Magnums

Counting Sets with Surreal Numbers

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Seminar, UCD, 23 April 2019

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

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Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Magnums and Subsets of $\mathbb{N}$

The aim of this work is to define a number

$$m(A)$$

for subsets A of $\mathbb{N}$ that corresponds to our intuition about the size or magnitude of A.

We call $m(A)$ the magnum of A.

Magnum = Magnitude Number

Magnums and Subsets of $\mathbb{N}$

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Magnum = Magnitude Number

"C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente."

It is by logic that we prove, but by intuition that we discover [Poincaré].

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

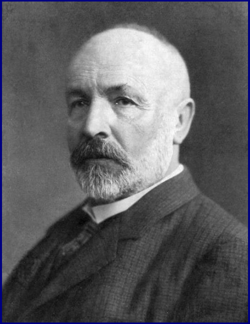
Finis

Galileo Galilei (1564–1642)

Every number n can be matched with its square n^2.

In a sense, there are as many squares as whole numbers.

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

Georg Cantor (1845–1918)

- Invented Set Theory.
- One-to-one Correspondence.
- Infinite and Well-ordered Sets.
- Cardinals and Ordinals.
- Proved $\mathrm{card}(\mathbb{Q}) = \mathrm{card}(\mathbb{N})$.
- Proved $\mathrm{card}(\mathbb{R}) > \mathrm{card}(\mathbb{N})$.
- Hierarchy of Infinities.

Set Theory: Controversy

Cantor was strongly criticized by

- Henri Poincaré.
- Leopold Kronecker.
- Ludwig Wittgenstein.

Set Theory is a "grave disease" (HP).
Cantor is a "corrupter of youth" (LK).
"Nonsense; laughable; wrong!" (LW).

Set Theory: A Difficult Birth

Set Theory brought into prominence several paradoxical results.

It was so innovative that many mathematicians could not appreciate its fundamental value and importance.

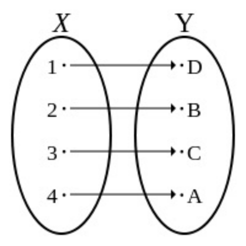
Gösta Mittag-Leffler was reluctant to publish it in his *Acta Mathematica.* He said the work was "100 years ahead of its time".

David Hilbert said:
"We shall not be expelled from the paradise that Cantor has created for us."

Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Infinite Sets

Now we consider sets that are infinite.

We take the natural numbers and the even numbers

$$\mathbb{N} = \{1, 2, 3, ...\}$$

$$\mathbb{E} = \{2, 4, 6, ...\}$$

By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$, we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[\mathbb{E}]$$

Again,

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[\mathbb{E}]$$

But this is paradoxical: The set of natural numbers contains all the even numbers

$$\mathbb{E} \subsetneqq \mathbb{N}\,.$$

But $\mathbb{N}$ also contains all the odd numbers.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

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Ordinal Numbers are used to describe the order type of well-ordered sets.

An ordinal may be defined as the set of ordinals that precede it. Thus 27 is the set $\{0, 1, 2, \ldots, 26\}$.

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The smallest infinite ordinal is ω, the order type of the set of natural numbers $\mathbb{N}$.

Indeed, ω can be identified with the set $\mathbb{N}$.

After ω come $\omega + 1, \omega + 2, \ldots, \omega \cdot 2$.

Then $\omega \cdot m + n$ and on to $\omega^2, \omega^3, \ldots, \omega^\omega$.

Diagram of Ordinals up to ω^2

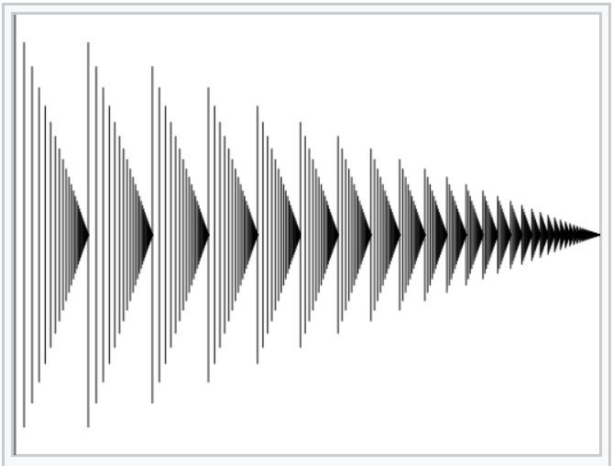

Figure: Each 'matchstick' is an ordinal $\omega \cdot m + n$.

