Pedro Nunes and the Retrogression of the Sun

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Outline

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Background

How I learned about this question.

- Recreational Maths Conference in Lisbon.
- Henriques Leitão gave a talk on Pedro Nunes.
- Claim: The Sun sometimes reverses direction.
- My reaction was one of scepticism.
- Initially, I could not prove the result.
- Later I managed to prove retrogression occurs.
- I hope that I can convince you of this, too.

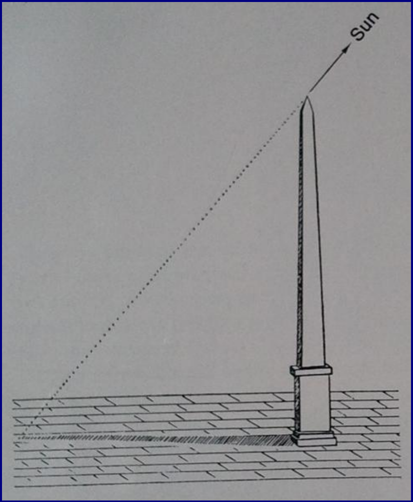
Things We All Know Well

- The Sun rises in the Eastern sky.
- It follows a smooth and even course.
- It sets in the Western sky.

The idea that the compass bearing of the Sun might reverse seems fanciful.

But that was precisely what Portuguese mathematician Pedro Nunes showed in 1537.

Obelisk serving as a Gnomon



Path of Sun traces a hyperbola

Nunes made an amazing prediction:

In certain circumstances, the shadow cast by the gnomon of a sun dial moves backwards.

Nunes' prediction was counter-intuitive:
We expect the azimuthal angle to increase steadily.

If the shadow on the sun dial moves backwards, the Sun must reverse direction or retrogress.

Nunes' discovery came long before Newton or Galileo or Kepler, and Copernicus had not yet published his heliocentric theory.

Mathematical Prediction

The retrogression had never been seen by anyone and it was a remarkable example of the power of mathematics to predict physical behaviour.

Nunes himself had not seen the effect, nor had any of the tropical navigators or explorers whom he asked.

Nunes was aware of the link between solar regression and the biblical episode of the sun dial of Ahaz (Isaiah 38:7–9). However, what he predicted was a natural phenomenon, requiring no miracle.

It was several centuries before anyone claimed to have observed the reversal (Leitão, 2017).

In a book published in Lisbon in 1537, Nunes showed how, under certain circumstances, the azimuth of the Sun changes direction twice during the day, moving first forwards, then backwards and finally forwards again.

To witness this, the observer must be located at a latitude lower than that of the Sun, that is, in the tropics with the Sun closer to the pole.

Nunes was completely confident about his prediction:

> *"This is something surprising but it cannot be denied because it is demonstrated with mathematical certainty and evidence."*
> *(Quoted from Leitão, 2017).*

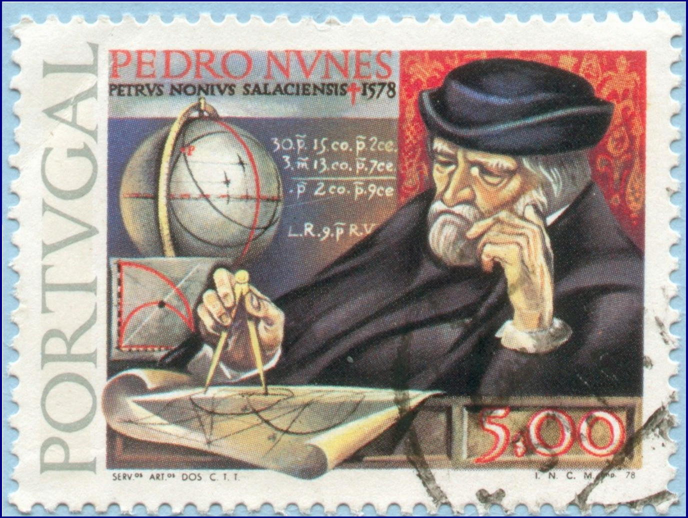
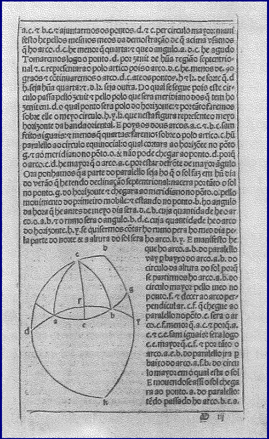


