Numerical Integration using Laplace Transforms

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Outline

Prehistory of NWP

ENIAC Integrations

Laplace Transform Scheme

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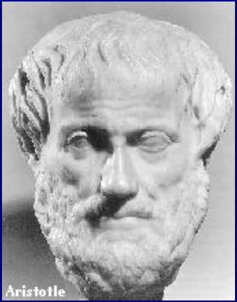
Aristotle's *Meteorologia*

Aristotle wrote the first book on Meteorology, the *Μετεωρολογια* (*μετεωρον*: Something in the air).

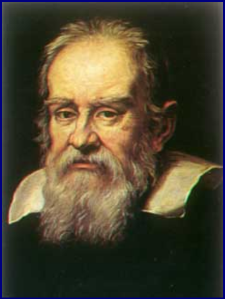
This work studied the causes of various weather phenomena.



Aristotle (384-322 BC)

Galileo Galilei (1564–1642)

Galileo formulated the basic law of falling bodies, which he verified by careful measurements.



He constructed a telescope, with which he studied lunar craters, and discovered four moons revolving around Jupiter.

Galileo is credited with the invention of the Thermometer.

Thus began quantitative meteorology.

Galileo's Star Student

Evangelista Torricelli (1608–1647), a student of Galileo, devised the first accurate **barometer**.

The link between pressure and the weather was soon noticed.

Torricelli inventing the barometer

Pascal and Puy de Dome

Pascal demonstrated the change of pressure with height.

Newton's Law of Motion

The rate of change of momentum of a body is equal to the sum of the forces acting on the body:

Force = Mass × Acceleration

A Tricky Question

If Astronomers can make accurate long-range forecasts, why can't Meteorologists do the same?

A Tricky Question

If Astronomers can make accurate long-range forecasts, why can't Meteorologists do the same?

- **Size of the Problem**

  Cometary motion is a relatively simple problem, with few degrees of freedom;
  Dynamics is enough.

  The atmosphere is a continuum with infinitely many variables;
  Thermodynamics is essential.

- **Order versus Chaos**

  The equations of the solar system are quasi-integrable and the motion is regular.

  The equations of the atmosphere are essentially nonlinear and the motion is chaotic.

Leonhard Euler (1707–1783)

- Born in Basel in 1707.
- Died 1783 in St Petersburg.
- Formulated the equations for incompressible, inviscid fluid flow:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla p = \mathbf{g} .$$

$$\nabla \cdot \mathbf{V} = 0$$

The Navier-Stokes Equations

Euler's Equations:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla p = \mathbf{g}^{\star} .$$

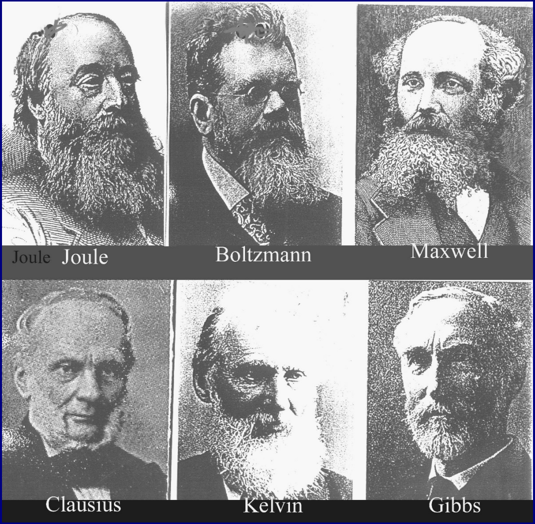
The Navier-Stokes Equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{V} + \mathbf{g}^{\star} .$$

Motion on the rotating Earth:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + 2\Omega \times \mathbf{V} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{V} + \mathbf{g} .$$

The Inventors of Thermodynamics



The Equations of the Atmosphere

GAS LAW (Boyle's Law and Charles' Law.)

Relates the pressure, temperature and density

CONTINUITY EQUATION

Conservation of mass

WATER CONTINUITY EQUATION

Conservation of water (liquid, solid and gas)

EQUATIONS OF MOTION: Navier-Stokes Equations

Describe how the change of velocity is determined by the pressure gradient, Coriolis force and friction

THERMODYNAMIC EQUATION

Determines changes of temperature due to heating or cooling, compression or rarefaction, etc.

Seven equations; seven variables (u, v, w, ρ, p, T, q).

