

Mathematics & Music: Symmetry & Symbiosis

Peter Lynch
School of Mathematics & Statistics
University College Dublin

RDS Library Speaker Series
Minerva Suite, Wednesday 14 March 2018



Outline

“The Two Cultures”

Pythagoras

Sinusoidal Waves

Musical Notation

Tuning

Canons & Fugues

Fascinating Rhythm

Symmetry

Musical Chords



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Three Sweeping Statements: No. 1

Everyone Loves Music



Three Sweeping Statements: No. 2

Everyone Hates Maths



Three Sweeping Statements: No. 3

Music and Mathematics are the Same Thing



Love/Hate Image

Given the close link between music and maths,
how can we love one and hate the other?

1959 Rede Lecture: *The Two Cultures*

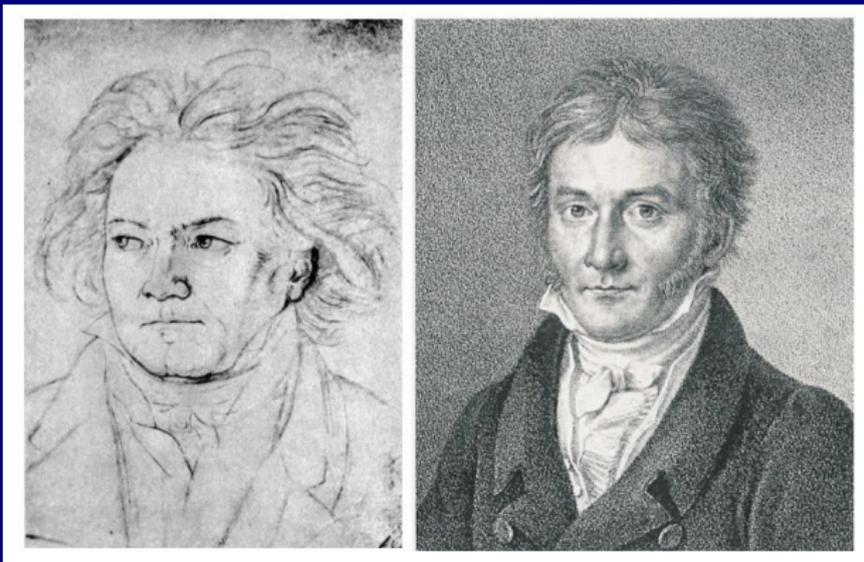


The concept of **The Two Cultures** was introduced by the British scientist and novelist **C. P. Snow**.

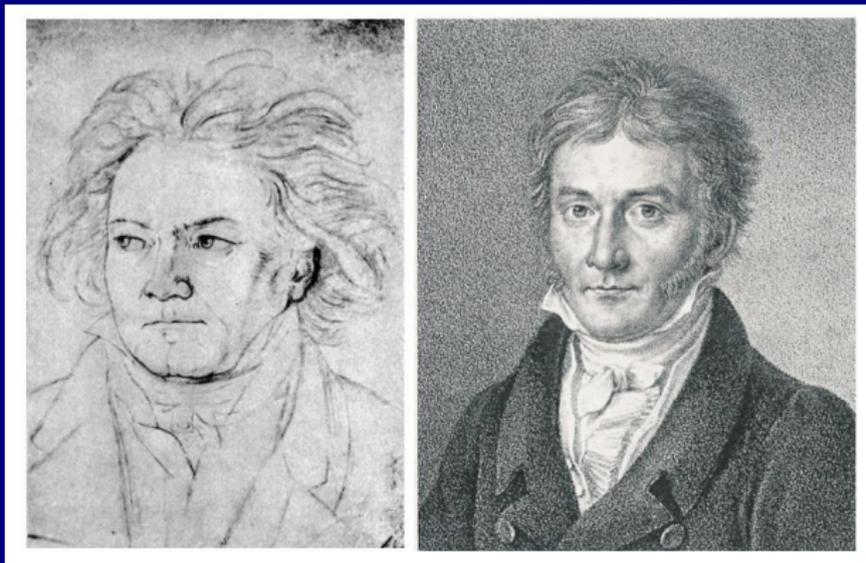
This concept is still relevant today.



Who's Who Here?



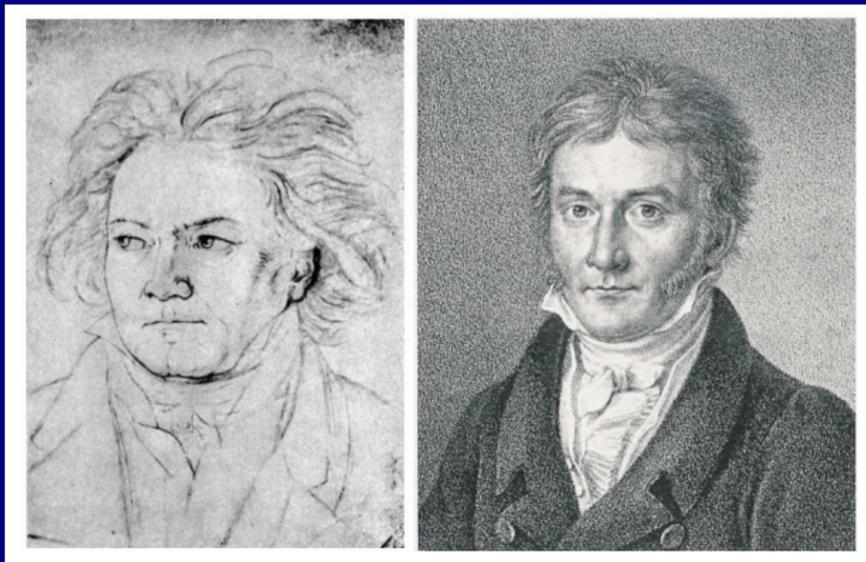
Who's Who Here?



LEFT: Ludwig van Beethoven (1770–1827)



Who's Who Here?



LEFT: Ludwig van Beethoven (1770–1827)
RIGHT: Carl Friedrich Gauss (1777–1855)



Beethoven / Gauss

Beethoven and Gauss were at the height of their creativity in the early 19th century.

The work of Gauss has a greater impact on our daily lives than the magnificent creations of Beethoven.

Yet, Beethoven is known to all, Gauss to only a few!

*“Of course I’ve heard of Beethoven,
but who is this Gauss dude?”*



Two Parallel Languages

MUSIC	MATHS
Pitch	Frequency
Scales	Modular Arithmetic
Intervals	Logarithms
Overtones	Integers
Octave Identification	Equivalence Relation
Equal Temperament	Exponents
Timbre	Harmonic Analysis
Canon Form	Group Theory
Chord Progressions	Orbifold Topology

You know more mathematics than you realize!



Music & Maths

There are many parallels between music and maths:

Structure

Symmetry

Pattern

etc.

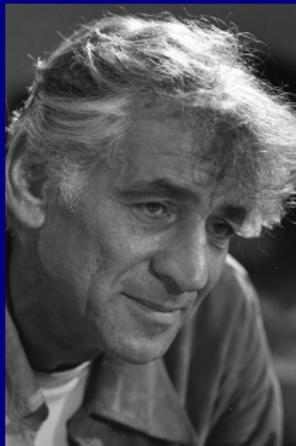
But music is **accessible to all** while maths is not.
Music gets into the soul through the **emotions**.