Von Neumann's Definition

Each ordinal number is the well-ordered set of all smaller ordinal numbers.

First few von Neumann ordinals

0 = { }		= ∅
1 = { 0 }		= {∅}
2 = { 0, 1 }		= { ∅, {∅} }
3 = { 0, 1, 2 }		= { ∅, {∅} , {∅, {∅}} }
4 = { 0, 1, 2, 3 }		= { ∅, {∅}, {∅, {∅}}, {∅, {∅}, {∅, {∅}}} }

For von Neumann, the successor of α is $\alpha \cup \{\alpha\}$.

Ernst Zermelo had used a slightly different (equivlent) definition of ordinals.

A World from Empty Bags

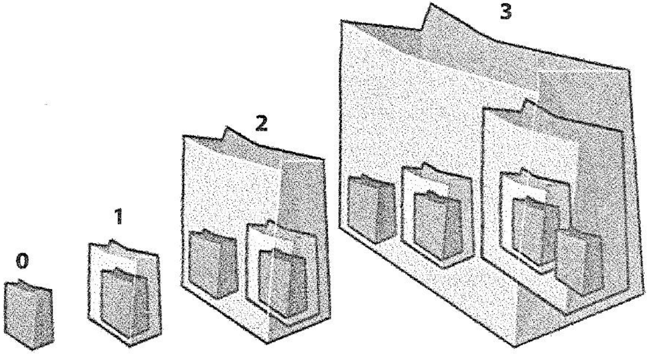

FIGURE 2.3 The empty set has no member, like an empty paper bag. But by putting the empty paper bag in a larger paper bag you can form big and bigger sets – the basis of our definition of number.

The Burali-Forti Paradox

The class of ordinal numbers is not a set.

If it were a set, it would be a member of itself, contradicting the strict ordering by membership.

Bertrand Russell noticed the contradiction. In 1903 he discussed it in his *Principles of Mathematics.*

The proper class of ordinals is variously denoted as

$$\mathrm{Ord} \quad \textbf{or} \quad \mathrm{ON} \quad \textbf{or} \quad \infty$$

Arithmetic on the Ordinals

Every well-ordered set has an ordinal number.

For infinite sets, there are many possible orderings:

$\mathrm{ord}(\{1, 2, 3, 4, \dots\}) = \omega$ **while** $\mathrm{ord}(\{2, 3, 4, \dots, 1\}) = \omega+1$

Arithmetic on the Ordinals

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The ordinals are non-commutative:

$$1 + \omega \neq \omega + 1$$

Worse still, $1 + \omega = \omega$**. One is tempted to subtract** ω **to get** $1 = 0$**.**

Not a good basis for a calculus of transfinites.

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Richard Dedekind (1831–1916)

Irrational Numbers

Richard Dedekind defined irrational numbers by means of cuts of the rational numbers $\mathbb{Q}$.

For example, $\sqrt{2}$ is defined as (L, R), where

$$\begin{aligned} L &= \{\text{All rationals less than } \sqrt{2}\} \\ R &= \{\text{All rationals greater than } \sqrt{2}\} \end{aligned}$$

More precisely, and avoiding self-reference,

$$\begin{aligned} L &= \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\} \\ R &= \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\} \end{aligned}$$

Irrational Numbers

For each irrational number there is a corresponding cut (L, R).

We can regard the cut as equivalent to the number.

There are rules to manipulate cuts that are equivalent to the arithmetical rules for numbers.

Irrational Numbers

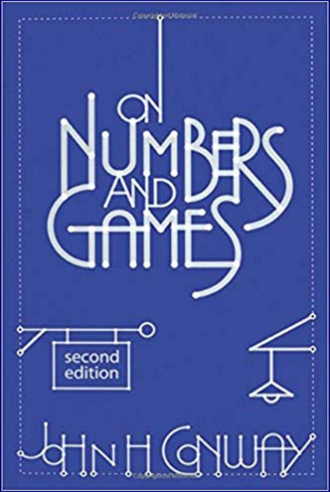
For each irrational number there is a corresponding cut (L, R).