Scientific Forecasting in a Nut-Shell

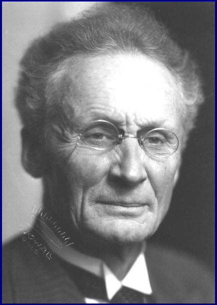
- The atmosphere is a physical system
- Its behaviour is governed by the laws of physics
- These laws are expressed quantitatively in the form of mathematical equations
- Using observations, we can specify the atmospheric state at a given initial time:
  "Today's Weather"
- Using the equations, we can calculate how this state will change over time:
  "Tomorrow's Weather"

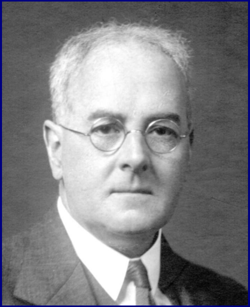
Scientific Forecasting in a Nut-Shell

Problems:

- The equations are very complicated (non-linear) and a powerful computer is required to do the calculations
- The accuracy decreases as the range increases; there is an inherent limit of predictibility.

Pioneers of Scientific Forecasting





Cleveland Abbe, Vilhelm Bjerknes, Lewis Fry Richardson

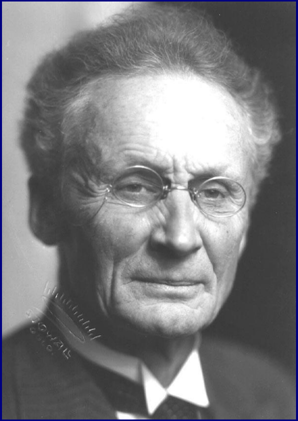
Cleveland Abbe

By 1890, the American meteorologist Cleveland Abbe had recognized that:

Meteorology is essentially the application of hydrodynamics and thermodynamics to the atmosphere.

Abbe proposed a mathematical approach to forecasting.

Vilhelm Bjerknes

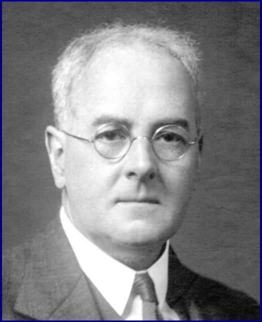


A more explicit analysis of weather prediction was undertaken by the Norwegian scientist Vilhelm Bjerknes

He identified the two crucial components of a scientific forecasting system:

- Analysis
- Integration

Lewis Fry Richardson



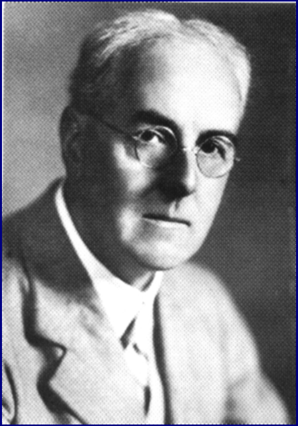
The English Quaker scientist Lewis Fry Richardson attempted a **direct solution of the equations of motion.**

He dreamed that numerical forecasting would become a practical reality.

Today, forecasts are prepared routinely using his methods . . .

. . . his dream has indeed come true.

Lewis Fry Richardson, 1881–1953.



During WW I, Richardson computed by hand the pressure change at a single point.

His 'forecast' was a catastrophic failure:

$\Delta p =$ 145 hPa in 6 hrs

But Richardson's method was scientifically sound.

Initialization of Richardson's Forecast

Richardson's Forecast has been repeated using a modern computer.

The atmospheric observations for 20 May, 1910, *were recovered from original sources.*

- **ORIGINAL:** $$\frac{dp_s}{dt} = +\mathbf{145}\,\mathrm{hPa}/6\,\mathrm{h}$$
- **INITIALIZED:** $$\frac{dp_s}{dt} = -\mathbf{0.9}\,\mathrm{hPa}/6\,\mathrm{h}$$

Observations: The barometer was steady!

Outline

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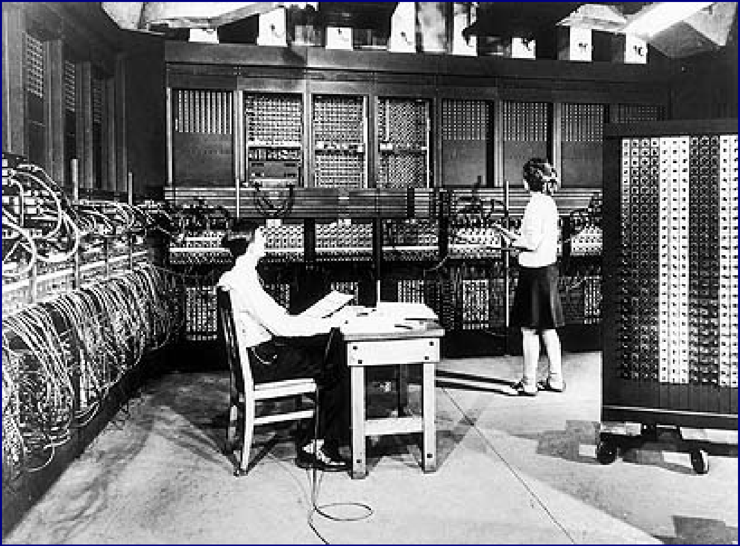
ENIAC Integrations

