Resonant Rossby-Haurwitz Triads

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International Summer School on Atmospheric and Oceanic Science
16–20 September, 2013

Outline

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Barotropic Vorticity Equation (BVE)

- Shallow, incompressible fluid on rotating sphere
- Horizontal velocity non-divergent
- Radius a, rotation rate Ω
- Longitude/latitude coordinates (λ, ϕ)

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The dynamics are governed by the conservation of absolute vorticity:

$$\frac{d}{dt}(\zeta + f) = 0 \,.$$

$$f = 2\Omega \sin\phi \qquad \zeta = \mathbf{k}\cdot\nabla\times\mathbf{V}$$

$$f = \begin{bmatrix}\textbf{planetary}\\ \textbf{vorticity}\end{bmatrix} \qquad \zeta = \begin{bmatrix}\textbf{relative}\\ \textbf{vorticity}\end{bmatrix} \qquad f+\zeta = \begin{bmatrix}\textbf{absolute}\\ \textbf{vorticity}\end{bmatrix}$$

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Conservation of absolute vorticity:

$$\frac{d}{dt}(\zeta + f) = 0$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} .$$

Introducing a stream-function ψ, we get:

$$\mathbf{V} = \mathbf{k} \times \nabla\psi \qquad\qquad \zeta = \nabla^2\psi$$

and the vorticity equation becomes:

$$\frac{\partial\nabla^2\psi}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} + \frac{1}{a^2}\frac{\partial(\psi, \nabla^2\psi)}{\partial(\lambda, \mu)} = 0$$

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where $\mu = \sin\phi$.

This is the non-divergent barotropic vorticity equation

The Jacobian term represents non-linear advection.

Omitting the nonlinear term, the BVE has solutions

$$\psi = \psi_0 Y_n^m(\lambda, \mu) \exp(-i\sigma t) = \psi_0 P_n^m(\mu) \exp[i(m\lambda - \sigma t)]$$

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The frequency σ is given by the dispersion formula

$$\sigma = \sigma_n^m \equiv -\frac{2\Omega m}{n(n+1)} .$$

- m **is the zonal wavenumber**
- n **is the total wavenumber**

Both m and n are integers.

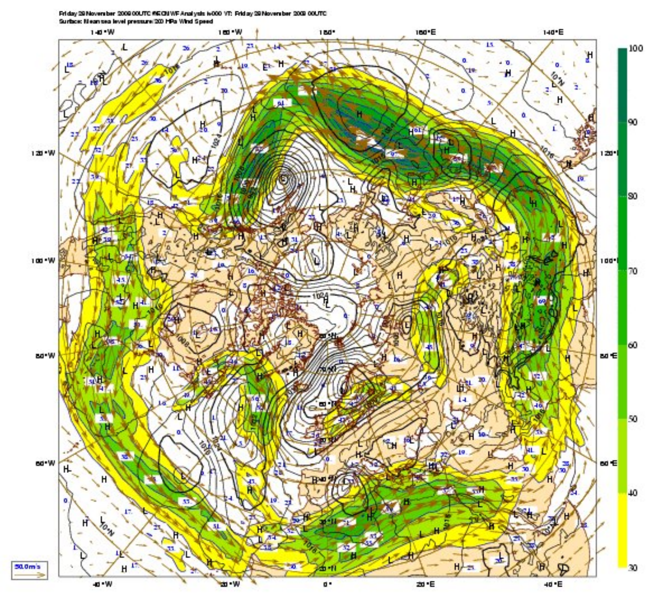
The functions $Y_n^m(\lambda, \mu)$ are eigenfunctions of the Laplacian operator on the sphere:

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m .$$

The solution

$$\psi = \psi_0 Y_n^m(\lambda, \mu) \exp(-i\sigma t)$$

is called a Rossby-Haurwitz wave.


Surface: Mean sea level pressure/300 HPa Wind Speed
140°W
160°W
180°
160°E
140°E
120°W
100°W
80°W
60°W
40°W
20°W
0°
20°E
40°E
50.0m/s
100
90
80
70
60
50
40
30

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This is not usually true for a combination of RH waves:

The velocity of one component advects the vorticity of another.

The waves interact and their amplitudes change.

The spherical harmonics $Y_n^m(\lambda, \mu)$ form an **orthonormal basis** on the sphere.

