Pedro Nunes and the Retrogression of the Sun

Peter Lynch
School of Mathematics & Statistics
University College Dublin

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Outline

Introduction

Pedro Nunes

Analysis of Solar Retrogression

Variation of the Azimuthal Angle

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Background

How I learned about this question.

- Recreational Maths Conference in Lisbon.
- Henriques Leitão gave a talk on Pedro Nunes.
- Claim: The Sun sometimes reverses direction.
- My reaction was one of scepticism.
- Initially, I could not prove the result.
- Later I managed to prove retrogression occurs.
- I hope that I can convince you of this, too.
Things We All Know Well

- The Sun rises in the Eastern sky.
- It follows a smooth and even course.
- It sets in the Western sky.

The idea that the compass bearing of the Sun might reverse seems fanciful.

But that was precisely what Portuguese mathematician Pedro Nunes showed in 1537.
Obelisk serving as a Gnomon

Path of Sun traces a hyperbola
Nunes made an amazing prediction:
In certain circumstances, the shadow cast by the gnomon of a sun dial moves backwards.

Nunes’ prediction was counter-intuitive: We expect the azimuthal angle to increase steadily.

If the shadow on the sun dial moves backwards, the Sun must reverse direction or retrogress.

Nunes’ discovery came long before Newton or Galileo or Kepler, and Copernicus had not yet published his heliocentric theory.
The retrogression had never been seen by anyone and it was a remarkable example of the power of mathematics to predict physical behaviour.

Nunes himself had not seen the effect, nor had any of the tropical navigators or explorers whom he asked.

Nunes was aware of the link between solar regression and the biblical episode of the sun dial of Ahaz (Isaiah 38:7–9). However, what he predicted was a natural phenomenon, requiring no miracle.

It was several centuries before anyone claimed to have observed the reversal (Leitão, 2017).
In a book published in Lisbon in 1537, Nunes showed how, under certain circumstances, the azimuth of the Sun changes direction twice during the day, moving first forwards, then backwards and finally forwards again.

To witness this, the observer must be located at a latitude lower than that of the Sun, that is, in the tropics with the Sun closer to the pole.

Nunes was completely confident about his prediction:

“This is something surprising but it cannot be denied because it is demonstrated with mathematical certainty and evidence.”

(Quoted from Leitão, 2017).
Figure: Pedro Nunes on a Portuguese postage stamp (1978).
Figure: Pedro Nunes book (1537) and Leitão’s interpretation.
Leitão, who has made a detailed study of Nunes’ works, reviewed the method used by him.

While Nunes’ arguments are mathematically sound, they are difficult to follow, so we will demonstrate the retrogression in a more transparent way.

But first, let us look at Pedro Nunes himself.
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Pedro Nunes (1502–1578)

Pedro Nunes (also known as Petrus Nonius), a Portuguese cosmographer and one of the greatest mathematicians of his time, is best known for his contributions to navigation and to cartography.

- Studied at University of Salamanca.
- Returned to Lisbon, where he taught.
- Later Professor of Mathematics, Univ. of Coimbra.
- 1533: Qualified as a doctor of medicine.
- 1547: Appointed Chief Royal Cosmographer.
Nunes had great skill in spherical trigonometry.

He introduced improvements to the Ptolemaic system of astronomy, which was still current at that time.

Copernicus did not publish his theory until just before his death in 1543.

Nunes also worked on problems in mechanics.
Much of Nunes’ research was in the area of navigation, a subject of great importance in Portugal during that period.

Sea trade was the main source of Portuguese wealth.

Nunes understood how a ship sailing on a fixed compass bearing would not follow a great circle route but a spiraling course called a loxodrome or rhumb line that winds in decreasing loops towards the pole.

Nunes taught navigation skills to some of the great Portuguese explorers.
Loxodrome Curve

Image from Wikimedia Commons
Nunes has a place of prominence on the Monument to the Portuguese Discoveries in Lisbon, which shows several famous navigators.
Figure: Pedro Nunes (1502–1578).
My name is Pedro Nunes: I was the most important mathematician in the history of Portugal.
“I was a very famous mathematician during my life, and some say that I was the most important Portuguese mathematician of all time.”

“We mathematicians are people just like everyone else. The only difference is that we like Mathematics a lot.”
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Nunes demonstrated the retrogression using **spherical trigonometry**.

We will derive a condition for retrogression using a simple transformation and elementary differential calculus.

An expression is found for the azimuth of the Sun as a function of the time.

For reversal to occur, the derivative of this function must vanish.

The condition follows immediately from this.
Frames of Reference

Cartesian frame \((x, y, z)\)
with \(x\)-axis through \((0^\circ, 0^\circ)\).

Origin at centre of Earth.

Frame rotating with Earth.

Polar frame \((r, \theta, \lambda)\).

Latitude is \(\phi = \frac{\pi}{2} - \theta\).
Frames of Reference

Assume Sun is at fixed latitude $\phi_S$.

If its longitude at Noon is $\lambda_O$, then

$$\lambda_S(t) = \lambda_O - \Omega(t - t_O)$$

where $\Omega$ is the angular velocity of Earth.

Given the distance $A$ from Earth to Sun, the cartesian coordinates of the Sun are

$$(x_S, y_S, z_S) = (A \cos \lambda_S \cos \phi_S, A \sin \lambda_S \cos \phi_S, A \sin \phi_S)$$.
The observation point $P_O$ is at $(x_O, y_O, z_O)$.

The polar coordinates are easily found: $(a, \theta_O, \lambda_O)$.

No loss of generality in assuming $\lambda_O = 0$.