Laplace Transform Scheme

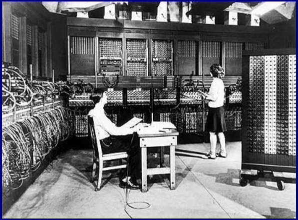
Crucial Advances, 1920–1950

- Dynamic Meteorology
  - Quasi-geostrophic Theory
- Numerical Analysis
  - CFL Criterion
- Atmopsheric Observations
  - Radiosonde
- Electronic Computing
  - ENIAC

The ENIAC



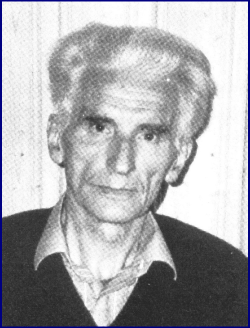
The ENIAC



The **ENIAC** was the first multi-purpose programmable electronic digital computer.
It had:

- 18,000 vacuum tubes
- 70,000 resistors
- 10,000 capacitors
- 6,000 switches
- Power: 140 kWatts

Charney Fjørtoft von Neumann



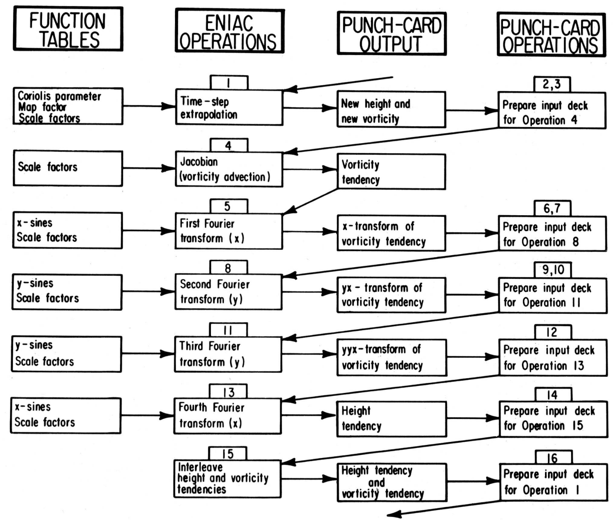
Numerical integration of the barotropic vorticity equation
***Tellus*, 2, 237–254 (1950).**

Charney, et al., *Tellus*, 1950.

- The atmosphere is treated as a single layer.
- The flow is assumed to be nondivergent.
- Absolute vorticity is conserved.

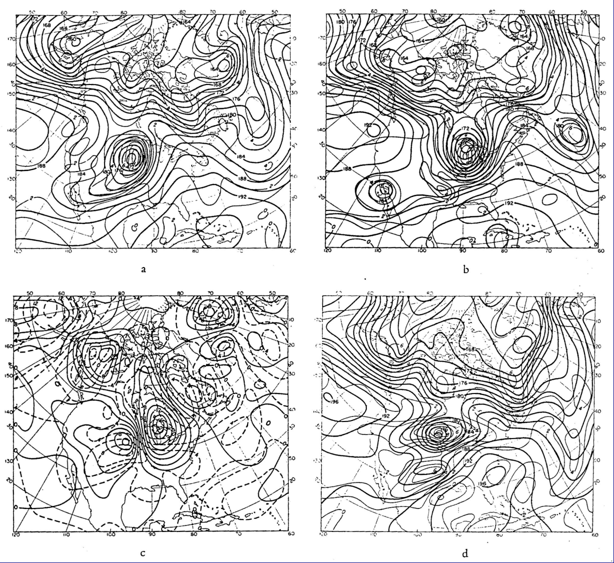
$$\frac{d(\zeta + f)}{dt} = 0.$$

The ENIAC Algorithm: Flow-chart



G. W. Platzman: The ENIAC Computations of 1950 — Gateway to Numerical Weather Prediction (BAMS, April, 1979).

ENIAC Forecast for Jan 5, 1949



NWP Operations

The Joint Numerical Weather Prediction Unit was established on July 1, 1954:

- Air Weather Service of US Air Force
- The US Weather Bureau
- The Naval Weather Service.