Maths is understood through the intellect.
Appreciation comes via a **rational route**.

Music has instant appeal. Maths takes time.



Leonard Bernstein



“Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?”



Some Relevant Mathematical Concepts

- ▶ **Integers. Rationals. Real Numbers**
- ▶ **Logarithms and Exponentials**
- ▶ **Equivalence Relationships**
- ▶ **Geometric Transformations**
- ▶ **Modular Arithmetic**
- ▶ **Groups and Rings**
- ▶ **Periodic Functions**
- ▶ **Orbifold Topology.**



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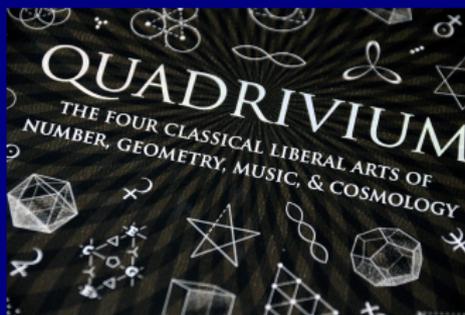
Fascinating Rhythm

Symmetry

Musical Chords



The Quadrivium



The Pythagoreans organized their studies into the **Quadrivium**, comprising four disciplines:

- ▶ **Arithmetic**
- ▶ **Geometry**
- ▶ **Music**
- ▶ **Astronomy**



Static/Dynamic Number. Pure/Applied

- ▶ **Arithmetic:** Static number
- ▶ **Music:** Dynamic number

Arithmetic represents numbers at rest.

Music is numbers in motion.

Arithmetic is **pure or abstract** in nature.

Music is **applied or concrete** in nature.



Static/Dynamic Space. Pure/Applied

- ▶ **Geometry: Static space**
- ▶ **Astronomy: Dynamic space**

Geometry represents space at rest.

Astronomy is space in motion.

Geometry is **pure or abstract** in nature.

Astronomy is **applied or concrete** in nature.



Discovery of Pythagoras

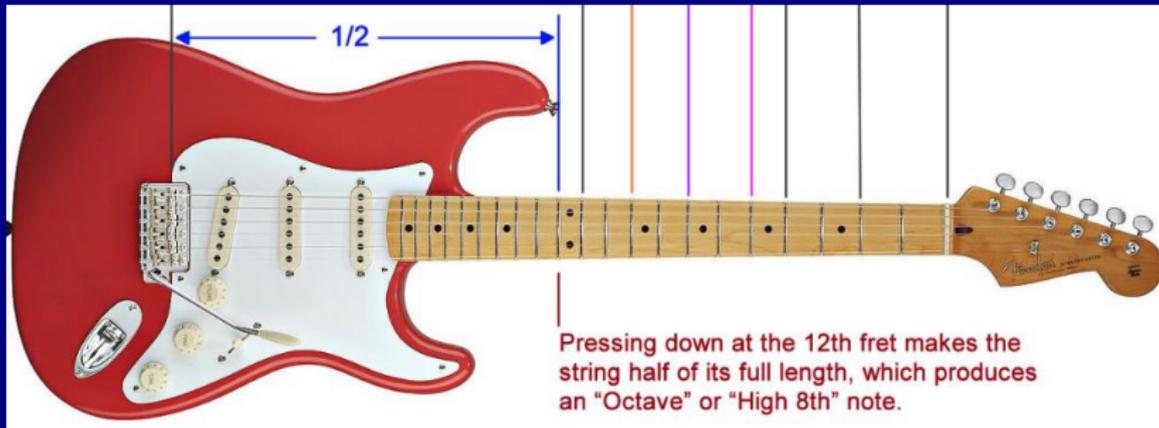
The Pythagoreans discovered
a remarkable connection:

**Ratios of small whole numbers are directly linked
with consonant or harmonically pleasing chords.**

*“There is geometry in the humming of the strings,
There is music in the spacing of the spheres.”*



Guitar Strings



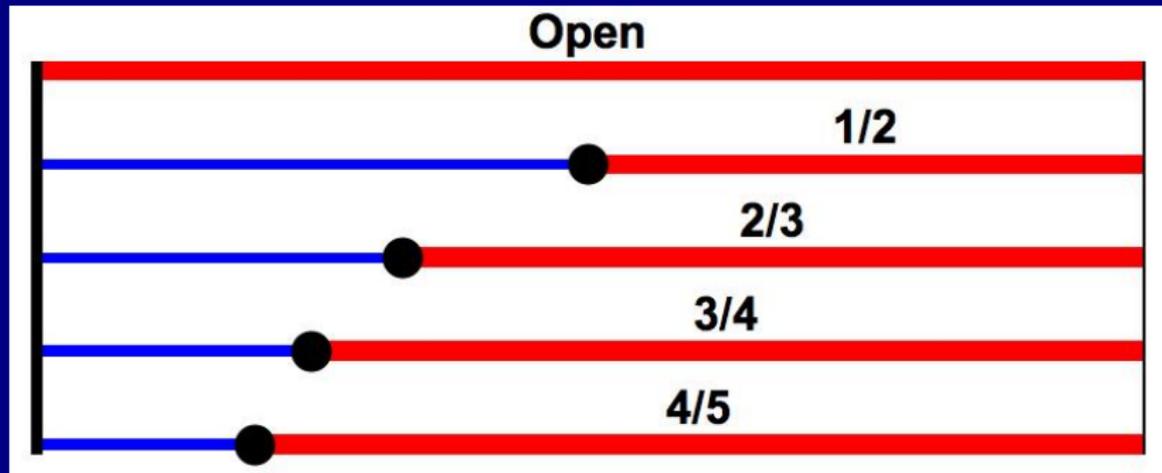
Open string vibrates at 264 Hz. Call this f .

- ▶ String of length $\frac{1}{2}$ vibrates at $\frac{2}{1} f$.
- ▶ String of length $\frac{2}{3}$ vibrates at $\frac{3}{2} f$.
- ▶ String of length $\frac{3}{4}$ vibrates at $\frac{4}{3} f$.

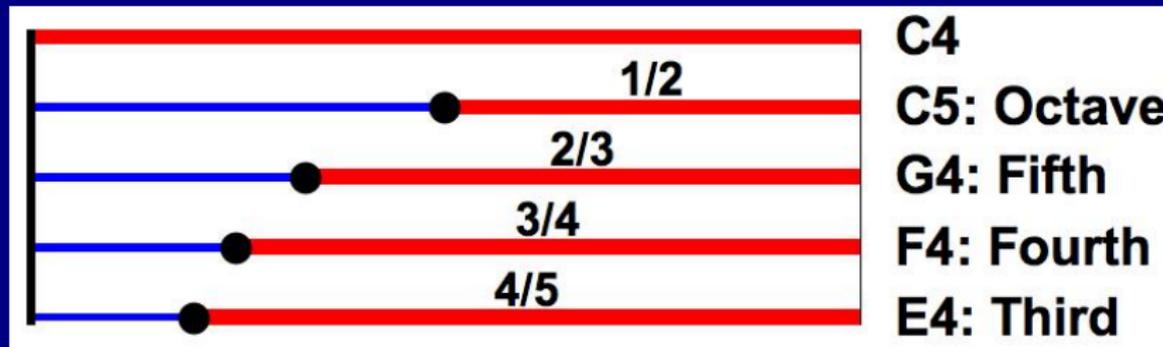
Demonstrate the pitches on the uke.