We can regard the cut as equivalent to the number.

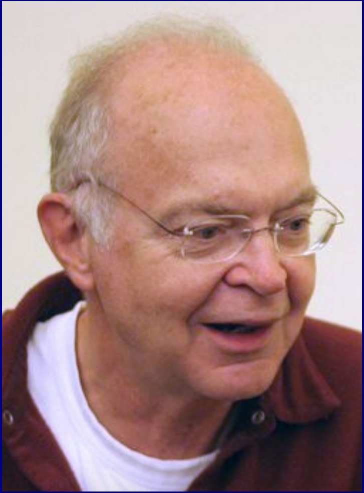
There are rules to manipulate cuts that are equivalent to the arithmetical rules for numbers.

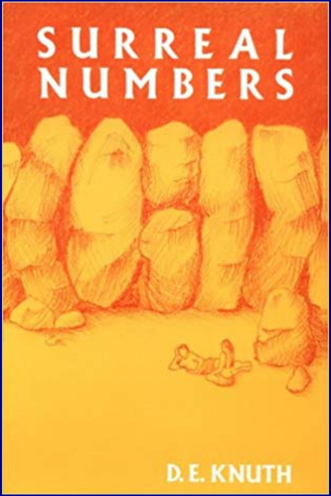
The surreal numbers are based upon a dramatic generalization of Dedekind's cuts.

John H. Conway's ONAG



Donald Knuth's *Surreal Numbers*





Constructing the Surreals

The Surreal numbers $\mathbb{S}$ are constructed **inductively.**

- Every number x is defined by a pair of sets, the left set and the right set:

$$x = \{ L \mid R \}$$

- No element of L is greater than or equal to any element of R.

x is the **simplest** number between L and R.

Constructing the Surreals

We start with 0**, defined as**

$$0 = \{\varnothing \mid \varnothing\} = \{\ \{\ \} \mid \{\ \}\ \} = \{\ \mid\ \}$$

Then 1, 2, 3 **and so on are defined as**

$$\{\,0 \mid\ \} = 1 \qquad \{\,1 \mid\ \} = 2 \qquad \{\,2 \mid\ \} = 3 \qquad \ldots$$

Negative numbers are defined inductively as

$$-x = \{-R \mid -L\,\}$$

so that

$$\{\ \mid 0\,\} = -1 \qquad \{\ \mid -1\,\} = -2 \qquad \{\ \mid -2\,\} = -3 \qquad \ldots$$

Constructing the Surreals

Dyadic fractions (of the form $m/2^n$) appear as

$$\{ 0 \mid 1 \} = \tfrac{1}{2} \quad \{ 1 \mid 2 \} = \tfrac{3}{2} \quad \{ 0 \mid \tfrac{1}{2} \} = \tfrac{1}{4} \quad \{ \tfrac{1}{2} \mid 1 \} = \tfrac{3}{4} \quad \dots$$

After an infinite number of stages,
all the dyadic fractions have emerged.

At the next stage, all other real numbers appear.

Infinite and infinitesimal numbers also appear.

Surreal Numbers

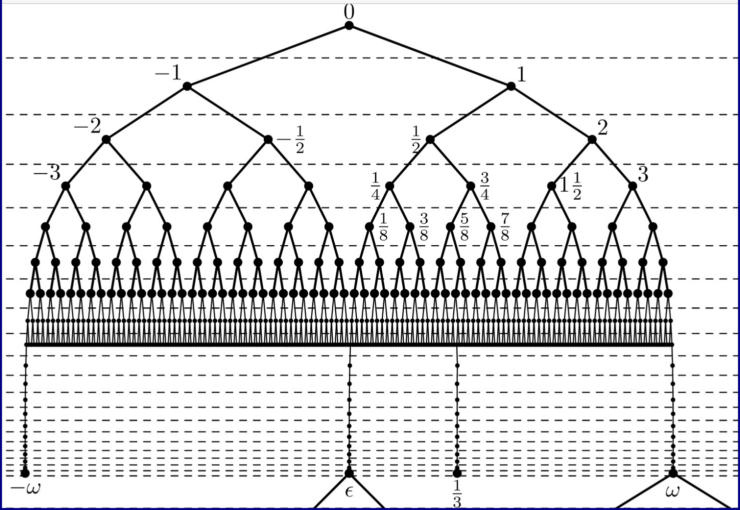


Figure: Surreal network from 0 to the first infinite number ω.