Figure : Pedro Nunes on a Portuguese postage stamp (1978).



a.c. ꝛ b.c. ꝛ ajuntarmos os pontos. d. ꝛ c. per circulo mayor: manifesto he pellos mesmos meos da demostração de q̄ acima usamos q̄ ho arco. d.c. he menor q̄ quarta: ꝛ que o angulo. a. d. c. he agudo Tomaremos logo o ponto. d. por zenit de hũa regiã septentrional ꝛ. c. representara o polo artico pois o arco. d.c. he menos de. 90 graos ꝛ cõtinuaremos o arco. d.c. ate os pontos. h ꝛ h. de sorte q̄. d. h. seja hũa quarta: ꝛ. d. h. seja outra. Do qual se segue pois este circulo passa pello zenit ꝛ pello polo que sera meridiano dos q̄ tem ho zenit em. d. o qual ponto sera polo do horizonte: ꝛ portanto faremos sobre elle o meyo circulo. h. y. h. que nesta figura representa o meyo horizonte da banda oriental. E poys os dous arcos. a. c. ꝛ. b. c. sam feitos iguaes: ꝛ menos q̄ quartas: faremos sobre o polo artico. c. hũ parallelo ao circulo equinocial: o qual cortara ao horizõte no põto g. ꝛ ao meridiano no põto. o. & não pode chegar ao ponto. d. porq̄ o arco. c. d. he mayor q̄ o arco. a. c. por estar defrõte de mayor angulo Ora ponhamos q̄ a parte do parallelo seja ho q̄ o sol faz em hũ dia do verão q̄ he tendo declinação septentrional: nacera portãto o sol no ponto. g. do horizonte: ꝛ chegara ao meridiano no põto. o. pello mouimento do primeiro mobile: ꝛ estando no ponto. b. ho angulo das horas q̄ he antes de meyo dia sera. d. c. b. cuja quantidade he o arco. o. a. b. ꝛ o rumo sera o angulo. b. d. c. cuja quantidade he o arco do horizonte. h. y. se quisermos cõtar ho rumo pera ho meo dia pella parte do norte & a altura do sol sera ho arco. b. y. E manifesto he que ho arco. a. b. do parallelo vay q̄ baixo do arco. a. b. do circulo da altura do sol porq̄ se partirmos ho arco. a. b. do circulo mayor pello meo no ponto. f. ꝛ deceremos ao arco perpendicular. c. f. q̄ chegue ao parallelo no põto. e. sera o arco. c. f. menor q̄. a. c. ꝛ porq̄. a. c. ꝛ c. e. sam iguais: sera logo c. e. mayor q̄. c. f. ꝛ por tãto o arco. a. e. b. do parallelo ira p baixo do arco. a. f. b. do circulo mayor em o qual esta o sol E mouendo se assi o sol chegara ao ponto. a. do parallelo: tẽdo passado ho arco. b. e. a.

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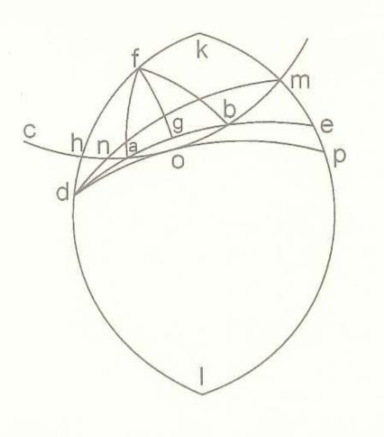


Figure : Pedro Nunes book (1537) and Leitão's interpretation.

Leitão, who has made a detailed study of Nunes' works, reviewed the method used by him.

While Nunes' arguments are mathematically sound, they are difficult to follow, so we will demonstrate the retrogression in a more transparent way.

But first, let us look at Pedro Nunes himself.

Outline

Pedro Nunes (1502–1578)

Pedro Nunes (also known as Petrus Nonius), a Portuguese cosmographer and one of the greatest mathematicians of his time, is best known for his contributions to navigation and to cartography.