The Monochord



The Monochord



Feynman on Pythagoras' Discovery



Richard Feynman

“The first example, outside geometry, of a numerical relationship in nature.”

Pythagoras made his discovery through **observation**. This aspect does not seem to have impressed him.

Had Pythagoras followed up on this idea, *“Physics might have had a much earlier start.”*



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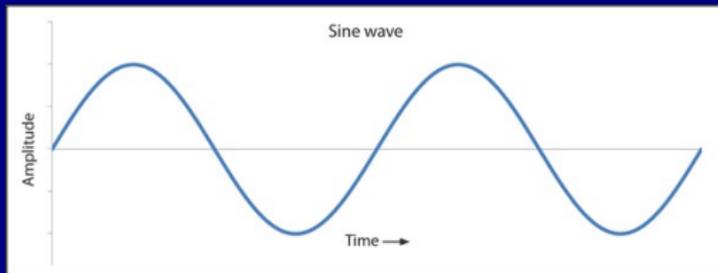
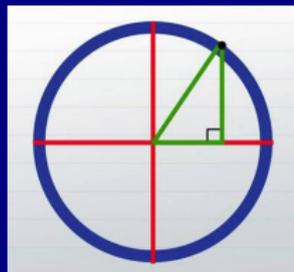
Fascinating Rhythm

Symmetry

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Sine Waves and Circles



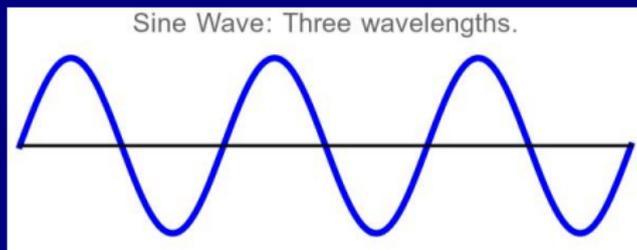
As the **black dot** moves around the circle,
its **height** traces a **sine wave** in time.

*Up and down and up and down and up and down
and up and down and up and down and ...*

Sine waves are a kind of **circular functions**.



Terminology

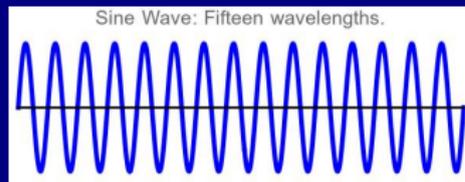
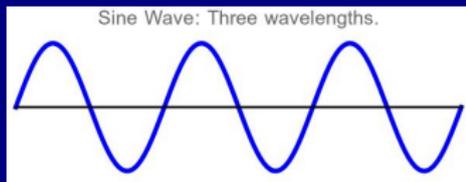


Music

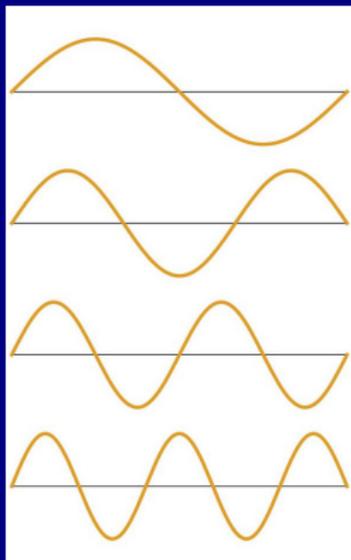
Physics

Loudness \Leftrightarrow Amplitude

Pitch \Leftrightarrow Frequency



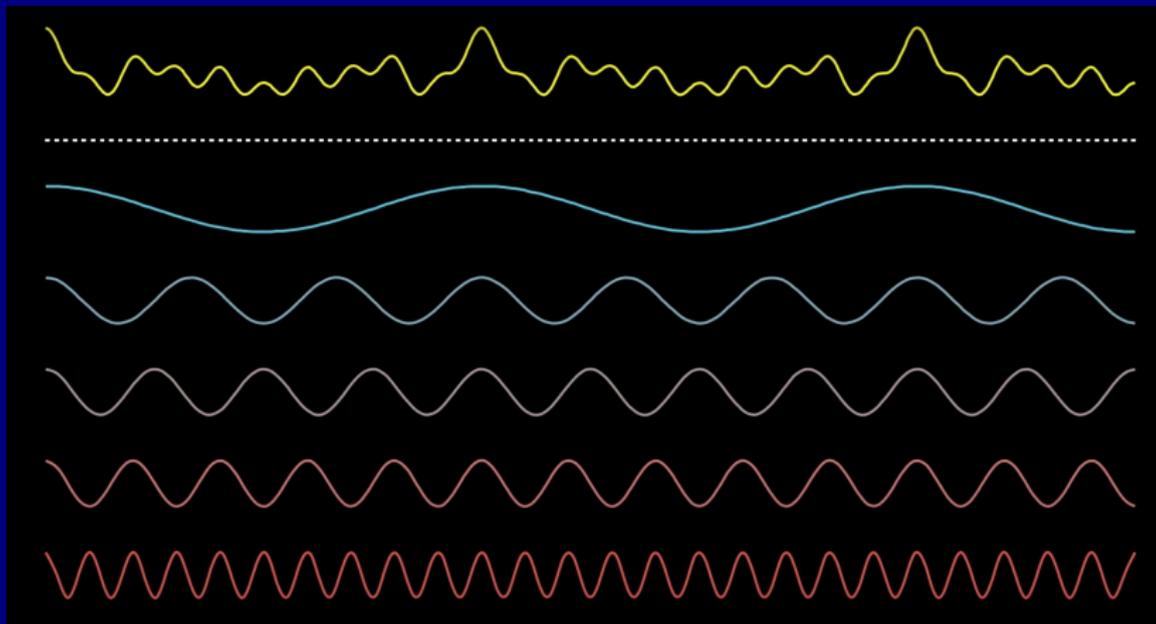
Simple Harmonics



A simple sine wave and its first three overtones.



Fourier Components



A Harmonic Generator



[https://meettechniek.info/
additional/additive-synthesis.html](https://meettechniek.info/additional/additive-synthesis.html)

A website for generating sine waves and harmonics.