The First Infinite Number

The first infinite number ω is defined as

$$\omega = \{0, 1, 2, 3, \dots \mid \; \}$$

We can also introduce

$$\omega + 1 = \{0, 1, 2, \dots \; \omega \mid \; \}, \qquad \omega - 1 = \{0, 1, 2, \dots \; \mid \omega\}$$

$$2\omega = \{0, 1, 2, \dots \; \omega, \omega{+}1, \dots \mid \; \} \quad \tfrac{1}{2}\omega = \{0, 1, 2, \dots \; \mid \omega, \omega{-}1, \dots\}$$

and many other more exotic numbers.

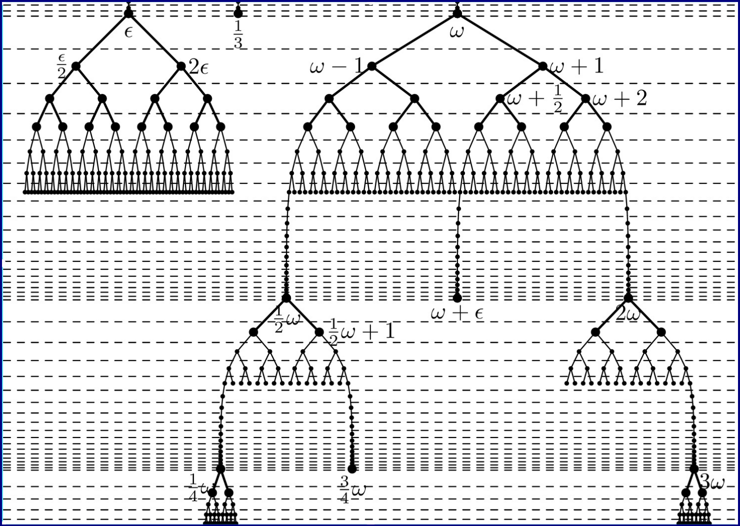


Figure: Network of early infinite and infinitesimal numbers.

Manipulating Infinite Numbers

The surreal numbers behave beautifully:
The class $\mathbb{S}$ is a totally ordered Field.

We can define quantities like

$$\omega^2 \qquad \omega^\omega \qquad \sqrt{\omega} \qquad \log \omega$$

and many even stranger numbers.

The First Infinitesimal Number $\epsilon = 1/\omega$

On day ω, the number $\epsilon = 1/\omega$ appears.

It can be shown that

$$\frac{\omega}{\omega} = \omega \times \epsilon = 1$$

Since we are interested in subsets of $\mathbb{N}$, we will consider surreals less than or equal to ω.

Closing Lines of Knuth's Book

B. Alice! Feast your eyes on this!

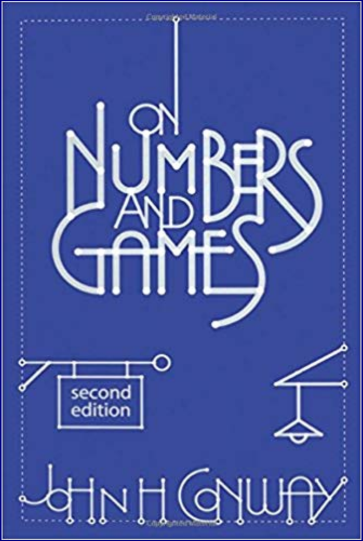
$$\sqrt{\omega} \equiv \left(\{1, 2, 3, 4, \ldots\}, \left\{ \frac{\omega}{1}, \frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega}{4}, \ldots \right\} \right);$$