Thus, the stream function has an expansion

$$\psi(\lambda, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \psi_n^m(t) Y_n^m(\lambda, \mu) \, .$$

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$$\psi = \sum_{\gamma} \psi_{\gamma}(t) Y_{\gamma}(\lambda, \mu) e^{-i\sigma_{\gamma} t} \qquad \zeta = \sum_{\gamma} \zeta_{\gamma}(t) Y_{\gamma}(\lambda, \mu) e^{-i\sigma_{\gamma} t}$$

with $\psi_{\gamma} = -a^2 \kappa_{\gamma} \zeta_{\gamma}$, where $\kappa_{\gamma} = 1/(n(n+1))$.

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with $\psi_{\gamma} = -a^2 \kappa_{\gamma} \zeta_{\gamma}$, where $\kappa_{\gamma} = 1/(n(n+1))$.

If the nonlinear interactions are weak, the coefficients will vary slowly with time compared to $\exp(-i\sigma_{\gamma} t)$.

This is the key to the two-timing perturbation technique.

Flows governed by the BVE conserve total energy and total enstrophy:

$$E = \frac{1}{4\pi a^2}\iint \tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\,d\lambda\,d\mu = -\frac{1}{4\pi a^2}\iint \tfrac{1}{2}\psi\zeta\,d\lambda\,d\mu$$

$$S = \frac{1}{4\pi a^2}\iint \tfrac{1}{2}\zeta^2\,d\lambda\,d\mu = -\frac{1}{4\pi a^2}\iint \tfrac{1}{2}\nabla\psi\cdot\nabla\zeta\,d\lambda\,d\mu$$

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In terms of the **spectral coefficients**, these are:

$$E = \tfrac{1}{2}\sum_\gamma \kappa_\gamma|\zeta_\gamma|^2\,, \qquad S = \tfrac{1}{2}\sum_\gamma |\zeta_\gamma|^2\,.$$

The constancy of energy and enstrophy profoundly influences the energetics of solutions of the BVE.

The equations for the spectral coefficients are:

$$\frac{d\zeta_\gamma}{dt} = \tfrac{1}{2} i \sum_{\alpha,\beta} I_{\gamma\beta\alpha} \zeta_\beta \zeta_\alpha \exp(-i\sigma t) \,,$$

where $\sigma = (\sigma_\alpha + \sigma_\beta - \sigma_\gamma)$.
The interaction coefficients are given by

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$$I_{\gamma\beta\alpha} = (\kappa_\beta - \kappa_\alpha) K_{\gamma\beta\alpha} \,.$$

The coupling integrals $K_{\gamma\beta\alpha}$ vanish unless $m_\alpha + m_\beta = m_\gamma$, when they they are given by

$$K_{\gamma\beta\alpha} = \tfrac{1}{2} \int_{-1}^{+1} P_\gamma \left(m_\beta P_\beta \frac{dP_\alpha}{d\mu} - m_\alpha P_\alpha \frac{dP_\beta}{d\mu} \right) d\mu \,.$$

Selection Rules (technical)

For non-vanishing interaction, the following selection rules must be satisfied:

$$
\begin{array}{rcl}
m_\alpha + m_\beta & = & m_\gamma \\
m_\alpha^2 + m_\beta^2 & \neq & 0 \\
n_\gamma n_\beta n_\alpha & \neq & 0 \\
n_\alpha & \neq & n_\beta \\
n_\alpha + n_\beta + n_\gamma & \text{is} & \text{odd} \\
(n_\beta - |m_\beta|)^2 + (n_\alpha - |m_\alpha|)^2 & \neq & 0 \\
|n_\alpha - n_\beta| < & n_\gamma & < n_\alpha + n_\beta \\
(m_\beta, n_\beta) \neq (-m_\gamma, n_\gamma) & \text{and} & (m_\alpha, n_\alpha) \neq (-m_\gamma, n_\gamma)
\end{array}
$$

Symmetries: $I_{\gamma\alpha\beta} = I_{\gamma\beta\alpha} \quad \text{and} \quad K_{\gamma\alpha\beta} = -K_{\gamma\beta\alpha}\,.$