Then the latitude and longitude of $P_O$ are

$$(\phi_O, \lambda_O) = \left(\frac{\pi}{2} - \theta_O, 0\right).$$
We define local cartesian coordinates \((X, Y, Z)\) at the observation point by rotating the \((x, y, z)\) frame about the \(y\)-axis through an angle equal to the colatitude \(\theta_O\).

The \(Z\)-axis then points vertically upward through \(PO\).

Moving the origin to \(PO\), the \((X, Y)\) plane is tangent to the Earth at this point.
The cartesian coordinates of the Sun in the new system are given by the affine transformation

\[
\begin{pmatrix}
X_S \\
Y_S \\
Z_S
\end{pmatrix} =
\begin{bmatrix}
\cos \theta_O & 0 & -\sin \theta_O \\
0 & 1 & 0 \\
\sin \theta_O & 0 & \cos \theta_O
\end{bmatrix}
\begin{pmatrix}
x_S \\
y_S \\
z_S
\end{pmatrix} -
\begin{pmatrix}
0 \\
0 \\
a
\end{pmatrix}
\]

Since $A \gg a$, we can omit $(0, 0, a)^T$. Then

\[
\begin{pmatrix}
X_S \\
Y_S \\
Z_S
\end{pmatrix} =
\begin{bmatrix}
\sin \phi_O & 0 & -\cos \phi_O \\
0 & 1 & 0 \\
\cos \phi_O & 0 & \sin \phi_O
\end{bmatrix}
\begin{pmatrix}
x_S \\
y_S \\
z_S
\end{pmatrix}
\]

The latitude and longitude of the Sun in the rotated system are

\[ \Phi_S = \arcsin\left(\frac{Z_S}{A}\right) \]
\[ \Lambda_S = \arctan\left(\frac{Y_S}{X_S}\right) \]

The azimuth and elevation (or altitude) are

\[ \alpha = \pi - \Lambda_S \]
\[ e = \Phi_S \]
Summary of Computations

Azimuth and elevation are \((\alpha, \ e)\).

We convert these to latitude and longitude \((\phi_s, \lambda_s)\) in the local frame.

We then get the cartesian coordinates \((X_s, \ Y_s, \ Z_s)\) in the local frame.

Then we transform to the original cartesian coordinates \((x, \ y, \ z)\).

Finally, we express \((\alpha, \ e)\) in terms of the geographic variables \(\{\lambda_s, \phi_s, \phi_o\}\).
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If the Sun is to retrogress, the time derivative of the azimuth $\Lambda_S$ must vanish.

$$\tan \Lambda_S = \frac{Y_S}{X_S} = \frac{sin \lambda_S \cos \phi_S}{cos \lambda_S \cos \phi_S \sin \phi_O - \sin \phi_S \cos \phi_O}$$

The vanishing of the derivative leads, after some manipulation, to the equation

$$\cos \lambda_S = \frac{\tan \phi_O}{\tan \phi_S}$$

This gives the point of retrogression $\lambda_S$ in terms of the solar latitude $\phi_S$ and observation latitude $\phi_O$. 
Again,

\[
\cos \lambda_S = \frac{\tan \phi_O}{\tan \phi_S}
\]

The derivative vanishes only if the right hand side is less than unity:

\[\phi_O < \phi_S.\]

Retrogression will be seen only if the observation point is between the Equator and the Sun's latitude.

In particular, it must be in the tropics.
Assume it is the Summer solstice: $\phi_S = 23.5^\circ \text{N}$.

We consider observations at ($\phi_O = 40^\circ \text{N}$) and within the tropics ($\phi_O = 20^\circ \text{N}$).

We plot the zenith angle $(\zeta = 90^\circ - e)$ versus azimuth.

The observation point is at the centre, and the course of the Sun is shown by a curve.
Extratropical Observation Point: $\phi_O = 40^\circ$

Figure: Path of the Sun for observation point at $40^\circ$N.
Tropical Observation Point: $\phi_O = 20^\circ$

Figure: Path of the Sun for observation point at $20^\circ$N.
Azimuth and Elevation: $\phi_O = 40^\circ$

Figure: Solar elevation and azimuth for observation at $40^\circ$N.
Azimuth and Elevation: $\phi_O = 20^\circ$

**Figure**: Solar elevation and azimuth for observation at $20^\circ$N.
Azimuth is $65^\circ$ at sunrise, at maximum $77^\circ$ by mid-morning and decreases to zero at Noon.
Another Angle on the Azimuth

Figure: Elevation versus Azimuth for observer at 40°.
Another Angle on the Azimuth

Figure: Elevation versus Azimuth for observer at $20^\circ$. 
Confirming the Analysis

For the specific values \( \phi_O = 20^\circ \) and \( \phi_S = 23.5^\circ \),

\[
\cos \lambda_S = \frac{\tan \phi_O}{\tan \phi_S}
\]

gives the turning longitude as \( \lambda_S = 33.17^\circ \).

This corresponds to an azimuth of \( 77.4^\circ \)
and an elevation of \( 59.1^\circ \).

This is in excellent agreement with the numerical solution shown in the figure above.
Initial and Maximum values of azimuth: 64.9273  77.3993
Maximum and minimum values of elevation: 86.5  0.0900745
Maximum and minimum values of zenith angle: 89.9099  3.5

Hour 5.4 9.8 12.
Azim 64.9273  77.3993  0.
Elev 0.0900745  59.2166  86.5
Zenith 89.9099  30.7834  3.5

Azimuth at turn: 77.3995
Elevation at turn: 59.063
What has been will be again, what has been done will be done again; there is nothing new under the Sun.

Ecclesiastes 1:9
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• Morrison, J, 1898: The sun dial of Ahaz. Popular Astronomy, 6 (10), 537–549.


Thank You