Magnums

Counting Sets with Surnatural Numbers

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Outline

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Magnums and Subsets of $\mathbb{N}$

The aim of this work is to define a number

$$m(A)$$

for subsets A of $\mathbb{N}$ that corresponds to our **intuition** about the size or magnitude of A.

We call $m(A)$ the **magnum of A**.

Magnum = Magnitude Number

Galileo Galilei (1564–1642)

Every number n can be matched with its square n^2.

In a sense, there are as many squares as whole numbers.

1	2	3	4	5	6	7	8	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	...
1	4	9	16	25	36	49	64	...

Outline

Infinite Sets

We take the natural numbers and the even numbers

$$\mathbb{N} := \{1, 2, 3, \ldots\}$$

$$2\mathbb{N} := \{2, 4, 6, \ldots\}$$

By associating each number with its double,

$$n \in \mathbb{N} \longleftrightarrow 2n \in 2\mathbb{N}$$

we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[2\mathbb{N}]\,.$$

Counterintuitive

But

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[2\mathbb{N}]\,.$$

is paradoxical: The set of natural numbers properly contains all the even numbers

$$2\mathbb{N} \subsetneq \mathbb{N}\,.$$

But $\mathbb{N}$ also contains all the odd numbers:

$$\mathbb{N} = 2\mathbb{N} \uplus (2\mathbb{N} - 1)\,.$$

In an intuitive sense, $\mathbb{N}$ is larger than $2\mathbb{N}$.

Review of Background

Cardinality is a *blunt instrument*:

The natural numbers, rationals and algebraic numbers all have the same cardinality.

So, $\aleph_0$ fails to discriminate between them.

Our aim is to define a number $m(A)$ for sets $A \subset \mathbb{N}$ that corresponds to our *intuition.*

Review of Background

Cardinality is a *blunt instrument*:

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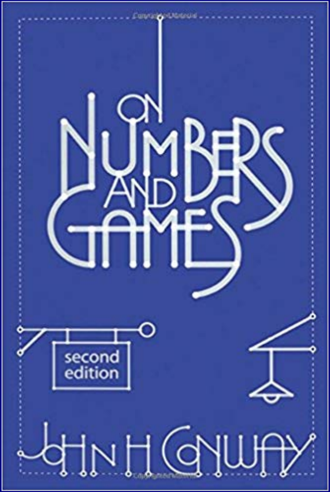
Our aim is to define a number $m(A)$ for sets $A \subset \mathbb{N}$ that corresponds to our *intuition.*

"It is by logic that we prove,
but by intuition that we discover."
[Henri Poincaré]

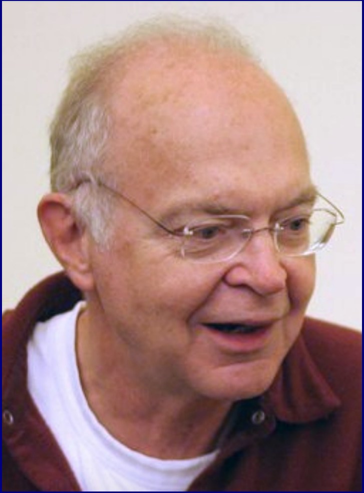
We will define $m(A)$ as a surreal number.

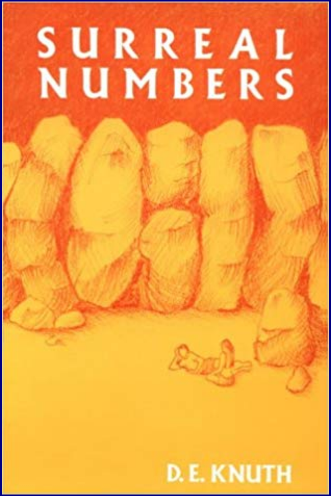
Outline

John H. Conway's ONAG [1976 / 2001]



Donald Knuth's *Surreal Numbers* [1974]





Constructing the Surreals

The Surreal numbers **No** are constructed **inductively**, using just **two simple rules:**

1. Every **new number** x is defined by a pair of sets of **old numbers**, the left set and the right set:

$$x = \{ \, L_x \mid R_x \, \}$$

2. No element of the left set L_x is greater than or equal to any element of the right set R_x.

Then x is the **simplest** number between L_x and R_x.