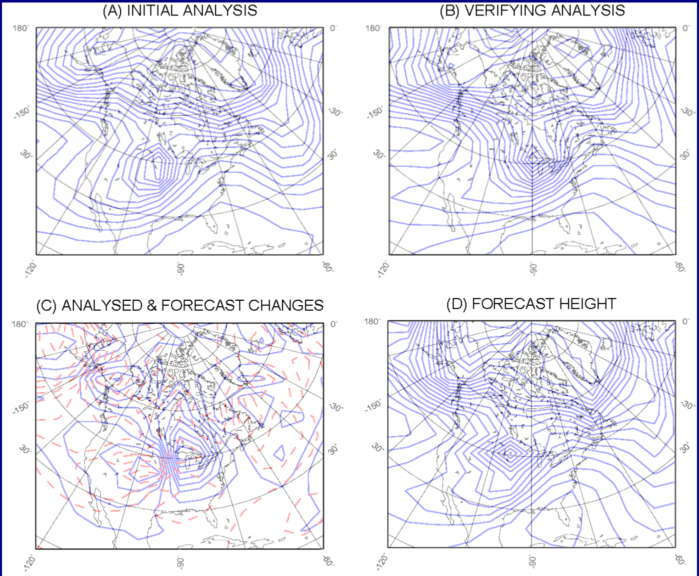
Operational numerical weather forecasting began in May 1955, using a 3-level quasi-geostrophic model.

Recreating the ENIAC Forecasts

The ENIAC integrations have been repeated using:

- **A MATLAB program to solve the BVE**
- **Data from the NCEP/NCAR reanalysis**

Recreation of the Forecast

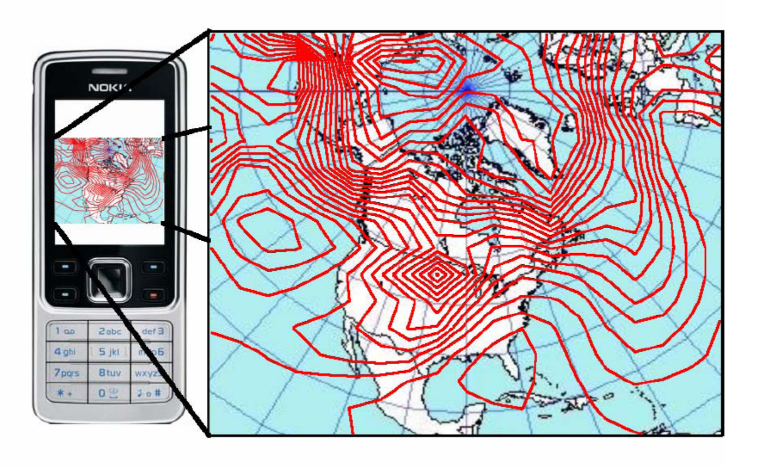


Computing Time for ENIAC Runs

- On ENIAC, a 24 hour forecast took about 24 hours computing time.
- The program ENIAC.M was run on a Sony Vaio (model VGN-TX2XP)
- The main loop of the 24-hour forecast ran in about 30 ms.
- More recently, run on a mobile phone: **PHONIAC**. Run time 75 ms.

Lynch, Peter, 2008: The ENIAC Forecasts: A Recreation. *Bull. Amer. Met. Soc.*, 89, 45–55.

PHONIAC: Portable Hand Operated Numerical Integrator and Computer



Notices of the AMS (the other AMS!)

Cover of the September 2013 issue of Notices of the American Mathematical Society.

See also Weather, November 2008.

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Laplace Transform Scheme

The Laplace Transform Scheme

LT scheme developed for initialization (c. 1984)

LT scheme applied to forecasting by J. Van Isacker and W. Struylaert (c. 1985).

Published in *WMO/IUGG Symposium Proceedings*, Tokyo, 1986.

Some References

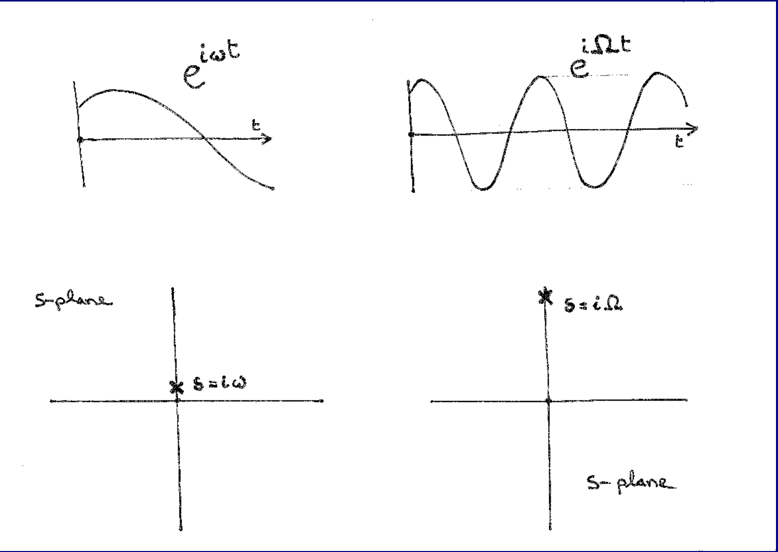
- Van Isacker J and Struylaert W., 1985:
  Numerical Forecasting using Laplace Transforms.
  Publications Serie A 115.
  Institut Royal Meteorologique de Belgique, Brussels.
- Van Isacker J and Struylaert W. 1986:
  Laplace Transform applied to a baroclinic model.
  In *Short- and Medium-Range Numerical Weather Prediction*, Proceedings of IUGG NWP Symposium.
  Matsuno T. (ed.) Meteorol. Soc. Japan, Tokyo.
  247–253.