*** If the technology fails, use the trusty Uke ***



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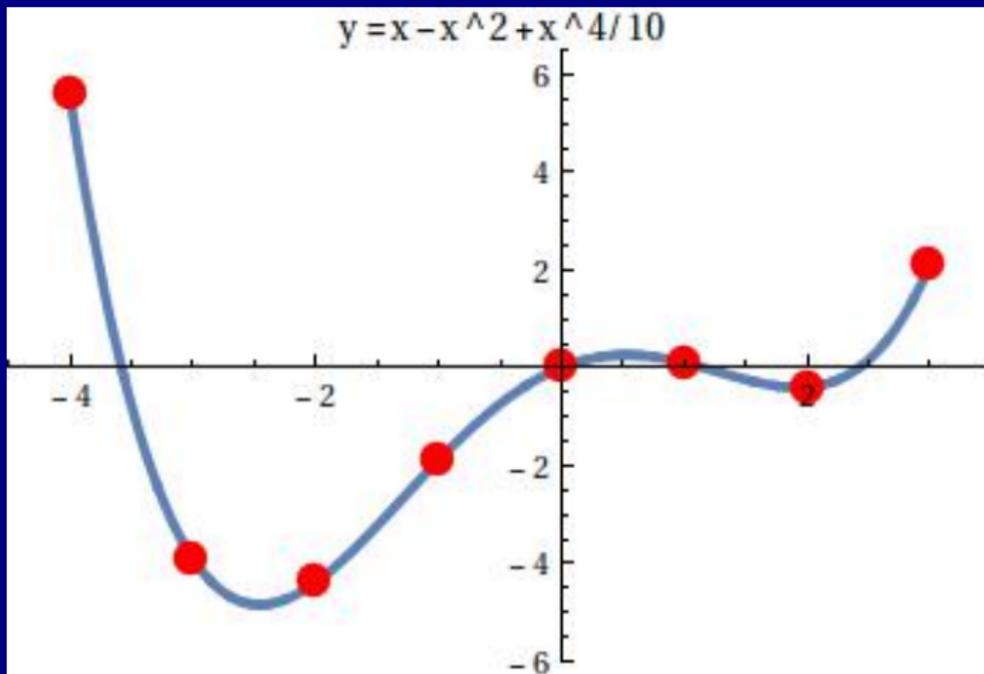
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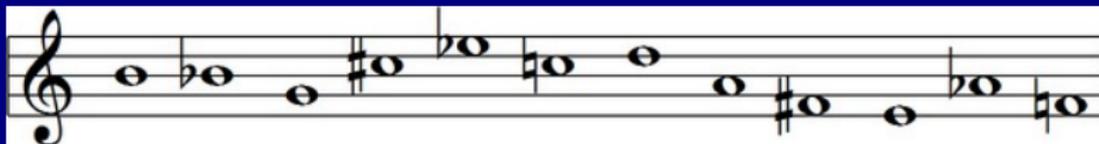
A Mathematical Graph: Joining the Dots



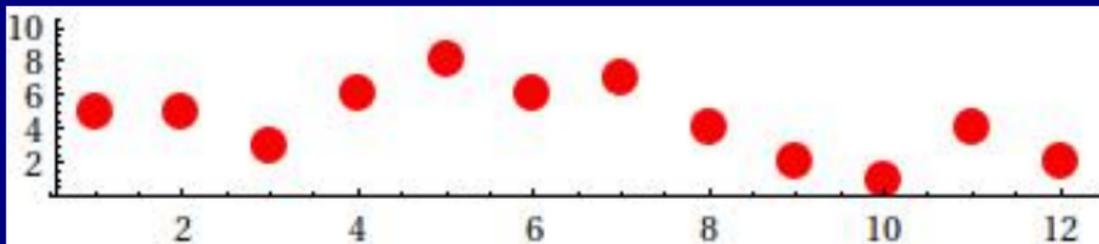
A plot of y versus x .



Music as a Graph



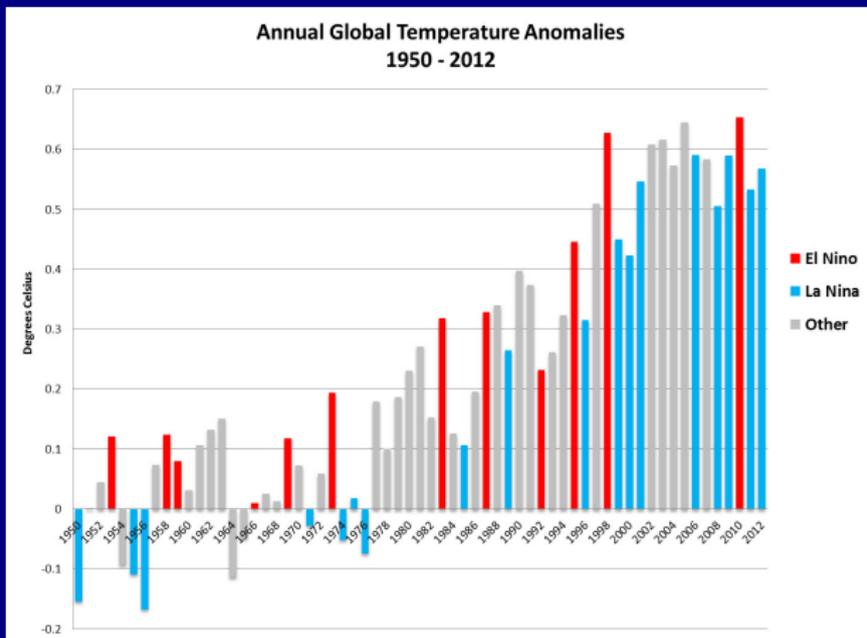
A sequence of musical notes: a simple tone row.



A musical score is just a graph of **pitch versus time!**



Symbiosis: Musing on a Graph



A new composition: *The Glow-Ball Gavotte.*



Beethoven's *Moonlight Sonata*

MOONLIGHT SONATA
Op. 27, No. 2
(First Movement)
LUDWIG VAN BEETHOVEN

Adagio sostenuto

sempre pp *simile*

A combination of **precision** and **vagueness**.



Precision and Vagueness



The standard **pitch** is A_4 ,
the A above Middle C.
Its frequency is **440 Hz**.

By contrast, the **time-scale** is
described only by the phrase
Adagio sostenuto.
It could be given like this:



The **loudness** is specified simply by *sempre pp*.



Beethoven's *Moonlight Sonata*

MOONLIGHT SONATA
Op. 27, No. 2
(First Movement)

LUDWIG VAN BEETHOVEN

Adagio sostenuto

The image displays the musical score for the first movement of Beethoven's Moonlight Sonata. It features two systems of music. The first system shows the beginning of the piece, with a treble clef and a bass clef. The treble clef part starts with a series of eighth notes, marked 'sempre pp' and 'simile'. The bass clef part has a single low note. The second system continues the treble clef part with a series of eighth notes, while the bass clef part has a series of low notes.

An ingenious and delightful combination
of precision (pitch) and vagueness (pp).



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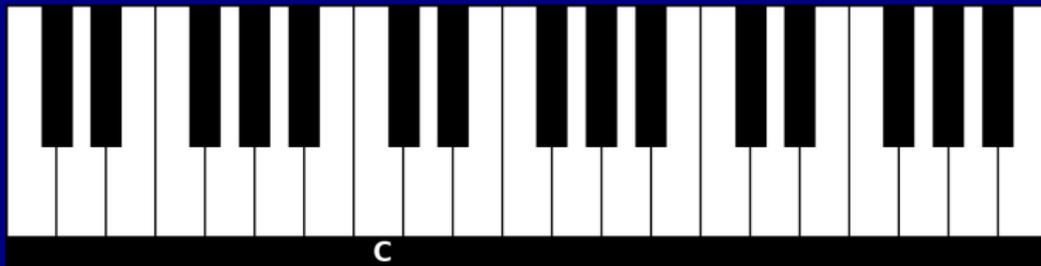
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The Piano Keyboard



Where do all the notes come from?