Constructing the Surreals

We start by defining the number zero as

$$0 = \{\varnothing \mid \varnothing\} = \{\ \mid\ \}$$

Then $1, 2, 3$ and so on are defined as

$$\{\,0 \mid\ \} = 1 \qquad \{\,1 \mid\ \} = 2 \qquad \{\,2 \mid\ \} = 3 \qquad \ldots$$

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Negative numbers are defined inductively as

$$-x = \{-R \mid -L\,\},$$

so that, for example,

$$\{\ \mid 0\,\} = -1 \qquad \{\ \mid -1\,\} = -2 \quad \ldots$$

Constructing the Surreals

Dyadic fractions (of the form $m/2^n$) appear as

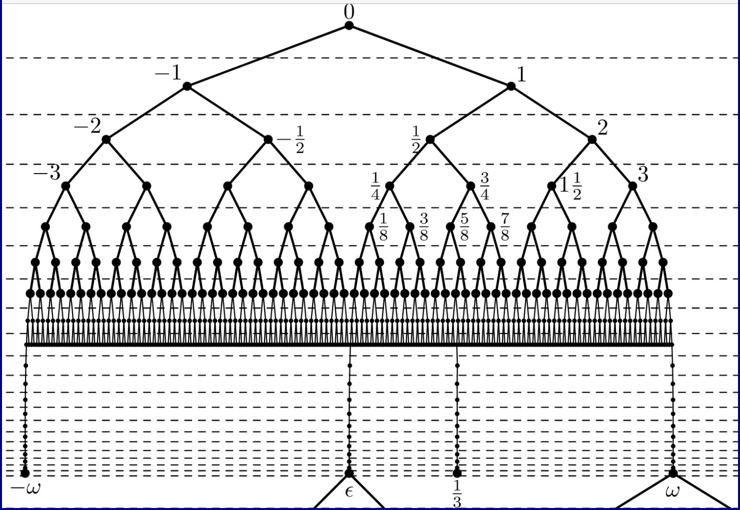
$$\{\,0 \mid 1\,\} = \tfrac{1}{2} \quad \{\,1 \mid 2\,\} = \tfrac{3}{2} \quad \{\,0 \mid \tfrac{1}{2}\,\} = \tfrac{1}{4} \quad \{\,\tfrac{1}{2} \mid 1\,\} = \tfrac{3}{4} \quad \ldots$$

Over an infinite number of stages,
all the dyadic fractions emerge.

At that stage, all other real numbers appear.

Infinite and infinitesimal numbers also appear.

Surreal Numbers



Surreal network from 0 to the first infinite number ω.

[Image: Wikimedia Commons]

The First Infinite Number

The first infinite number ω appears on Day ω:

$$\omega = \{0, 1, 2, 3, \ldots \mid \ \}$$

On following days, we get

$$\begin{aligned} \omega + 1 &= \{0, 1, 2, \ldots \ \omega \mid \ \} \\ \omega - 1 &= \{0, 1, 2, \ldots \mid \omega\} \\ 2\omega &= \{0, 1, 2, \ldots \ \omega, \omega + 1, \ldots \mid \ \} \\ \tfrac{1}{2}\omega &= \{0, 1, 2, \ldots \mid \omega, \omega - 1, \ldots\}, \end{aligned}$$

and many other more exotic numbers.

Manipulating Infinite Numbers

The **Class of Surreal Numbers** is denoted **No**.

Conway defined arithmetic operations on **No** such that surreal numbers behave beautifully:

The Class No is a totally ordered Field.

We can define quantities like

$$\omega^2 \qquad \omega^\omega \qquad \sqrt{\omega} \qquad \log \omega$$

and many even stranger numbers.

The Omnific Integers Oz

Conway (ONAG, Ch. 5) defined the Class **Oz** of **omnific integers**: $x \in$ No is an omnific integer if

$$x = \{x - 1 | x + 1\} .$$

So x is the **simplest number** between $x - 1$ and $x + 1$.

The omnifics greatly extend the real integers $\mathbb{Z}$:

$$\mathbb{Z} \subset \textbf{Oz}$$

Omnifics Oz are the appropriate integers for No.

The Surnatural Numbers Nn

The positive omnific numbers are called the surnatural numbers:

$$\mathbf{Nn} := \mathbf{Oz}^{+}\,.$$

The magnum m maps sets to the surnatural numbers:

$$m : \mathscr{P}(\mathbb{N}) \to \mathbf{Nn}\,.$$

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$$m : \mathscr{P}(\mathbb{N}) \to \mathbf{Nn} \, .$$

Since $\omega/2 \in \mathbf{Nn}$, ω is an even number. Moreover, ω is a multiple of 3, of 4, of k.

Since $\sqrt[k]{\omega} \in \mathbf{Nn}$, ω is a perfect square, a perfect cube, and a perfect k-th power.

Outline

Two Approaches to Defining Magnums

We will develop **two distinct approaches**
to the definition of set magnums:

- The Incremental or **Genetic Approach**,
- Extension of the **Counting Function**.

The two approaches are compatible,
and yield identical values for $m(A)$.

The Magnum Form

We seek a general expression in the form

$$m(A) = \{m(B) : B \subset A \mid m(C) : A \subset C\} ,$$

where

- All the subsets B of A are on the left and
- All the supersets C of A are on the right.