The Laplace Transform: Definition

For a function of time $f(t),\ t \geq 0$, the LT is defined as

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t \,.$$

Here, s is complex and $\hat{f}(s)$ is a complex function of s.



LF and HF oscillations and their transforms

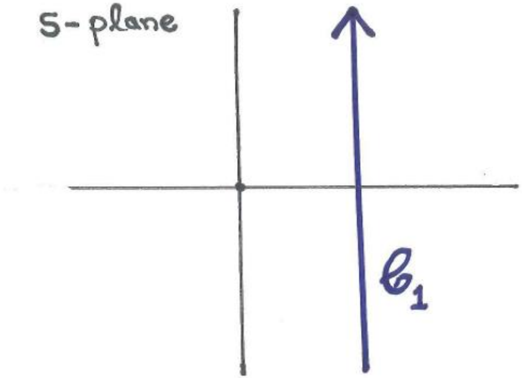
The Laplace Transform: Inversion

The **inversion formula** is

$$f(t) = \frac{1}{2\pi i} \int_{\mathcal{C}_1} e^{st} \hat{f}(s) \, \mathrm{d}s \, .$$

where $\mathcal{C}_1$ is a contour in the s-plane.

Contour for inversion of Laplace Transform



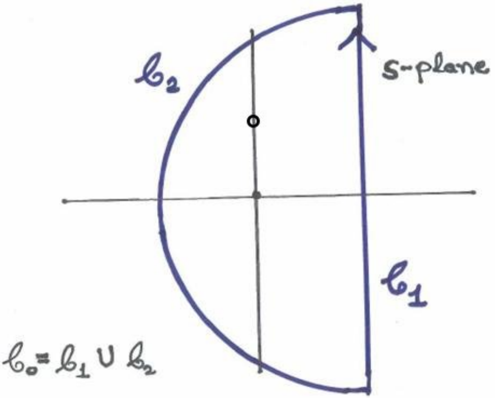
For an integral around a closed contour,

$$f(t) = \frac{1}{2\pi i} \oint_{\mathcal{C}_0} \frac{\alpha \exp(st)}{s - i\omega} \, \mathrm{d}s \,,$$

we can apply the residue theorem:

$$f(t) = \sum_{\mathcal{C}_0} \left[\textbf{Residues of } \left(\frac{\alpha \exp(st)}{s - i\omega} \right) \right]$$

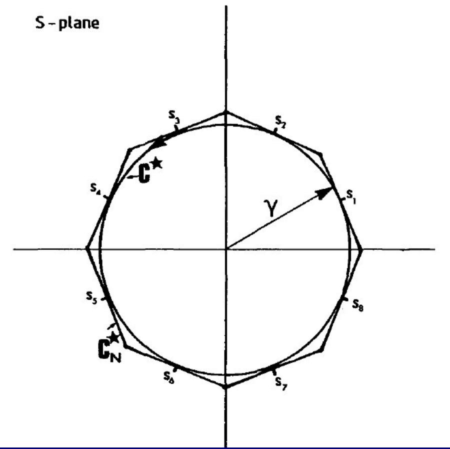
Closed Contour



Contribution from $\mathcal{C}_2$ vanishes in limit of infinite radius

Approximating the Contour $\mathcal{C}^\star$

We replace the circle $\mathcal{C}^\star$ by an N-gon $\mathcal{C}_N^\star$:



Numerical approximation: the inverse

$$\mathcal{L}^{\star}\{\hat{f}(s)\} = \frac{1}{2\pi i}\oint_{\mathcal{C}^{\star}} \exp(st)\,\hat{f}(s)\,\mathrm{d}s$$

is approximated by the summation

$$\mathcal{L}^{\star}_{N}\{\hat{f}(s)\} = \frac{1}{2\pi i}\sum_{n=1}^{N} \exp(s_n t)\,\hat{f}(s_n)\,\Delta s_n$$

(For details, see Clancy and Lynch, 2011a)

Van Isacker's Inversion Formula

$$\mathcal{L}_N^{\star}\{\hat{f}(s)\} = \frac{1}{2\pi i} \sum_{n=1}^{N} \exp(s_n t)\, \hat{f}(s_n)\, \Delta s_n$$

We introduce a correction factor, and arrive at:

$$\mathcal{L}_N^{\star}\{\hat{f}(s)\} = \frac{1}{N} \sum_{n=1}^{N} \exp_N(s_n t)\, \hat{f}(s_n)\, s_n$$

Here $\exp_N(z)$ is the N-term Taylor expansion of $\exp(z)$.