Middle C

Middle C is the 'central note' on the piano.
It is commonly pitched at **261.63 Hz**.

The standard frequency of the note **A4** is 440 Hz.

$$261.63 = 440 \div 2^{9/12}$$

Where does the peculiar factor $2^{9/12}$ come from?
It arises from the **well-tempered scale**.



Pythagorean Tuning

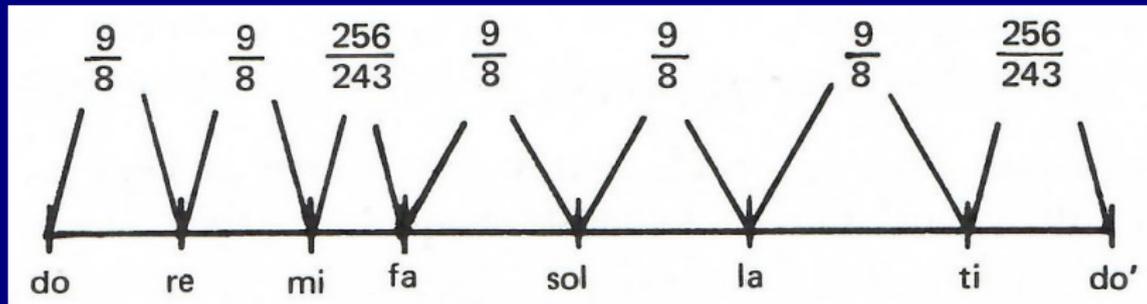
Pythagoras discovered that a **perfect fifth** — with frequency ratio $3:2$ — is especially harmonious.

The entire musical scale can be constructed using only the ratios $2:1$ (octaves) and $3:2$ (fifths).

In the **tonic sol-fa** scale the eight notes of the major scale are **Do, Re, Mi, Fa, So, La, Ti, Do**.



Pythagorean Tuning



Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1



The Pythagorean Comma

The Pythagoreans noticed that $2^{19} \approx 3^{12}$.

Going up twelve fifths, with ratio $(3/2)^{12}$,
and down seven octaves, with ratio $(1/2)^7$
gets us back (almost) to our starting point.

The number

$$3^{12}/2^{19} \approx 1.01364$$

is called the **Pythagorean comma**.



The Pythagorean Comma

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is called the **Pythagorean comma**.

The 13th note is not quite the same as the starting note. The two notes are **enharmonics**:

$$F\sharp \neq G\flat$$



Triads and Just Intonation

The triad — three notes separated by 4 and 3 semitones, such as **C–E–G** — is of central importance in western music.

In the tuning scheme of Pythagoras, the third (**C–E**) has a frequency ratio

$$\left(\frac{9}{8}\right)^2 = \frac{81}{64}$$

Substituting

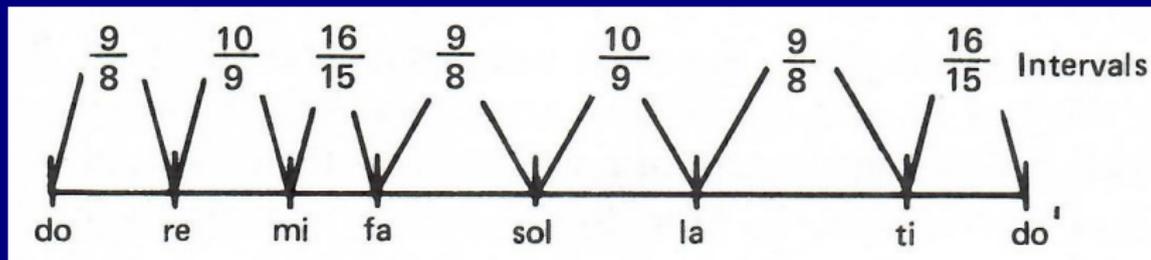
$$\frac{81}{64} \approx \frac{80}{64} = \frac{5}{4}$$

the three notes of the triad **C–E–G** are in the ratio

$$4 : 5 : 6$$



Just Intonation



Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1



Pythagorean and Just Intonation

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Pythagorean intonation.

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Just intonation.



The Well-Tempered Scale

It is impossible to tune a piano so that all fifths have perfect frequency ratios of 3:2.

Idea: Make all semitone intervals equal.



The Well-Tempered Scale

An octave has ratio 2:1. We need a number that yields 2 when multiplied by itself 12 times:

$$\sqrt[12]{2} \approx 1.059$$

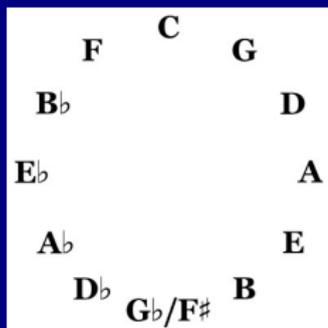
In the tempered scale, all intervals are imperfect, but are **close enough to be acceptable to the ear.**

★ ★ ★

Johann Sebastian Bach's **Well-Tempered Clavier** is a collection of preludes and fugues in all 24 keys.



Organizing Scheme: the Circle of Fifths



The **Circle of Fifths** represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

The Circle is useful in harmonising melodies and building chords.



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The Geometry of Canons

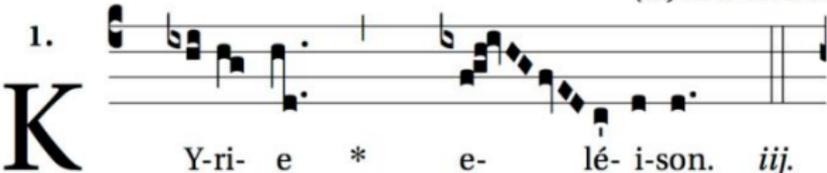
Gregorian chant is a **monophonic**, unaccompanied chant, developed during the 9th and 10th centuries.

There are no fixed measures and no time signature.

Kyrie XI.
(Orbis Factor)

(X.) XIV.-XVI. s.

1.



K Y-ri- e * e- lé- i-son. iij.

Gradually, regular division into **bars or measures**, each of the same fixed length, emerged.



The Geometry of Canons

Frè - re Jac - ques, Dor - mez - vous ? Son - nez les ma - ti - nes, Ding, daing, dong ! *etc.*

Frè - re Jac - ques, Dor - mez - vous ? Son - nez les ma - ti - nes, *etc.*

Frè - re Jac - ques, Dor - mez - vous ? *etc.*

Frè - re Jac - ques, *etc.*

The image shows a musical score for the canon 'Frère Jacques' in 2/4 time. It consists of four staves. The first staff begins with the melody: Frère Jacques, Dormez-vous? Sonnez les matines, Ding, daing, dong! etc. The second staff enters after one measure. The third staff enters after two measures. The fourth staff enters after three measures. The time signature is 2/4 and the key signature has one sharp (F#).

A manuscript at the French National Library contains 86 canons by J.-P. Rameau, including Frère Jacques.