This form guarantees **The Euclidean Principle.**

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This form guarantees The Euclidean Principle.

However, ...

Definition of $m(A)$ Step-by-step

We will construct $m(A)$ in **incremental fashion.**

We use the magnums of 'old' sets to generate the magnums of 'new' sets!

For each ordinal number α, we define three families:

- $\mathscr{M}_\alpha$: **Made sets** magnumbered on or before Day α,
- $\mathscr{N}_\alpha$: **New sets,** magnumbered on Day α, and
- $\mathscr{O}_\alpha$: **Old sets,** magnumbered before Day α.

Definition of $m(A)$ Step-by-step

For each ordinal γ, on Day γ we define a premagnum:

$$m_\gamma(A) = \{m(B) : B \in \mathscr{O}_\gamma, B \subset A \mid m(C) : C \in \mathscr{O}_\gamma, A \subset C\}\,.$$

The proper subsets B and supersets C range over all sets magnumbered prior to Day γ.

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The proper subsets B and supersets C range over all sets **magnumbered** prior to Day γ.

When a stage $\gamma = \alpha$ is reached where $m_\gamma(A)$ cannot undergo further changes, we define

$$m(A) := m_\alpha(A)$$

and call α the **Birthday** of $m(A)$.

Birthdays of the Magnums

When is the magnum of a subset of $\mathbb{N}$ first defined?
To answer, we consider the ordinals as they arise:

Day 0: The magnum of $\varnothing$ is defined to be 0.

Day 1: Magnums of all singletons $\{n\}$ defined to be 1.

Day 2: Magnums of all doubletons $\{m, n\}$ equal to 2.

Day n: All sets with n elements have magnum n.

Finite subsets of $\mathbb{N}$ are magnumbered on finite days.
Their magnums are all the finite ordinal numbers.

Day ω: The set $\mathbb{N}$ is given a magnum on this day:
$m(\mathbb{N}) = \omega$, the first infinite magnum.

Calendar for Magnumbering Sets

Day #	Magnum	Sets
0	0	$\varnothing$
1	1	$\{k\}$, $\cong$
2	2	$\{k, \ell\}$, $\cong$
3	3	$\{k, \ell, m\}$, $\cong$
$\dots$	$\dots$	$\dots$
n	n	$\{m_1, m_2, \dots, m_n\}$, $\cong$
$\dots$	$\dots$	$\dots$
ω	ω	$\mathbb{N}$
$\omega + 1$	$\omega - 1$	$\mathbb{N} \setminus \{k\}$, $\cong$
$\omega + 2$	$\omega - 2$	$\mathbb{N} \setminus \{k, \ell\}$, $\cong$
$\dots$	$\dots$	$\dots$
$\omega + n$	$\omega - n$	$\mathbb{N} \setminus \{m_1, m_2, \dots, m_n\}$, $\cong$
$\dots$	$\dots$	$\dots$
2ω	$\frac{\omega}{2}$	$2\mathbb{N}$, $2\mathbb{N} - 1$, $\cong$

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Defining $m(A)$ using Density

The density of a set $A \subset \mathscr{P}(\mathbb{N})$ is

$$\rho_A = \lim_{n \to \infty} \frac{\kappa_A(n)}{n}$$

We might attempt to define the magnum of A as

$$m(A) := \rho_A \cdot \omega \,.$$

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$$m(A) := \rho_A \cdot \omega \,.$$

There are serious limitations with this:

For example, for $A = \{n^2 : n \in \mathbb{N}\}$ we have

$$\rho_A = 0 \qquad \text{so} \qquad m(A) = 0.$$

We must consider other ways to evaluate $\kappa_A(\omega)$.

Outline

Difficulties with Limits

Conway states (ONAG, page 43) that we cannot assume the limit of $(1, 2, 3, \dots)$ is ω.

Therefore, we cannot conclude that $m(\mathbb{N}) = \omega$.

Limits don't work for the surreal numbers.

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Limits don't work for the surreal numbers.

Nonstandard analysis depends on a Transfer Axiom.

In a nut-shell, this states that (first-order) properties of real numbers also hold for hyper-real numbers.

There is no Transfer Axiom for the surreals.

Example: $\sqrt{2}$ is a rational number in No.

Extending Functions from $\mathbb{N}$ to Nn

We define the counting function $\kappa_A : \mathbb{N} \to \mathbb{N}$ thus:

$\kappa_A(n) =$ Number of terms of A less than or equal to n.

Sometimes, the extension to Nn is obvious:

$\kappa : n \mapsto n^2\ ,\ n \in \mathbb{N}$ to $\widehat{\kappa} : \nu \mapsto \nu^2\ ,\ \nu \in$ Nn .

so we have $\widehat{\kappa}(\omega) = \omega^2$.

The Extension Axiom generalizes this idea.