Redundancy rules: $K_{\alpha\bar{\beta}\gamma} = K_{\gamma\beta\alpha} \quad \text{and} \quad K_{\beta\gamma\bar{\alpha}} = K_{\gamma\beta\alpha}\,.$

Outline

Crucial Advances, 1920–1950

- Dynamic Meteorology
  - Rossby Waves (and RH Waves)
  - Quasi-geostrophic Theory
  - Baroclinic Instability
- Numerical Analysis
  - CFL Criterion
- Atmopsheric Observations
  - Radiosonde
- Electronic Computing
  - ENIAC

Charney, et al., *Tellus*, 1950.

$$\begin{bmatrix}\text{Absolute}\\\text{Vorticity}\end{bmatrix} = \begin{bmatrix}\text{Relative}\\\text{Vorticity}\end{bmatrix} + \begin{bmatrix}\text{Planetary}\\\text{Vorticity}\end{bmatrix} \qquad \eta = \zeta + f\,.$$

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- The atmosphere is treated as a single layer.
- The flow is assumed to be nondivergent.
- Absolute vorticity is conserved.

$$\boxed{\frac{\mathrm{d}(\zeta + \mathrm{f})}{\mathrm{dt}} = 0.}$$

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This equation looks simple. But it is nonlinear:

$$\frac{\partial}{\partial t}[\nabla^2\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0\,,$$

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We investigate highly truncated solutions of the BVE.

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Under certain circumstances, the interactions are so weak that the simple low-order structure persists.

We consider the case where there are just three non-vanishing components:

$$\psi = \Re\{\psi_\alpha Y_\alpha \exp(-i\sigma_\alpha t) + \psi_\beta Y_\beta \exp(-i\sigma_\beta t) + \psi_\gamma Y_\gamma \exp(-i\sigma_\gamma t)\}\,.$$

The selection rules then imply that the only non-vanishing interaction coefficients are:

$$I_{\gamma\beta\alpha} = I_{\gamma\alpha\beta} \qquad I_{\beta\bar{\alpha}\gamma} = I_{\beta\gamma\bar{\alpha}} \qquad I_{\alpha\bar{\beta}\gamma} = I_{\alpha\gamma\bar{\beta}}\,.$$

Using the symmetries and redundancy rules, all the coefficients can be expressed in terms of one, K:

$$
\begin{aligned}
i\dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma) K \zeta_\beta^* \zeta_\gamma \exp(+i\sigma t) \\
i\dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha) K \zeta_\gamma \zeta_\alpha^* \exp(+i\sigma t) \\
i\dot{\zeta}_\gamma &= +(\kappa_\alpha - \kappa_\beta) K \zeta_\alpha \zeta_\beta \exp(-i\sigma t)
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In general, the right-hand sides of these equations vary rapidly in time. If the equations are averaged over a time $\tau = 2\pi/\sigma$, **the right hand sides vanish . . .**

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. . . unless $\sigma = 0$: this is the case of resonance.

We consider only the resonant case below.

The condition for resonance, $\sigma = 0$, may be written

$$m_\alpha \kappa_\alpha + m_\beta \kappa_\beta = m_\gamma \kappa_\gamma \, .$$

We consider the generic case:

$$\kappa_\alpha > \kappa_\gamma > \kappa_\beta \, .$$

Thus, $n_\alpha < n_\gamma < n_\beta$, so that the component ζ_γ is of a scale intermediate between the others (Fjørtoft, '53).

The equations may now be written

$$\begin{aligned} i\dot{\zeta}_\alpha &= k_\alpha \zeta_\beta^* \zeta_\gamma \\ i\dot{\zeta}_\beta &= k_\beta \zeta_\gamma \zeta_\alpha^* \\ i\dot{\zeta}_\gamma &= k_\gamma \zeta_\alpha \zeta_\beta \end{aligned}$$

where, assuming $K > 0$, the coefficients

$$k_\alpha = (\kappa_\gamma - \kappa_\beta)K\,, \quad k_\beta = (\kappa_\alpha - \kappa_\gamma)K\,, \quad k_\gamma = (\kappa_\alpha - \kappa_\beta)K$$

are all positive and $k_\alpha + k_\beta = k_\gamma$.