The Geometry of Canons

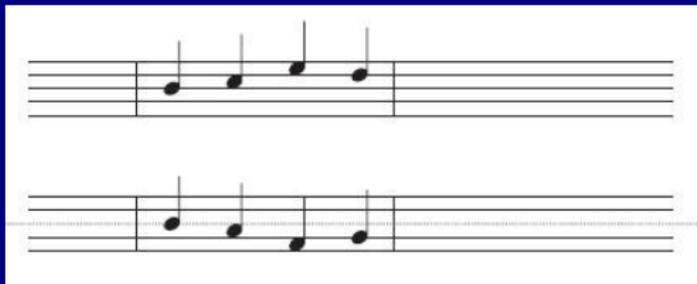
- ▶ Canon at the Unison
- ▶ Canon at an Interval
- ▶ Canon *contrario motu*
- ▶ Retrograde Canon
- ▶ Perpetual Canon

Johann Sebastian Bach was the grand master of canon form.

The transformations used in Canon Form are described by mathematical groups.



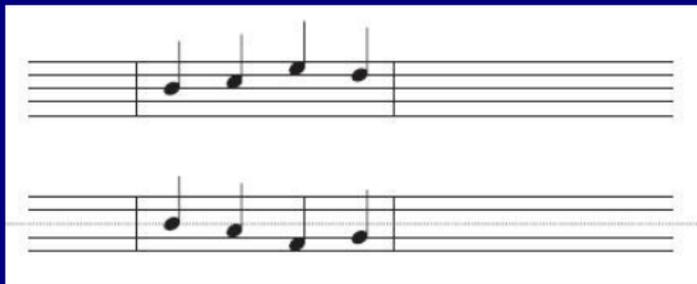
Simple Transformations



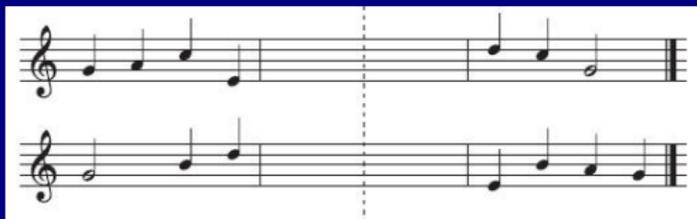
Inversion, or rotation about a horizontal line.



Simple Transformations



Inversion, or rotation about a horizontal line.



Retrogression, or rotation about a vertical line.



“A Musical Offering”

Frederick the Great provided Bach with a theme:



A “Graphical” Offering

Bach was deeply familiar with symmetry.



This is illustrated by the seal he designed in 1722.



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Leibniz on Music



Gottfried Wilhelm Leibniz
(1646–1716)

“Music is the pleasure the human mind experiences from counting without being aware that it is counting.”

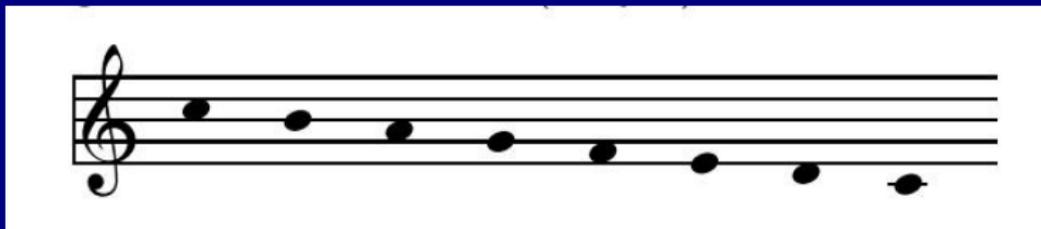


Beating the Time

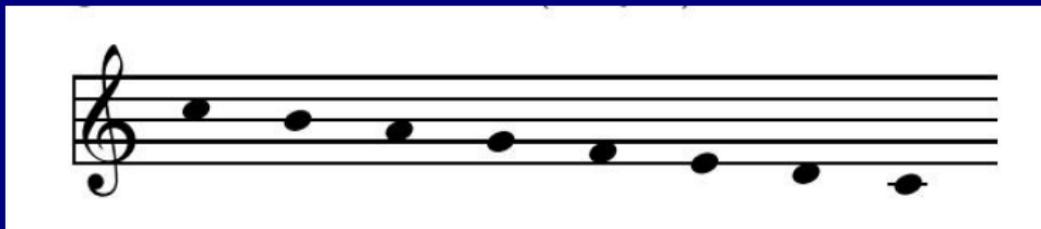
The diagram illustrates four staves of musical notation. The first staff contains four red quarter notes. The second staff contains four blue eighth notes, with the first two and last two beamed together. The third staff contains four green eighth notes, with the first and third beamed together. The fourth staff contains two purple eighth notes beamed together, a single purple quarter note, and a black squiggle.



Rhythm in Music



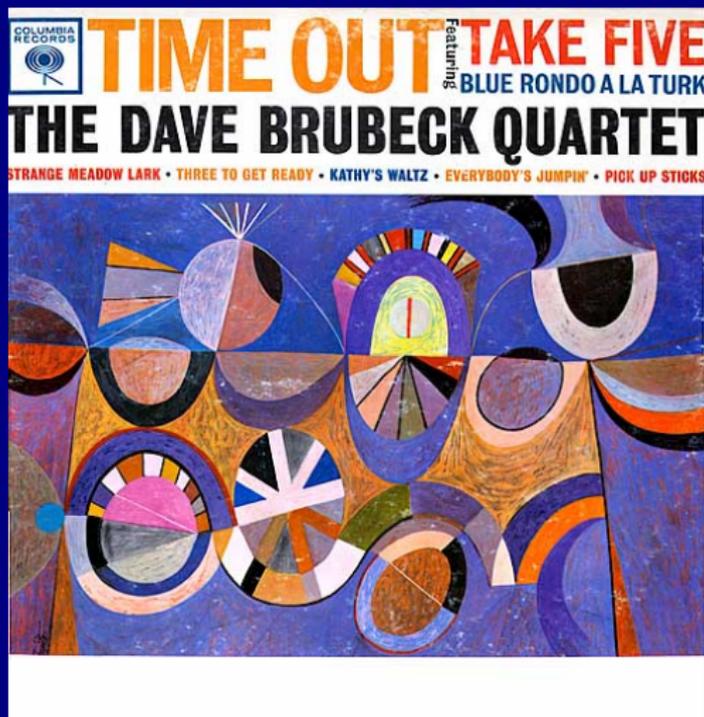
Rhythm in Music



Joy to the World.



Time Out (1959)



Take Five: 5/4 Time

Take Five

Music by Paul Desmond

Moderately fast $\text{♩} = 176$

The image displays the first two systems of the musical score for 'Take Five' by Paul Desmond. The music is written in 5/4 time and features a key signature of three flats (B-flat major or D-flat minor). The first system consists of two measures, with the first measure marked 'mf'. The second system consists of two measures, with the first measure containing a first ending bracket and a second measure containing a second ending bracket. The notation includes treble and bass clefs, a common time signature, and various musical symbols such as notes, rests, and accidentals.

Based upon use of time signatures that were unusual for jazz: $\frac{5}{4}$, $\frac{9}{8}$, etc.