- Studied at University of Salamanca.
- Returned to Lisbon, where he taught.
- Later Professor of Mathematics, Univ. of Coimbra.
- 1533: Qualified as a doctor of medicine.
- 1547: Appointed Chief Royal Cosmographer.

Nunes had great skill in spherical trigonometry.

He introduced improvements to the Ptolemaic system of astronomy, which was still current at that time.

Copernicus did not publish his theory until just before his death in 1543.

Nunes also worked on problems in mechanics.

Navigation

Much of Nunes' research was in the area of navigation, a subject of great importance in Portugal during that period.

Sea trade was the main source of Portuguese wealth.

Nunes understood how a ship sailing on a fixed compass bearing would not follow a great circle route but a spiraling course called a loxodrome or rhumb line that winds in decreasing loops towards the pole.

Nunes taught navigation skills to some of the great Portuguese explorers.

Loxodrome Curve

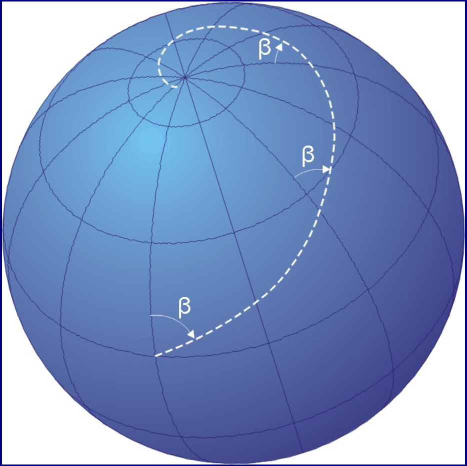


Image from Wikimedia Commons

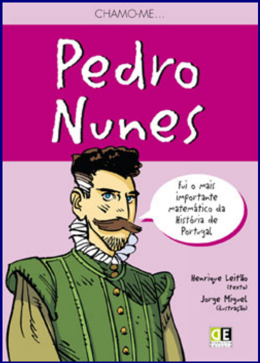
Monument to the Portuguese Discoveries

Nunes has a place of prominence on the Monument to the Portuguese Discoveries in Lisbon, which shows several famous navigators.

Figure : Pedro Nunes (1502–1578).

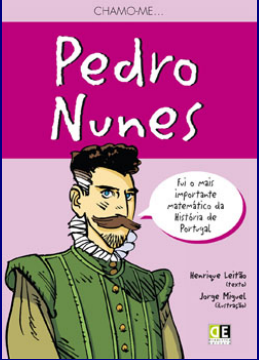
Chamo-me Pedro Nunes

Fui o mais importante matemático da história de Portugal.



My name is Pedro Nunes: I was the most important mathematician in the history of Portugal.

“I was a very famous mathematician during my life, and some say that I was the most important Portuguese mathematician of all time.”



“We mathematicians are people just like everyone else. The only difference is that we like Mathematics a lot.”

Outline

Nunes demonstrated the retrogression using spherical trigonometry.

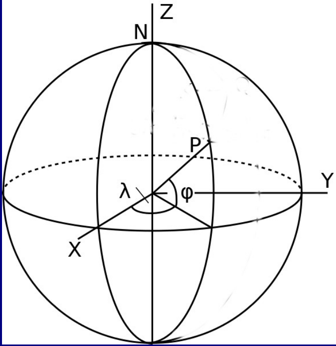
We will derive a condition for retrogression using a simple transformation and elementary differential calculus.

An expression is found for the azimuth of the Sun as a function of the time.

For reversal to occur, the derivative of this function must vanish.

The condition follows immediately from this.

Frames of Reference



Cartesian frame (x, y, z) with x-axis through $(0^\circ, 0^\circ)$.

Origin at centre of Earth.

Frame rotating with Earth.

Polar frame (r, θ, λ).

Latitude is $\phi = \frac{\pi}{2} - \theta$.

Frames of Reference

Assume Sun is at fixed latitude ϕ_S.

If its longitude at Noon is λ_O, then

$$\lambda_S(t) = \lambda_O - \Omega(t - t_O)$$

where Ω is the angular velocity of Earth.