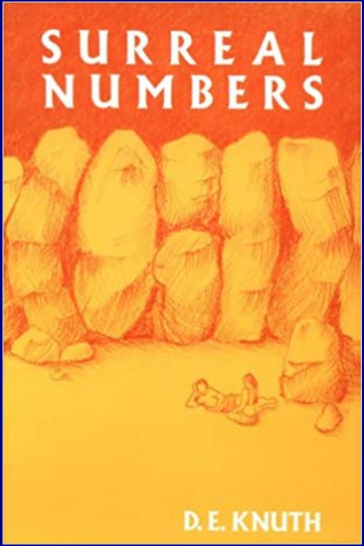
$$\sqrt{\epsilon} \equiv \left(\{\epsilon, 2\epsilon, 3\epsilon, 4\epsilon, \ldots\}, \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\} \right).$$

A. (falling into his arms) Bill! Every discovery leads to more, and more!

B. (glancing at the sunset) There are infinitely many things yet to do … and only a finite amount of time …

Books about Surreal Numbers





Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

BACKGROUND

Cardinality is a *blunt instrument*:
The natural numbers, rationals and algebraic numbers all have the same cardinality.

So, $\aleph_0$ fails to discriminate between them.

Our aim is to define a number $m(A)$ for subsets A of $\mathbb{N}$ that corresponds to our intuition about the size or magnitude of A.

We define $m(A)$ as a surreal number.

Desiderata

- For a finite subset A we have $m(A) = \mathrm{card}(A)$
- For a proper subset A of B we have

$$A \subsetneqq B \implies m(A) < m(B) \,.$$

- For the odd and even natural numbers

$$\mathbb{N}_O = \{1, 3, 5, \ldots\} \quad \implies \quad m(\mathbb{N}_O) \approx \tfrac{1}{2} m(\mathbb{N})$$
$$\mathbb{N}_E = \{2, 4, 6, \ldots\} \quad \implies \quad m(\mathbb{N}_E) \approx \tfrac{1}{2} m(\mathbb{N})$$

The Goal: A Genetic Definition

The ultimate aim is to construct a genetic definition of the magnum.

That is, for a given $A \subset \mathbb{N}$, to define two sets, L_A and R_A such that

$$m(A) = \{\ L_A \mid R_A\ \}$$

We have not been able to do this yet.

Difficulties with Limits

In ONAG (page 43), Conway states that we cannot assume the limit of the sequence $(1, 2, 3, \dots)$ is ω.

We cannot conclude that $m(\mathbb{N}) = \omega$.
Therefore, we will write $m(\mathbb{N}) = \varpi$.

The precise specification of ϖ as a surreal number in the form $\{ L \mid R \}$ remains to be done.

Euler's Number

The usual definition of Euler's number is

$$e = \lim_{n\to\infty} f(n)\,, \qquad \textbf{where} \qquad f(n) = \left(1 + \frac{1}{n}\right)^n .$$

Evaluating $f(n)$ for $n = \varpi$ we obtain a surreal number

$$e_\varpi = f(\varpi) = \left(1 + \frac{1}{\varpi}\right)^\varpi$$

which is not equal to e.

Extending Functions from $\mathbb{R}$ to $\mathbb{S}$

The extension of many functions from $\mathbb{R}$ to $\mathbb{S}$ can be done without difficulty.

$$f : x \mapsto x^2\,,\ \ x \in \mathbb{R} \qquad \textbf{to} \qquad f : x \mapsto x^2\,,\ \ x \in \mathbb{S}$$

so we have $f(\varpi) = \varpi^2$ and so on.

This is fine for polynomials, rational functions, the logarithm and trigonometric functions.

Some Examples

$$f(n) = \left(\frac{n-1}{n}\right) = 1 - \frac{1}{n} \qquad \textbf{so} \qquad f(\varpi) = 1 - \frac{1}{\varpi}$$

The value of $f(\varpi)$ may not be defined in all cases:

$$f(n) = (-1)^n \qquad \textbf{extends to} \qquad f(\varpi) = (-1)^\varpi$$

and it is not clear what the value of this should be.

We introduce the notation

$$\Lambda \equiv (-1)^\varpi$$

without (yet) defining the value to be assigned to Λ.