The energy and enstrophy of the triad may be written:

$$
\begin{aligned}
E &= \tfrac{1}{2}(\kappa_\alpha|\zeta_\alpha|^2 + \kappa_\beta|\zeta_\beta|^2 + \kappa_\gamma|\zeta_\gamma|^2) \\
S &= \tfrac{1}{2}(|\zeta_\alpha|^2 + |\zeta_\beta|^2 + |\zeta_\gamma|^2) \,.
\end{aligned}
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We now introduce the transformation

$$\eta_\alpha = \sqrt{k_\beta k_\gamma}\,\zeta_\alpha\,, \qquad \eta_\beta = \sqrt{k_\gamma k_\alpha}\,\zeta_\beta\,, \qquad \eta_\gamma = \sqrt{k_\alpha k_\beta}\,\zeta_\gamma\,,$$

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$$

The equations then assume the standard form:

$$
\begin{aligned}
i\dot{\eta}_\alpha &= \eta_\beta^*\eta_\gamma \\
i\dot{\eta}_\beta &= \eta_\gamma\eta_\alpha^* \\
i\dot{\eta}_\gamma &= \eta_\alpha\eta_\beta
\end{aligned}
$$

These are the three-wave equations.

Energy and enstrophy are conserved for the triad.

The Manley-Rowe quantities are defined as

$$
\begin{aligned}
N_1 &= |\eta_\alpha|^2 + |\eta_\gamma|^2 \\
N_2 &= |\eta_\beta|^2 + |\eta_\gamma|^2 \\
J &= |\eta_\alpha|^2 - |\eta_\beta|^2 \,.
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They are all constants of the motion.

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They are all constants of the motion.

The system may be shown to be the canonical equations arising from the Hamiltonian $H = \Re\{\eta_\alpha \eta_\beta \eta_\gamma^*\}$

Numerical Example

We integrated the BVE with initial conditions dominated by mode RH(4,5).

This is the mode that Hoskins (1973) said was stable but that Thuburn & Li (2000) found to be unstable.

The triad $(4, 5)$, $(1, 3)$ $(3, 7)$ comes close to satisfying the frequency criterion for resonance.

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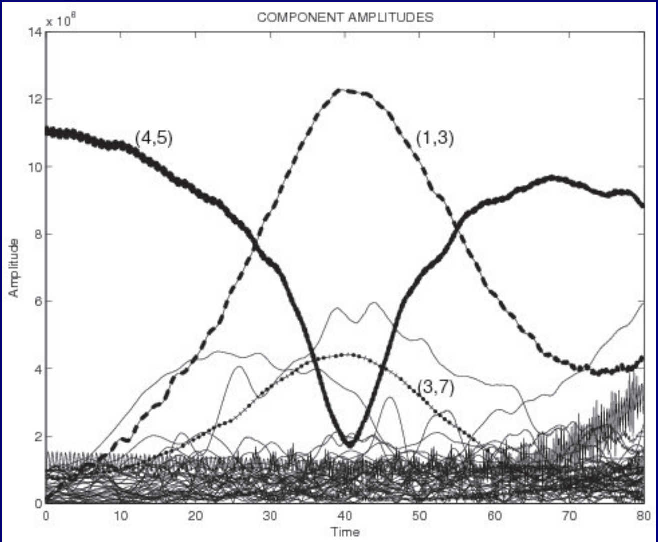
The triad $(4,5)$, $(1,3)$ $(3,7)$ comes close to satisfying the frequency criterion for resonance.

The respective frequencies (normalized by 2Ω) are

$$\sigma_5^4 = -0.13333 \qquad \sigma_3^1 = -0.08333 \qquad \sigma_7^3 = -0.05357$$

$$\sigma_5^4 = -0.13333 \qquad \sigma_3^1 + \sigma_7^3 = -0.13690 \approx \sigma_5^4$$



Evolution of component amplitudes over 80 days.

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Forced Planetary Waves

We now include forcing by orography and damping towards a reference state with potential vorticity f/H.

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The BPVE may be written

$$\frac{d}{dt}\left(\frac{\zeta+f}{H-h_0}\right) = -\nu\left(\frac{\zeta+f}{H-h_0} - \frac{f}{H}\right)$$

where H is the mean height, h_0 the elevation of the orography and ν is the damping coefficient.

The flow is separated into a zonal super-rotation $\bar{u} = a\cos\phi\,\bar{\omega}$ with constant $\bar{\omega}$, and a perturbation (u, v).