A General NWP Equation

We write the general NWP equations symbolically as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

where $\mathbf{X}(t)$ is the state vector at time t.

We apply the Laplace transform to get

$$(s\,\hat{\mathbf{X}} - \mathbf{X}_0) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_0 = \mathbf{0}$$

where $\mathbf{X}_0$ is the initial value of $\mathbf{X}$ and $\mathbf{N}_0 = \mathbf{N}(\mathbf{X}_0)$ is held constant at its initial value.

Now we take $n\Delta t$ to be the **initial time**:

$$(s\,\hat{\mathbf{X}} - \mathbf{X}^n) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}^n = \mathbf{0}$$

The solution can be written formally:

$$\hat{\mathbf{X}}(s) = (s\,\mathbf{I} + i\,\mathbf{L})^{-1}\left[\mathbf{X}^n - \frac{1}{s}\mathbf{N}^n\right]$$

We recover the filtered solution at time $(n+1)\Delta t$ by applying $\mathcal{L}^\star$ at time Δt beyond the initial time:

$$\mathbf{X}^\star((n+1)\Delta t) = \mathcal{L}^\star\{\hat{\mathbf{X}}(s)\}\Big|_{t=\Delta t}$$

The procedure may now be iterated to produce a forecast of any length.

Quarterly Journal of the Royal Meteorological Society — Q. J. R. Meteorol. Soc. 00: 1–7 (0000)

Laplace transform integration of the shallow water equations. Part 1: Eulerian formulation and Kelvin waves

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Laplace transform integration of the shallow water equations. Part 2: Lagrangian formulation and orographic resonance

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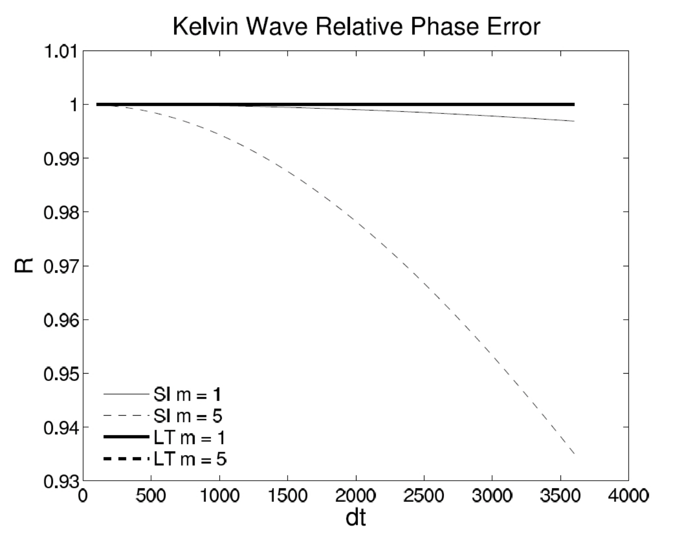
(Clancy and Lynch, QJRMS, 137, 2011)



Relative phase errors for semi-implicit (SI) and Laplace transform (LT) schemes for Kelvin waves $m = 1$ and $m = 5$.

Lagrangian Formulation

We now consider how to combine the Laplace transform approach with Lagrangian advection.

The general form of the equation is

$$\frac{\mathrm{D}\mathbf{X}}{\mathrm{D}t} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

where advection is now included in the time derivative.