Numerical Examples

For the real numbers, $0.999\cdots = 1$**.**
For the surreals, this is not the case:

$$f(n) = \underbrace{0.999\ldots9}_{n \text{ terms}} = 1-10^{-n}\,, \qquad \textbf{so} \qquad f(\varpi) = 1-10^{-\varpi} < 1\,.$$

Many more examples could be given, such as

$$\begin{aligned} 0.\overline{142857} &= \frac{142{,}857}{1{,}000{,}000}\left[1 + 10^{-6} + 10^{-12} + \ldots\right] \\ &= \frac{1}{7}\left[1 - 10^{-6\varpi}\right]\,. \end{aligned}$$

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Counting Sequence

We define the characteristic function of $A \subset \mathbb{N}$ by

$$\chi_A(n) = \begin{cases} 1, & n \in A \\ 0, & \textbf{otherwise} \end{cases}$$

We assume that $a_1 < a_2 < a_3 < \cdots < a_n < \ldots$.

Definition

We define the *counting sequence* κ_A to be the sequence of partial sums of the sequence $\{\chi_A(n)\}$:

$$\kappa_A(n) = \sum_{k=1}^{n} \chi_A(k)$$

Clearly, $\kappa(n) \leq n$ and $\kappa_A(n)$ counts the number of elements of A less than or equal to n.

The Magnum of A

Definition

If $\kappa_A(x)$ is defined for $x = \varpi$, the *magnum* of $A \subset \mathbb{N}$ is

$$m(A) = \kappa_A(\varpi)$$

Note that the magnum is a surreal number.

If A is a finite set, $m(A)$ is just $\mathrm{card}(A)$.

Principal Part of $m(A)$

We denote by $M(A)$ the infinite part of $m(A)$.

We write $m(A)$ in its ***normal form.*** Then

$$m(A) = \underbrace{M(A)}_{\text{Infinite}} + \underbrace{\bigl(m(A) - M(A)\bigr)}_{\text{Finite}}$$

This can be done in a canonical manner.

To compute the magnum, we write

$$\kappa_A(n) = \pi_A(n) + \bigl(\kappa_A(n) - \pi_A(n)\bigr)$$

Then $M(A) = \pi_A(\varpi)$ (if this exists).

A Set without a Magnum

Let U be the set of natural numbers with an odd number of decimal digits.

$$\chi_U(n) = \begin{cases} 1 \text{ if } n \text{ has an odd number of decimal digits}\,, \\ 0 \text{ if } n \text{ has an even number of decimal digits}\,. \end{cases}$$

If the density of U is $\rho_U(n) = \kappa_U(n)/n$ then

$$\begin{aligned} \rho_U(1) &= 0.0 \\ \rho_U(10) &= 0.9 \\ \rho_U(100) &= 0.09 \\ \rho_U(1000) &= 0.909 \\ \rho_U(10000) &= 0.0909 \end{aligned}$$

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Intuition about Sizes

How do we 'know' that $\mathbb{N}_E$ is half the size of $\mathbb{N}$.

We do not. But we have a 'feeling' about it.

Intuition about Sizes

How do we ‘know’ that $\mathbb{N}_E$ is half the size of $\mathbb{N}$.

We do not. But we have a ‘feeling’ about it.

Why?

For any large but finite N, about half the numbers less than N are odd and about half are even.

The Odd Numbers

The characteristic sequence for the *odd numbers* is

$$\chi_O(n) = (1, 0, 1, 0, 1, 0, \dots)$$

and the counting sequence for the odd numbers is

$$\kappa_O(n) = (1, 1, 2, 2, 3, 3, \dots)$$

We can write $\chi_O(n)$ and $\kappa_O(n)$ as

$$\chi_O(n) = \frac{1-(-1)^n}{2} \qquad \text{and} \qquad \kappa_O(n) = \tfrac{1}{2}\left[n + \frac{1-(-1)^n}{2}\right]$$

Evaluating the counting function at ϖ we get

$$m(\mathbb{N}_O) = \kappa_O(\varpi) = \frac{\varpi}{2} + \frac{1}{4}\left[1-(-1)^\varpi\right] = \frac{\varpi}{2} + \frac{1}{4} - \frac{\Lambda}{4}\,.$$

The Even Numbers

We repeat this procedure for the even numbers.