Blue Rondo a la Turk: 9/8 Time

Blue Rondo A La Turk

Music by Dave Brubeck

Vivace $\text{♩} = 148$

Con spirito

Chord progression: $Fmaj^7$ F^7 $F^{\#}$ $Faug$ F $Faug$ $F^{\#}$ F^7

The first system of musical notation is in 9/8 time, indicated by a 9 over an 8. The time signature is written as (2+2+2+3) over 8. The music is written for piano in two staves. The right hand features a melodic line with eighth and quarter notes, and the left hand provides a harmonic accompaniment with chords. The chords are labeled as Fmaj7, F7, F#, Faug, F, Faug, F#, and F7.

Chord progression: $Fmaj^7$ F^7 $F^{\#}$ $Faug$ F $Faug$ $F^{\#}$ Am^7 $D^{\#}$ Dm^7 Am

The second system of musical notation continues the piece. It features a double bar line followed by a repeat sign. The chord progression includes Fmaj7, F7, F#, Faug, F, Faug, F#, Am7, D#, Dm7, and Am. The notation shows the continuation of the melodic and harmonic lines from the first system.



The Stranglers: *Golden Brown*

THE STRANGLERS - EMI

GOLDEN BROWN

Tekst & muziek:
JEAN JACQUES BURNELL/HUGH CORNWELL/
JET BLACK & DAVID GREENFIELD

Moderately
Play 3 times

B♭m Fm G♭

Can you work out the time signature?

[Listen to the video clip.](#)



The Stranglers: *Golden Brown*

THE STRANGLERS - EMI

GOLDEN BROWN

Tekst & muziek:
JEAN JACQUES BURNELL/HUGH CORNWELL/
JET BLACK & DAVID GREENFIELD

Moderately
Play 3 times

B♭m Fm G♭ D♭

It looks like $(3+3+3+4)/8 = 13/8$.

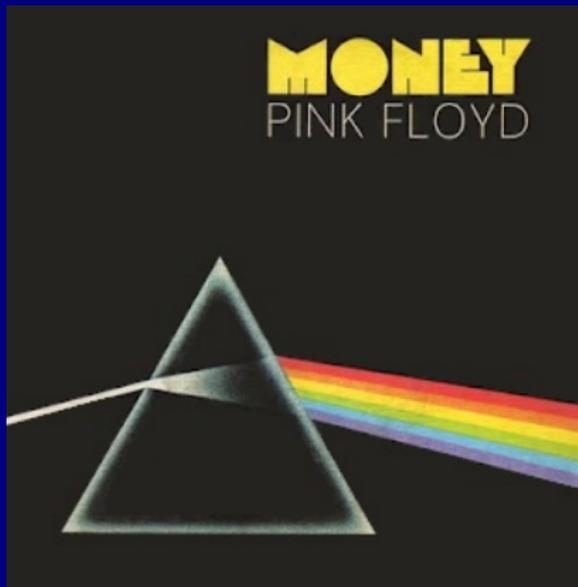
[Listen to the video clip.](#)



Pink Floyd: *Money*



Pink Floyd: *Money*



***Money* is composed mainly in 7/4 time.**



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Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- ▶ Many **buildings** are symmetric.
- ▶ **Our bodies** have bilateral symmetry.
- ▶ **Crystals** have great symmetry.
- ▶ **Viruses** can display stunning symmetries.
- ▶ At the **sub-atomic scale**, symmetry reigns.
- ▶ **Galaxies** have many symmetries.



The Taj Mahal



A Face with Symmetry: Halle Berry



Halle Berry

Berry Halle



An Asymmetric Face: You know Who!



Symmetry and Group Theory

Symmetry is an essentially **geometric** concept.

The mathematical theory of symmetry is **algebraic**.
The key concept is that of a **group**.

A group is a set of elements such that any two elements can be combined to produce another.

Instead of giving the mathematical **definition**,
I give an **example** to make things clear.



The Klein 4-Group

The four orientations of a book can be described in terms of four simple rotations:

- ▶ **P**: Place book upright with front cover upright
- ▶ **R**: Rotate 180° about vertical through centre
- ▶ **I**: Rotate 180° about horizontal through centre
- ▶ **RI**: Rotate 180° about perp. through centre

These operations make up the **Klein 4-Group**.



Twelve-tone Music

Table: Klein 4-Group.

	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

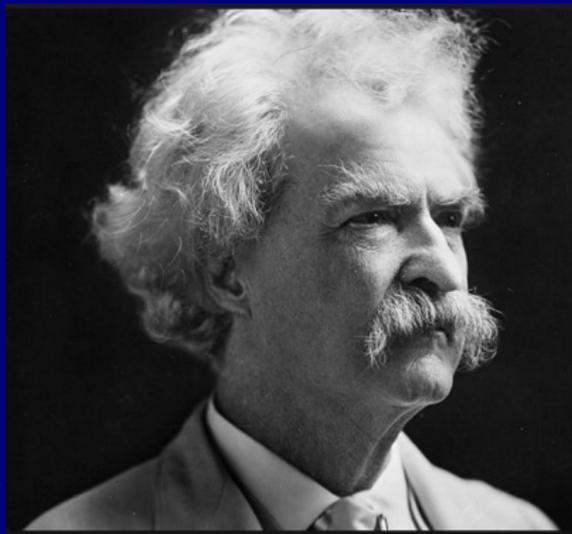
The Klein 4-group is the basic group of transformations in **twelve tone music**.

The operations are retrogression (R), inversion (I) and the rotation (RI).

The image shows a musical score with two staves. The top staff is labeled 'P' and 'R', and the bottom staff is labeled 'I' and 'RI'. The notation consists of eighth and sixteenth notes with various accidentals.



Mark Twain on Wagner



*“Wagner’s music
is much better
than it sounds.”*

**I am not aware whether Mark Twain ever
commented on Arnold Schoenberg’s music !**



Paganini: Caprice 24

Caprice no. 24 in A minor for violin

NICCOLO PAGANINI (1782-1840)

THEME

The image displays the musical notation for the theme of Caprice no. 24 in A minor for violin by Niccolò Paganini. The notation is presented on two staves. The first staff begins with a treble clef and a 2/4 time signature. The melody starts with a quarter note G4, followed by eighth notes A4 and B4, and a quarter note C5. The second staff continues the melody with eighth notes D5, E5, and F5, followed by a quarter note G5. The notation includes various rhythmic values, accidentals, and phrasing slurs, characteristic of Paganini's style.