Given the distance A from Earth to Sun, the cartesian coordinates of the Sun are

$$(x_S, y_S, z_S) = (A \cos\lambda_S \cos\phi_S, A \sin\lambda_S \cos\phi_S, A \sin\phi_S)\,.$$

The observation point P_O is at (x_O, y_O, z_O).

The polar coordinates are easily found: $(a,\ \theta_O, \lambda_O)$.

No loss of generality in assuming $\lambda_O = 0$.

Then the latitude and longitude of P_O are

$$(\phi_O, \lambda_O) = (\frac{\pi}{2} - \theta_O, 0)\,.$$

We define local cartesian coordinates (X, Y, Z) at the observation point by rotating the (x, y, z) frame about the y-axis through an angle equal to the colatitude θ_O.

The Z-axis then points vertically upward through P_O.

Moving the origin to P_O, the (X, Y) plane is tangent to the Earth at this point.

The cartesian coordinates of the Sun in the new system are given by the **affine transformation**

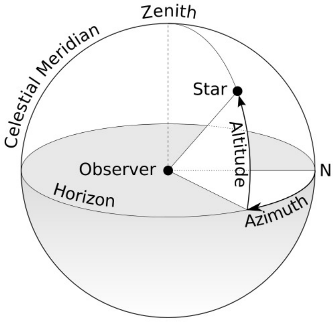
$$\begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} = \begin{bmatrix} \cos\theta_O & 0 & -\sin\theta_O \\ 0 & 1 & 0 \\ \sin\theta_O & 0 & \cos\theta_O \end{bmatrix} \begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

Since $A \gg a$, we can omit $(0, 0, a)^{\mathrm{T}}$. Then

$$\begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} = \begin{bmatrix} \sin\phi_O & 0 & -\cos\phi_O \\ 0 & 1 & 0 \\ \cos\phi_O & 0 & \sin\phi_O \end{bmatrix} \begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix}$$

The latitude and longitude of the Sun in the rotated system are

$$\Phi_S = \arcsin[Z_S/A]$$
$$\Lambda_S = \arctan[Y_S/X_S]$$



The azimuth and elevation (or altitude) are

$$\alpha = \pi - \Lambda_S$$
$$e = \Phi_S$$

Summary of Computations

Azimuth and elevation are (α, e).

We convert these to latitude and longitude (Φ_S, Λ_S) in the local frame.

We then get the cartesian coordinates (X_S, Y_S, Z_S) in the local frame.

Then we transform to the original cartesian coordinates (x, y, z).

Finally, we express (α, e) in terms of the geographic variables $\{\lambda_S, \phi_S, \phi_O\}$.

Outline

If the Sun is to retrogress, the time derivative of the azimuth Λ_S must vanish.

$$\tan\Lambda_S = \frac{Y_S}{X_S} = \frac{\sin\lambda_S\cos\phi_S}{\cos\lambda_S\cos\phi_S\sin\phi_O - \sin\phi_S\cos\phi_O}$$

The vanishing of the derivative leads, after some manipulation, to the equation

$$\cos\lambda_S = \frac{\tan\phi_O}{\tan\phi_S}$$

This gives the point of retrogression λ_S in terms of the solar latitude ϕ_S and observation latitude ϕ_O.

Again,

$$\cos \lambda_S = \frac{\tan \phi_O}{\tan \phi_S}$$

The derivative vanishes only if the right hand side is less than unity:

$$\phi_O < \phi_S \, .$$

Retrogression will be seen only if the observation point is between the Equator and the Sun’s latitude.

In particular, it must be in the tropics.

Numerical Results

Assume it is the Summer solstice: $\phi_S = 23.5^\circ$N.

We consider observations at ($\phi_O = 40^\circ$N)
and within the tropics ($\phi_O = 20^\circ$N).

We plot the zenith angle ($\zeta = 90^\circ - e$) versus azimuth.

The observation point is at the centre, and
the course of the Sun is shown by a curve.