$$\chi_E(n) = (0, 1, 0, 1, 0, 1, \dots)$$

$$\kappa_E(n) = (0, 1, 1, 2, 2, 3, \dots)$$

We can write these sequences as

$$\chi_E(n) = \frac{1 + (-1)^n}{2} \qquad \textbf{and} \qquad \kappa_E(n) = \tfrac{1}{2}\left[n - \frac{1 - (-1)^n}{2}\right]$$

Evaluating the counting function at ϖ we get

$$m(\mathbb{N}_E) = \kappa_E(\varpi) = \frac{\varpi}{2} - \frac{1}{4}\left[1 - (-1)^{\varpi}\right] = \frac{\varpi}{2} - \frac{1}{4} + \frac{\Lambda}{4}\,.$$

All Together

$$m(\mathbb{N}_O) = \frac{\varpi}{2} + \frac{1}{4} - \frac{\Lambda}{4}$$

$$m(\mathbb{N}_E) = \frac{\varpi}{2} - \frac{1}{4} + \frac{\Lambda}{4}$$

Assuming ϖ is an 'even number' $\Lambda = (-1)^{\varpi} = 1$ so

$$m(\mathbb{N}_O) = \frac{\varpi}{2}$$

$$m(\mathbb{N}_E) = \frac{\varpi}{2}$$

Since $\mathbb{N}_E$ and $\mathbb{N}_O$ are disjoint and $\mathbb{N}_E \cup \mathbb{N}_O = \mathbb{N}$, it is refreshing to observe that

$$m(\mathbb{N}_O) + m(\mathbb{N}_E) = \varpi = m(\mathbb{N}) \, .$$

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Zeros at the Beginning

Theorem: **Suppose the set A has magnum $m(A)$. Then the shifted sequence B defined by**

$$\chi_B(1) = 0\,, \qquad \chi_B(n) = \chi_A(n-1)\,, \quad n > 1$$

has magnum

$$m(B) = m(A) - \chi_A(\varpi)\,.$$

Corollary: **If the sequence B is shifted from A by k places, we have**

$$m(B) = m(A) - \sum_{j=1}^{k} \chi_A(\varpi + 1 - j)$$

General Arithmetic Sequence

Theorem: **The magnum of the arithmetic sequence** $A = \{a, a+d, a+2d, a+3d, \dots\}$ **is**

$$m(A) = \frac{\varpi}{d} + \left(\frac{d+1-2a}{2d}\right)$$

Squares of Natural Numbers

We now consider the set of squares of natural numbers $S = \{1, 4, 9, 16, \ldots\}$. The characteristic sequence is

$$\chi_S(n) = (1, \underbrace{0, 0}_{\text{2 zeros}}; 1, \underbrace{0, 0, 0, 0}_{\text{4 zeros}}; 1, \underbrace{0, 0, 0, 0, 0, 0}_{\text{6 zeros}}; 1, \ldots)$$

and the sequence of partial sums of this sequence is

$$\kappa(n) = (\underbrace{1, 1, 1}_{\text{3 terms}}, \underbrace{2, 2, 2, 2, 2}_{\text{5 terms}}, \underbrace{3, 3, 3, 3, 3, 3, 3}_{\text{7 terms}}, \ldots)$$

Theorem: **The magnum of the sequence of squares is**

$$m(S) = \sqrt{\varpi} - \tfrac{1}{2} + \text{HOT}\,.$$

General Geometric Sequence

We now consider the general geometric sequence

$$G = \{\beta r, \beta r^2, \beta r^3, \dots\}$$

Theorem: The magnum of the geometric sequence $G = \{\beta r, \beta r^2, \beta r^3 \dots\}$ is

$$m(G) = \frac{\ln \varpi}{\ln r} - \left(\frac{\ln \beta}{\ln r} + \frac{1}{2}\right) .$$

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Analysis on $\mathbb{S}$. Paper of RSS

Journal of Logic & Analysis 6:5 (2014) 1–39
ISSN 1759-9008

Analysis on Surreal Numbers

SIMON RUBINSTEIN-SALZEDO
ASHVIN SWAMINATHAN

Analysis on $\mathbb{S}$

This paper [RSS] attempts to extend the application of surreals to functions, limits, derivatives, power series and integrals.