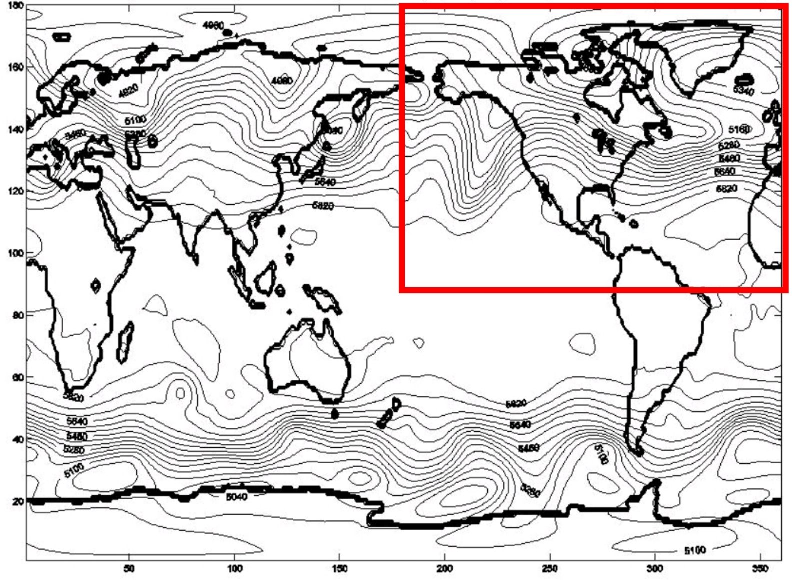
We *re-define* the Laplace transform to be the integral in time *along the trajectory of a fluid parcel*:

$$\hat{\mathbf{X}}(s) \equiv \int_{\mathcal{T}} e^{-st}\,\mathbf{X}(t)\,\mathrm{d}t$$

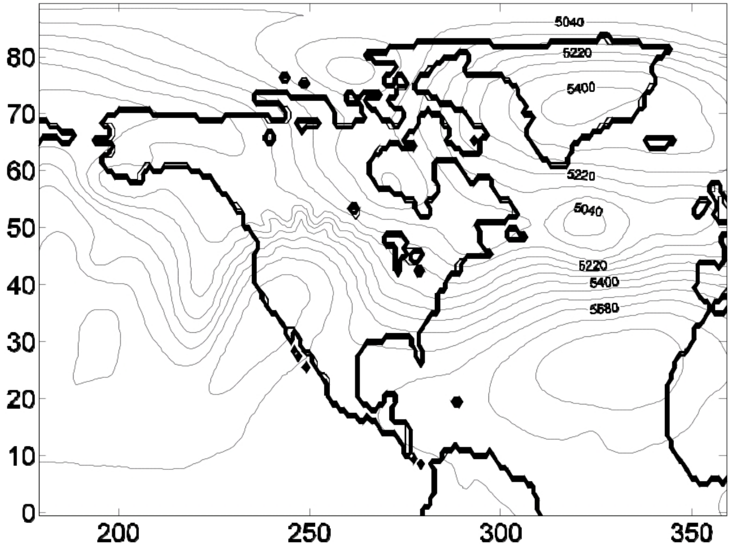
Orographic Resonance

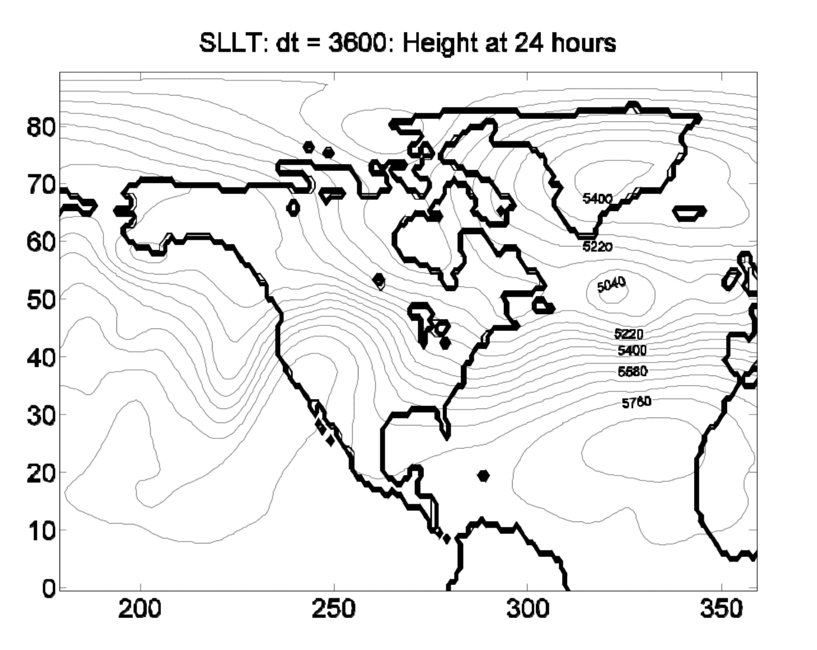
- Spurious resonance arises from coupling the semi-Lagrangian and semi-implicit methods
- Linear analysis of orographically forced stationary waves confirms this
- This motivates an investigating of orographic resonance in a full model.

Initial Height (m)



SLSI: dt = 3600: Height at 24 hours




SLLT: dt = 3600: Height at 24 hours
80
70
60
50
40
30
20
10
0
200
250
300
350
5400
5220
5040
5220
5400
5580
5760

Analytical Inversion

We now consider the LT scheme with the inverse computed analytically.

