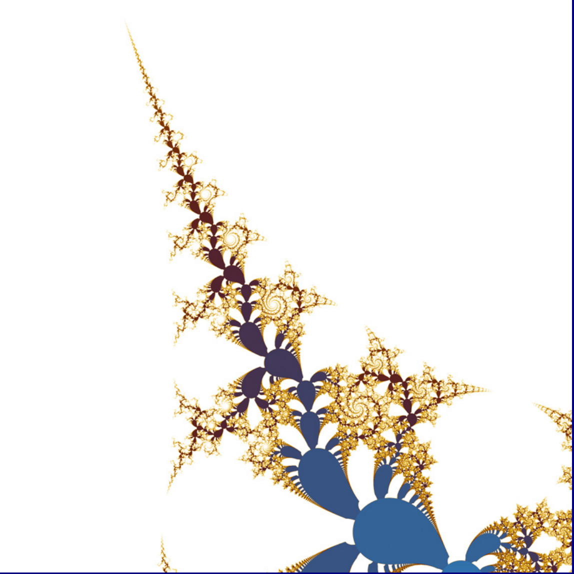

Figure : Center = (2.15,-0.91), range = 0.5

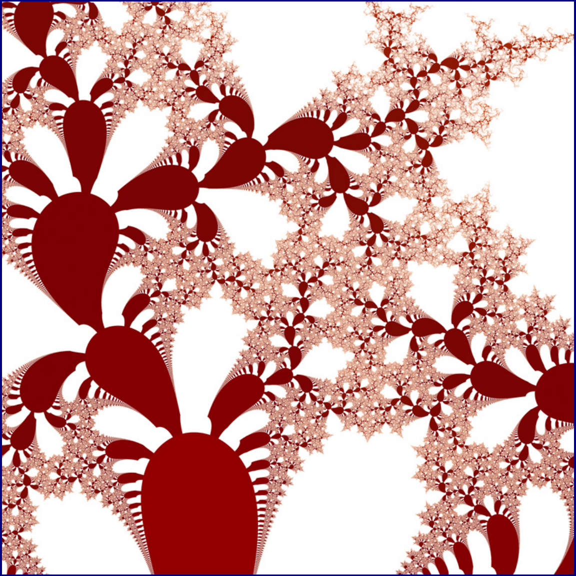

Figure : Center = (-2.37,-0.38), range = 0.5

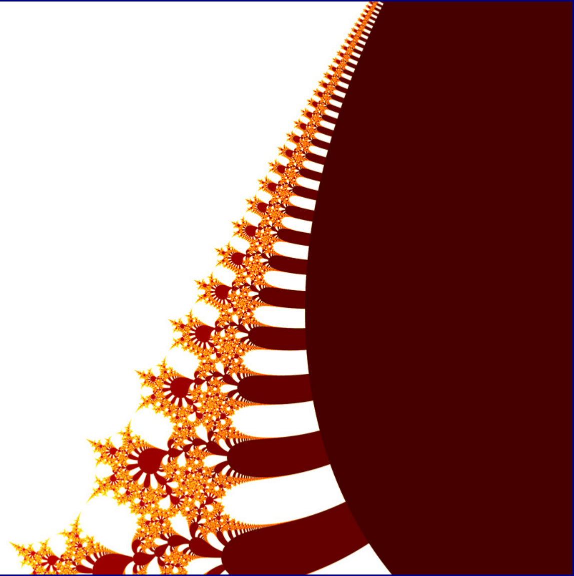

Figure : Center = (-0.94,0.41), range = 0.2

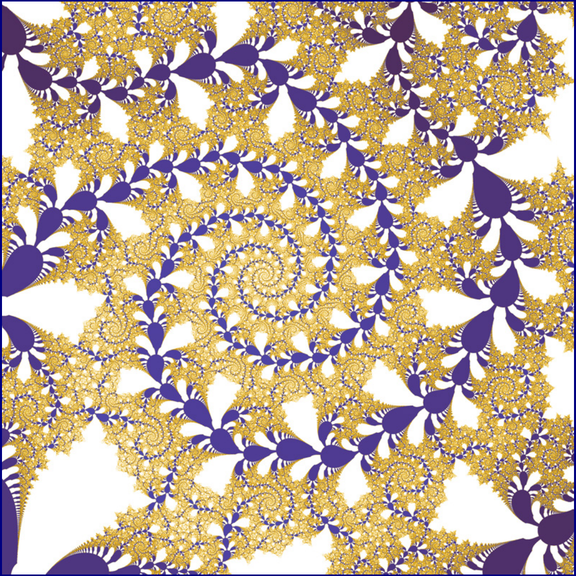

Figure : Center = (-0.95,2.4), range = 0.1

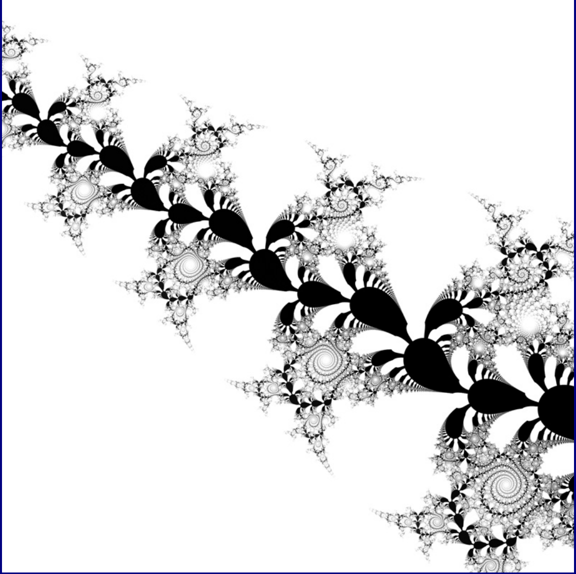

Figure : Center = (0.4,2.0), range = 0.2

Outline

Conclusion

Zooming in can be continued indefinitely, revealing ever more sturcture.

The fine details at any resolution are not reliable.

Structures that appear to be disjoint may be connected by fine filaments that are visible only at higher resolution.

It is necessary to set the escape radius to a very large value (e.g. $r_{max} = 10^{48}$) and allow many iterations.

Much more may be said about the power tower fractal.

Fixed points, for which ${}^{\infty}z = z$. Clearly, $z = 1$ and $z = -1$ are fixed points.

Periodic orbits (see `http://www.tetration.org/`)

Sarkovskii's Theorem implies that a map containing period three must contain all periods from one to infinity.

Many other interesting questions to be answered.

Thank you