- **A new definition of surreal numbers.**
- **A formula for the limit of a sequence.**
- **Characterization of convergent sequences.**
- **A new topology on $\mathbb{S}$.**
- **An Intermediate Value Theorem proved (even though $\mathbb{S}$ is not Cauchy complete).**

Background

The arithmetic and algebraic properties of $\mathbb{S}$ are now well understood:

- **Harry Gonshor found a definition of $\exp(x)$.**
- **Martin Kruskal found a definition of $1/x$.**
- **Clive Bach found a definition of $\sqrt{x}$.**

Analysis on $\mathbb{S}$ is the next big step.

Notation and Basic Properties

- $\mathbb{S}_{<a}$ is the class of surreals less than a.
- $\mathbb{S}_{>a}$ is the class of surreals greater than a.

Representations of the form $\{\ L \mid R\ \}$, where L and R are sets, are known as *genetic formulae.*

Notation and Basic Properties

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Representations of the form $\{ L \mid R \}$, where L and R are sets, are known as *genetic formulae.*

Ordinals are numbers of the form $\{ L \mid \ \}$.
The right hand set is empty!

Every surreal x can be uniquely expressed in *normal form* as a sum over ordinals:

$$x = \sum_{i \in \mathbb{S}_{<\beta}} r_i \cdot \omega^{y_i}$$

r_i are real numbers and y_i a decreasing sequence.

Gaps

The surreal number line is riddled with gaps.
Gaps are Dedekind sections on $\mathbb{S}$.

All gaps are born on day **On**.

The **Dedekind completion of $\mathbb{S}$**, denoted $\mathbb{S}^{\mathfrak{D}}$ contains all numbers and gaps.

Noteworthy gaps include

- **On** $= \{\mathbb{S} | \quad \}$, the gap larger than all surreals
- **Off** $= -$**On**, the gap smaller than all surreals
- $\infty = \{$neg. and finite pos. nums. $|$ inf. pos. nums.$\}$

A sequence is of length **On** if its elements are indexed over the proper class of ordinals **On**.

Open Sets. Topology

RSS define open sets:

- The empty set is open
- A nonempty subinterval of $\mathbb{S}$ is open if
  - It has endpoints in $\mathbb{S} \cup \{\textbf{On}, \textbf{Off}\}$
  - It does not contain its endpoints.
- A subclass $A \subset \mathbb{S}$ is open if it is a union of open intervals A_i indexed over a proper set I.

This definition produces a topology on $\mathbb{S}$.

Now we can define limits and continuity for $f : a \to \mathbb{S}$.

Sequences and Limits

For any formula $\{\ L \mid R\ \}$ for the limit of an **On**-length sequence at least one of L and R is a proper class.

Since this is not a number as defined by Conway, a new definition is needed.

Definition: For any $x \in \mathbb{S}^{\mathfrak{D}}$, the Dedekind representation of x is $\{\mathbb{S}_{<x}|\mathbb{S}_{>x}\}$.

All the usual properties of $\mathbb{S}$ still hold.

Limits of sequences and functions are now defined as certain Dedekind representations.

They are equivalent to the usual ϵ-δ definitions for sequences or functions that approach numbers.

Derivatives and Integrals

Limits are defined generically as numbers (or gaps):

$$\{ \text{ Left Class } \mid \text{ Right Class } \}.$$

Derivatives can be evaluated using the definitions.

A genetic definition or Dedekind representation
of Riemann integration is still outstanding.

Open Questions

RSS provide a list of open issues that includes:

- Sums of general series.
- Genetic formula for definite integrals.
- Definitions of other transcendental functions.
- Theory of differential equations.
- Surreal version of Stokes' Theorem.
- Genetic definition of the magnum.

Outline

Introduction

Georg Cantor

Ordinal Numbers

Surreal Numbers

Magnums: Counting Sets with Surreals

Definitions

Odd and Even Numbers

Some Simple Theorems

Analysis on $\mathbb{S}$

Finis

Opportunities

Many open challenges in analysis on $\mathbb{S}$.

May be crucial in physics.

Good projects for students.

Thank you