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Assuming that the orography is small, $h_0 \ll H$, we can write the equation as

$$\left(\frac{\partial}{\partial t} + \bar{\omega}\frac{\partial}{\partial \lambda}\right)\zeta + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} + \frac{1}{a^2}\frac{\partial(\psi,\zeta)}{\partial(\lambda,\mu)} - \frac{\bar{\omega} f}{H}\frac{\partial h_0}{\partial \lambda} = -\nu\left(\zeta - \frac{f h_0}{H}\right)$$

The flow is separated into a zonal super-rotation $\bar{u} = a\cos\phi\,\bar{\varpi}$ with constant $\bar{\varpi}$, and a perturbation (u, v).

Assuming that the orography is small, $h_0 \ll H$, we can write the equation as

$$\left(\frac{\partial}{\partial t} + \bar{\varpi}\frac{\partial}{\partial\lambda}\right)\zeta + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} + \frac{1}{a^2}\frac{\partial(\psi,\zeta)}{\partial(\lambda,\mu)} - \frac{\bar{\varpi}f}{H}\frac{\partial h_0}{\partial\lambda} = -\nu\left(\zeta - \frac{fh_0}{H}\right)$$

The linear normal modes have eigen-frequencies

$$\sigma_n^m = \bar{\varpi} - \frac{(2\Omega + \bar{\varpi})m}{n(n+1)} .$$

Bounded response to forcing

If $\bar{\omega}$ is such that σ_n^m vanishes for some (m, n), the orographic forcing leads to a solution that grows linearly with time until equilibrated by the damping.

In the absence of damping, it grows without limit.

Bounded response to forcing

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In the absence of damping, it grows without limit.

However, as the amplitude increases, nonlinear interactions transfer energy to other modes and it is possible to have a bounded response to constant orographic forcing.

This is the case we study below.

We seek a solution in the form of a resonant triad

$$\psi = \Re\{\psi_\alpha Y_\alpha \exp(-i\sigma_\alpha t) + \psi_\beta Y_\beta \exp(-i\sigma_\beta t) + \psi_\gamma Y_\gamma \exp(-i\sigma_\gamma t)\}$$

with $\sigma_\alpha + \sigma_\beta = \sigma_\gamma$.

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with $\sigma_\alpha + \sigma_\beta = \sigma_\gamma$.

Assuming that the solution is of small amplitude ϵ we expand the streamfunction as

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \dots$$

The nonlinear term involving $J(\psi, fh_0/H)$ does not enter at $O(\epsilon^2)$.

The damping coefficient ν is $O(\epsilon)$.

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We assume that the orography (actually, fh_0) has the same spatial structure $Y_\gamma(\lambda, \phi)$ as the γ-term, and

$$\bar{\omega} = \frac{(2\Omega + \bar{\omega})m_\gamma}{n_\gamma(n_\gamma + 1)} \qquad \text{or} \qquad \bar{\omega} = \frac{2\Omega m_\gamma \kappa_\gamma}{1 - m_\gamma \kappa_\gamma} .$$

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Thus, the γ-term resonates with the orography.

At order ϵ, the equations are linear and unforced, so the three components evolve independently.

At order ϵ^2, the forcing, damping and nonlinearity enter, and the equations at this level are

$$\begin{aligned}
\dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma) K \zeta_\beta^* \zeta_\gamma^* - \nu \zeta_\alpha \\
\dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha) K \zeta_\gamma^* \zeta_\alpha^* - \nu \zeta_\beta \\
\dot{\zeta}_\gamma &= +(\kappa_\alpha - \kappa_\beta) K \zeta_\alpha^* \zeta_\beta^* - \nu \zeta_\gamma + F
\end{aligned}$$

where the coefficient F is a constant proportional to the magnitude of the orographic forcing.

At order ϵ^2, the forcing, damping and nonlinearity enter, and the equations at this level are

$$
\begin{aligned}
\dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma) K \zeta_\beta^* \zeta_\gamma^* - \nu \zeta_\alpha \\
\dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha) K \zeta_\gamma^* \zeta_\alpha^* - \nu \zeta_\beta \\
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\end{aligned}
$$

where the coefficient F is a constant proportional to the magnitude of the orographic forcing.