Extratropical Observation Point: $\phi_O = 40°$

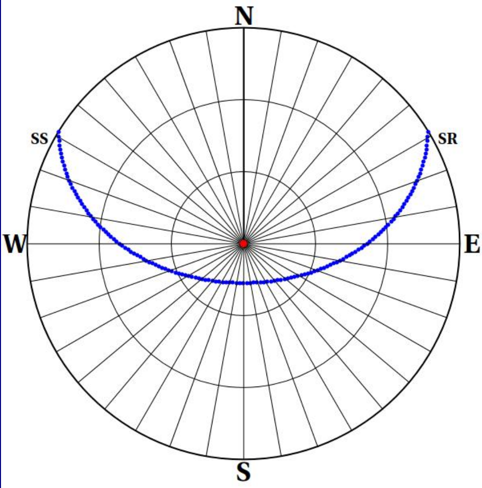


Figure : Path of the Sun for observation point at 40°N.

Tropical Observation Point: $\phi_O = 20^\circ$

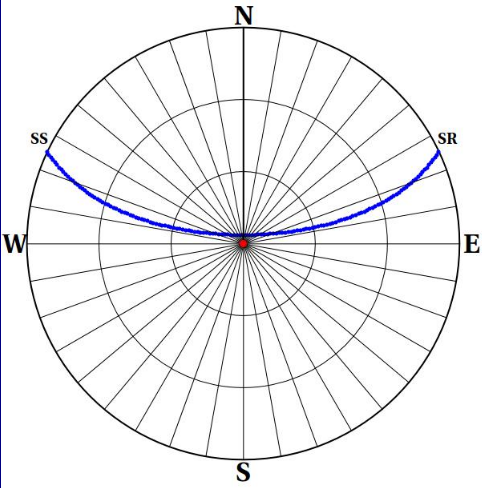


Figure : Path of the Sun for observation point at 20°N.

Azimuth and Elevation: $\phi_O = 40^\circ$

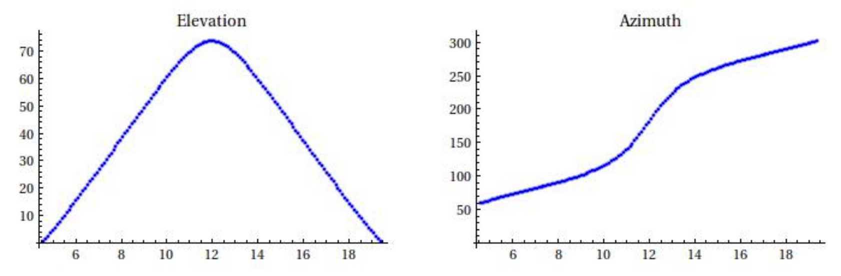


Figure : Solar elevation and azimuth for observation at 40°N.

Azimuth and Elevation: $\phi_O = 20^\circ$

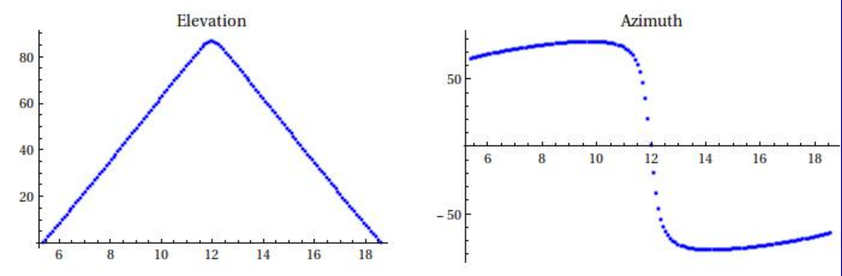


Figure : Solar elevation and azimuth for observation at 20°N.

More Fine Detail

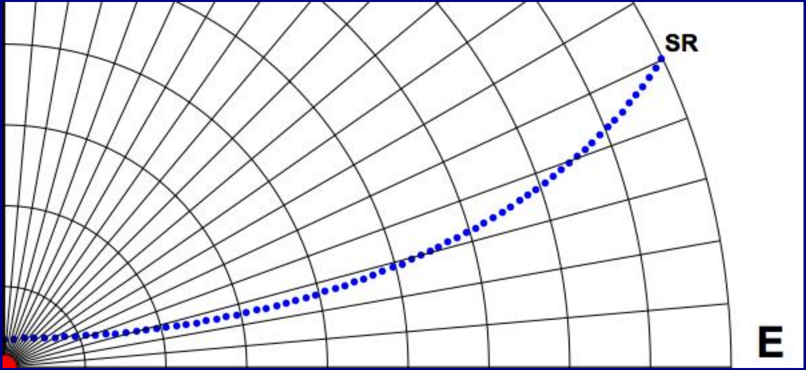


Figure : Path of the Sun for tropical observation point.