Paganini: Caprice 24

Caprice no. 24 in A minor for violin

NICCOLO PAGANINI (1782-1840)

THEME

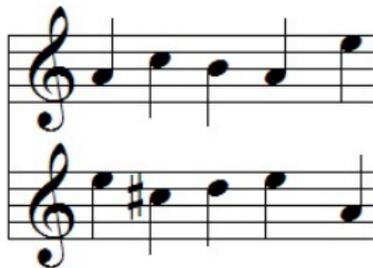


Rachmaninov's *Rhapsody*



Paganini (part of 24th capriccio theme)

Rachmaninoff (inversion: 18th variation theme)



<file:///home/peter/Dropbox/Music/RDS-MusicClips/Rachmaninov-Clip1.mp4>



Outline

“The Two Cultures”

Pythagoras

Sinusoidal Waves

Musical Notation

Tuning

Canons & Fugues

Fascinating Rhythm

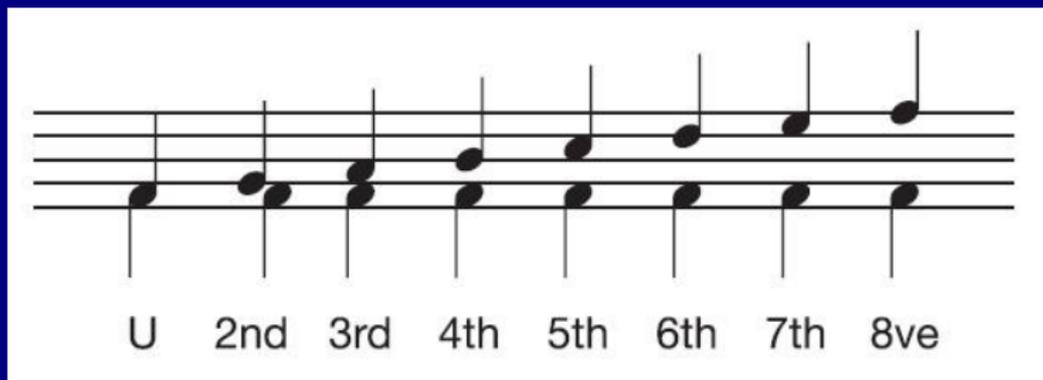
Symmetry

Musical Chords

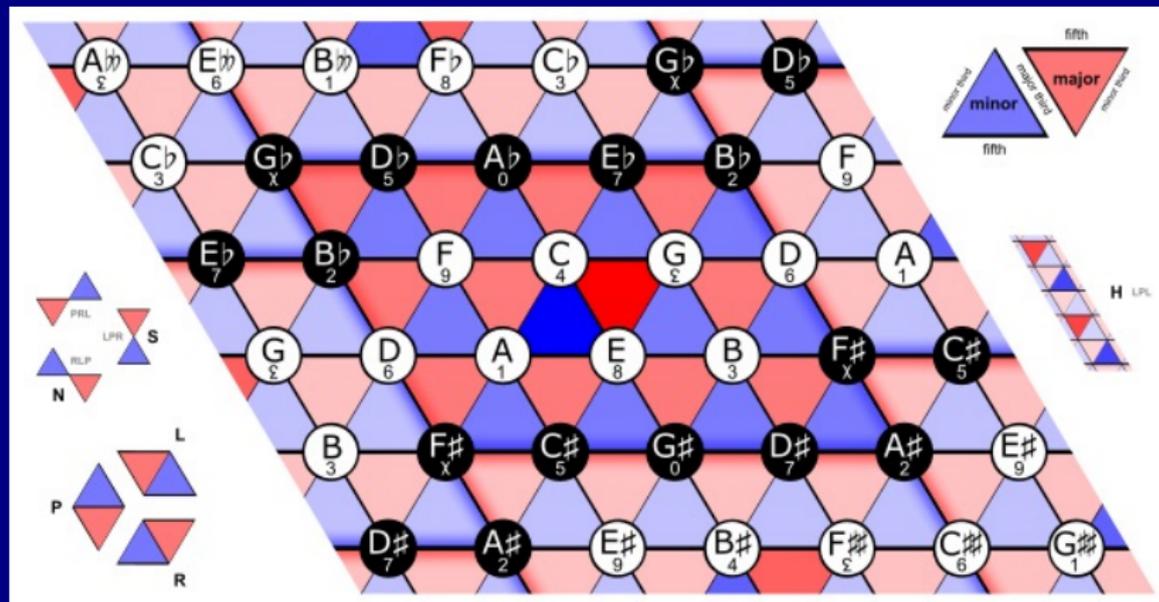


Musical Intervals. Chords

**An Interval is two notes sounded together.
Interval distances are counted inclusively.**



The Tonnetz

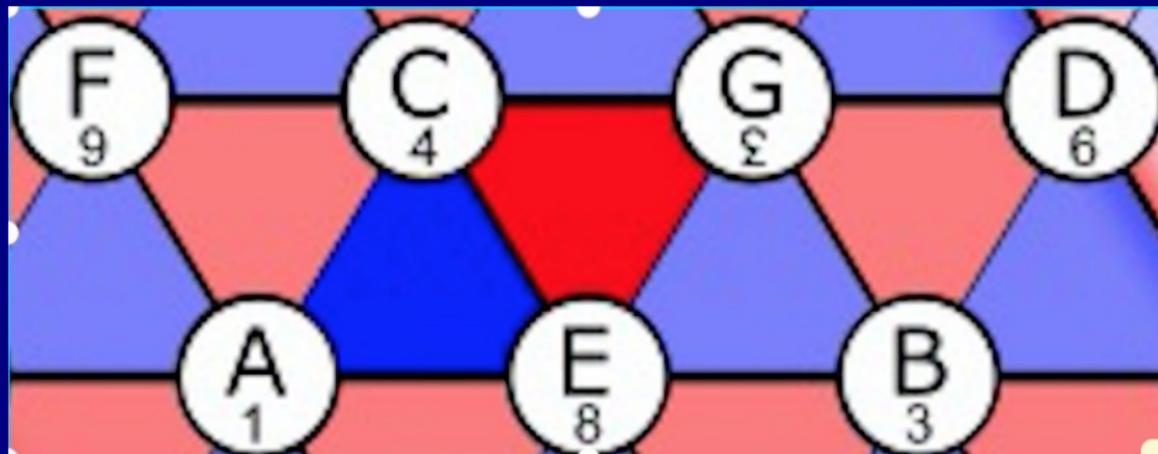


This has the topology of a torus or doughnut.

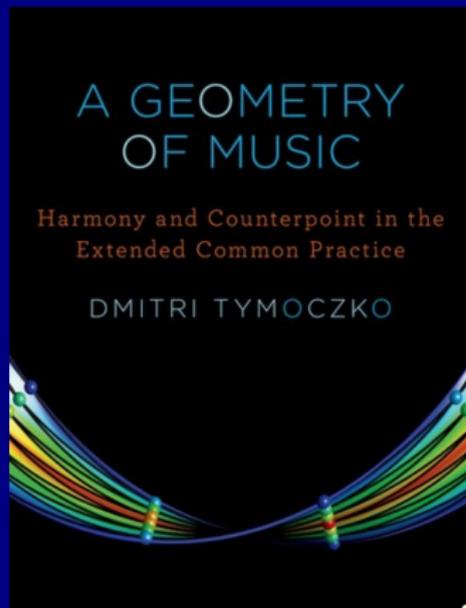
[Animated gif of Tonnetz on a Torus.](#)



The Tonnetz — Detail



A Geometry of Music



<https://vimeo.com/20300784>



Chopin's Prelude Op. 28, No. 4

Prelude

Largo F. Chopin, Op. 28, No. 4

p *espressivo*
tenuto sempre

5



Leonard Bernstein



“Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?”

“... music is not only a mysterious and metaphorical art; it is also born of science.

“It is made of mathematically measurable elements.

*... any explication of music must
combine mathematics with aesthetics.”*



Rachmaninov's *Rhapsody*

<file:///home/peter/Dropbox/Music/RDS-MusicClips/Rachmaninov-Clip1.mp4>



Thank you