This yields a filtered system. We relate it to the filtering schemes of Daley (1980).

The procedure requires explicit knowledge of the positions of the poles of the function to be inverted.

For the Eulerian model, this is simple.

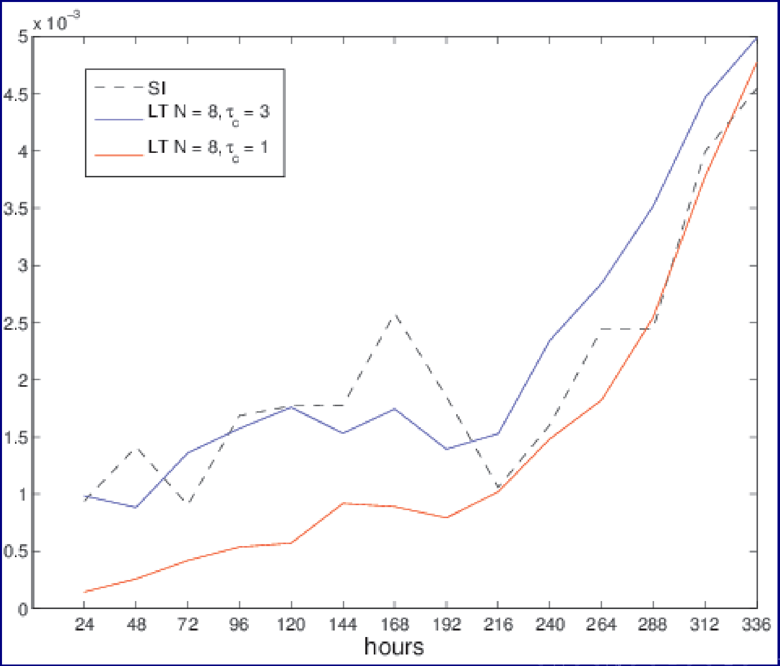
Eulerian Model: Rossby-Haurwitz (Case 6)

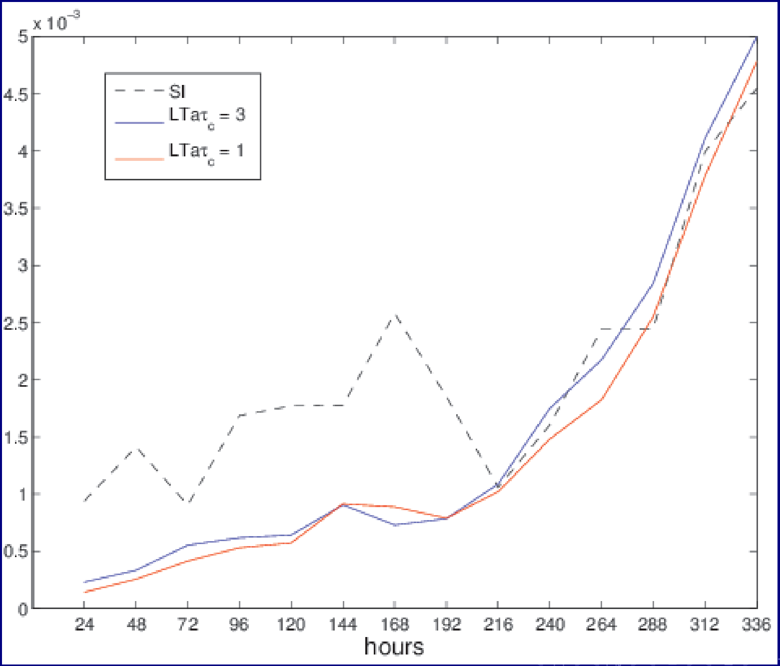
First plot:

- Reference SI scheme
- Numerical LT with N = 8, cutoff period 3 hours
- Numerical LT with N = 8, cutoff period 1 hour.

Second plot:

- Reference SI scheme
- Analytical LT with cutoff period 3 hours
- Analytical LT with cutoff period 1 hour.


x 10^-3
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
SI
LT N = 8, τ_c = 3
LT N = 8, τ_c = 1
24
48
72
96
120
144
168
192
216
240
264
288
312
336
hours


x 10^-3
SI
LTaτ_c = 3
LTaτ_c = 1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
24
48
72
96
120
144
168
192
216
240
264
288
312
336
hours

Conclusion

Advantages

- LT scheme effectively filters HF waves
- LT scheme more accurate than SI scheme
- LT scheme has no orographic resonance.

New Results

- Analytical LT more accurate than numerical
- Lagrangian scheme: more work needed
- Some problems remain with Coriolis terms.

Thank you

Thank you