Azimuth is 65° at sunrise, at maximum 77° by mid-morning and decreases to zero at Noon.

Another Angle on the Azimuth

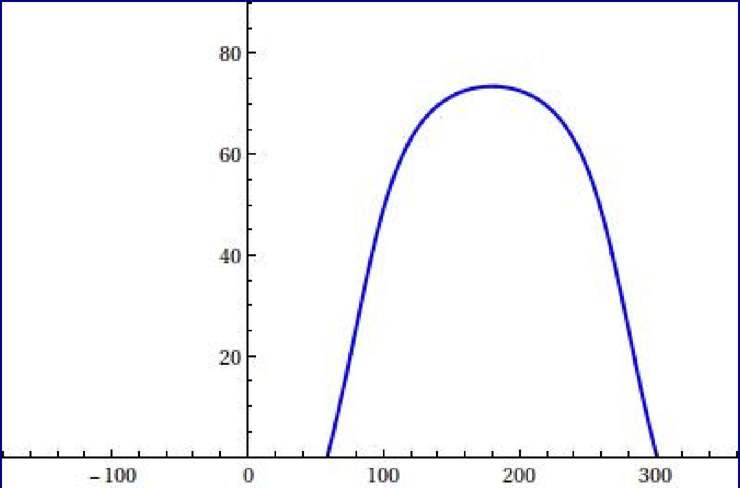


Figure : Elevation versus Azimuth for observer at 40°.

Another Angle on the Azimuth

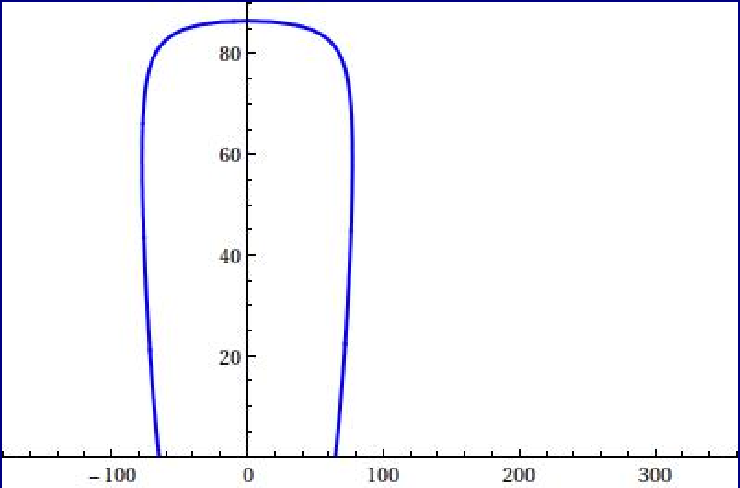


Figure : Elevation versus Azimuth for observer at 20°.

Confirming the Analysis

For the specific values $\phi_O = 20^\circ$ and $\phi_S = 23.5^\circ$,

$$\cos\lambda_S = \frac{\tan\phi_O}{\tan\phi_S}$$

gives the turning longitude as $\lambda_S = 33.17^\circ$.

This corresponds to an azimuth of 77.4° and an elevation of 59.1°.

This is in excellent agreement with the numerical solution shown in the figure above.

Data

Initial and Maximum values of azimuth: 64.9273 77.3993
Maximum and minimum values of elevation: 86.5 0.0900745
Maximum and minimum values of zenith angle: 89.9099 3.5

Hour 5.4 9.8 12.
Azim 64.9273 77.3993 0.
Elev 0.0900745 59.2166 86.5
Zenith 89.9099 30.7834 3.5

Azimuth at turn: 77.3995
Elevation at turn: 59.063

The Obligatory Biblical Quotation

What has been will be again,
what has been done will be done again;
there is nothing new under the Sun.

Ecclestiastes 1:9

Outline

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Thank You