The Axiom of Extension:

A function $f : \mathbb{N} \to \mathbb{N}$ is a recipe, rule or algorithm;
Given an <u>input</u> in $\mathbb{N}$, f produces an <u>output</u> in $\mathbb{N}$.

The Axiom of Extension states that it is possible to extend the domain of f to Nn.

For functions with a "natural" extension to Nn — for example, polynomials and logarithms — the Extension Axiom is superfluous.

In view of that, we omit technicalities.

The Axiom of Extension [OMIT]

For any functions $f : \mathbb{N} \to \mathbb{N}_0$ and $g : \mathbb{N} \to \mathbb{N}_0$, there exist extensions $\widehat{f} : \mathbf{Nn} \to \mathbf{Nn}$ and $\widehat{g} : \mathbf{Nn} \to \mathbf{Nn}$ such that

$$
\begin{aligned}
f(n) \overset{\rightarrow}{=} g(n) \quad &\Longrightarrow \quad \widehat{f}(\nu) = \widehat{g}(\nu) \text{ for } \nu \in \mathbf{Nn} \setminus \mathbb{N} \\
f(n) \overset{\rightarrow}{<} g(n) \quad &\Longrightarrow \quad \widehat{f}(\nu) < \widehat{g}(\nu) \text{ for } \nu \in \mathbf{Nn} \setminus \mathbb{N}
\end{aligned}
$$

and the extension preserves sums and products:

$$
(\widehat{f+g})(\nu) := \widehat{f}(\nu) + \widehat{g}(\nu) \quad \text{and} \quad (\widehat{f \cdot g})(\nu) := \widehat{f}(\nu) \cdot \widehat{g}(\nu).
$$

Defining the Magnum of A

The **defining function** of the sequence $A = (a_n)_n$ is

$$\alpha_A(n) := a_n \,.$$

The **counting function** may be expressed as

$$\kappa_A(n) = \lfloor \alpha_A^{-1}(n) \rfloor \,.$$

If κ_A is extended to **Nn**, we can define the magnum of A to be:

$$m(A) := \widehat{\kappa}_A(\omega)$$

Outline

Some Theorems

We have proved several useful theorems:

- $A \subset B \implies m(A) < m(B)$ **(Euclidean Principle).**
- $m(A \uplus B) = m(A) + m(B)$ **(Finite Additivity).**
- **A Density Theorem relates** $m(A)$ **to** ρ_A**.**
- $\{m(B) \,|\, m(C)\} = \widehat{\kappa}_A(\omega)$ **(Methods are Consistent).**
- **The General Isobary Theorem.**
- $m(U \times V) = m(U) \cdot m(V)$**.**

Some Theorems

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- $\{m(B) \mid m(C)\} = \widehat{\kappa}_A(\omega)$ **(Methods are Consistent).**
- **The General Isobary Theorem.**
- $m(U \times V) = m(U) \cdot m(V)$.
- **For Larger Sets:**
  - $m(\mathbb{N}) = \omega \implies m(\mathbb{Z}) = 2\omega + 1$.
  - $m(\mathbb{N} \times \mathbb{N}) = \omega^2$.
  - **With banded ordering of $\mathbb{N}^2$, $m(\mathbb{Q}) = O(\omega^{4/3})$.**

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Examples of Magnums

$A = k\mathbb{N} = \{kn : n \in \mathbb{N}\}$	$m(A) = \omega / k$
$A = \mathbb{N}^{(k)} = \{n^k : n \in \mathbb{N}\}$	$m(A) = \sqrt[k]{\omega}$
Arit. Seq. $A = \{k^n + \ell : n \in \mathbb{N}\}$	$m(A) = \left\lfloor \frac{\omega}{k} - \frac{\ell}{k} \right\rfloor$
Geom. Seq. $A = \{r^n : n \in \mathbb{N}\}$	$m(A) = \lfloor \log_r \omega \rfloor$
Prime Numbers $\{p_n : n \in \mathbb{N}\}$	$m(A) \approx \lfloor \omega / \log \omega \rfloor$
Fibonacci Numbers $\lfloor \varphi^n / \sqrt{5} \rceil$	$m(A) \approx \lfloor \log_\varphi (\sqrt{5}\omega) \rfloor$

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Conclusions

We have found magnums for a wide range of sets.

But there are many sets for which we
are unable to calculate the magnums.

- Does every subset of $\mathbb{N}$ have a magnum?
- Does every countable set have a magnum?

These questions remain to be answered.

Opportunities

- Great projects for students.
- Many open problems and challenges.
- Analysis over surreals is far from complete.
- Surreals must eventually be of value in physics!

Slides of Talk

Magnums: Counting Sets with Surnatural Numbers

https://maths.ucd.ie/~plynch/Talks/

Google for "Peter Lynch UCD" and click on "Talks"

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Thank you