Introducing a transformation as before, we get the **forced-damped three-wave equations:**

$$
\begin{aligned}
i\dot{\eta}_\alpha &= \eta_\beta^* \eta_\gamma - i\nu\eta_\alpha \\
i\dot{\eta}_\beta &= \eta_\gamma \eta_\alpha^* - i\nu\eta_\beta \\
i\dot{\eta}_\gamma &= \eta_\alpha \eta_\beta - i\nu\eta_\gamma + iF
\end{aligned}
$$

The quantities J, $N(= N_1 + N_2)$ and H are no longer conserved quantities, but obey the equations

$$
\begin{aligned}
\dot{J} &= -2\nu J \,, \\
\dot{N} &= -2\nu N + 2\Re\{F^*\eta_\gamma\} \,, \\
\dot{H} &= -3\nu N + 2\Re\{F^*\eta_\alpha\eta_\beta\} \,.
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Note that the energy quantity N may increase or decrease in response to the forcing F, depending on the phase relationship between F and η_γ.

Numerical Example

We integrated the BVE with orographic forcing of a single spectral component, RH(3,9).

The mean flow $\bar{\omega}$ is set so that this mode is stationary.

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$$\frac{m_\gamma}{n_\gamma(n_\gamma+1)} = \frac{m_\alpha}{n_\alpha(n_\alpha+1)} + \frac{m_\beta}{n_\beta(n_\beta+1)}$$

$$\left(\frac{3}{9\times 10}\right) = \left(\frac{1}{6\times 7}\right) + \left(\frac{2}{14\times 15}\right)$$

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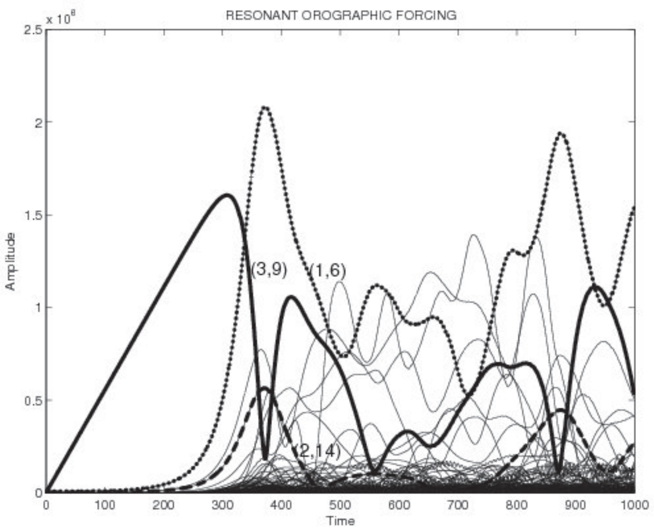
Initially, all modes have very small amplitudes, representing background noise.



Evolution of component amplitudes over 80 days.

In the figure above, we showed the component amplitudes for weak orographic forcing.

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Extended integrations confirm this behaviour.

Outline

Free Rossby wave triads in the atmosphere can be modelled by an elastic pendulum or swinging spring.

At a certain level of approximation, the equations of the two systems are mathematically isomorphic.

Thus, behaviour such as the precession of successive horizontal excursions of the spring indicated similar behaviour in the atmosphere.

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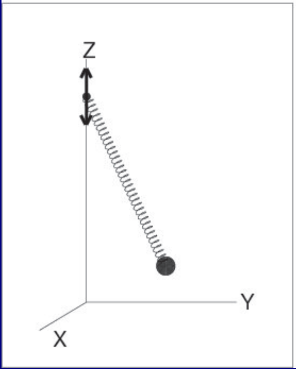
We extend this correspondence here to include forcing and damping.

Forced-damped swinging spring



We consider a swinging spring whose point of suspension oscillates vertically with period ω_Z.

ℓ_0 is unsteretched length
ℓ length at equilibrium
k is spring constant
$m = 1$ is unit mass.

The Lagrangian, approximated to cubic order, is

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}[\dot{x}^2 + \dot{y}^2 + (\dot{z}^2 + 2\dot{z}\dot{\zeta} + \dot{\zeta}^2)] \\ &\qquad -\tfrac{1}{2}[\omega_R^2(x^2+y^2) + \omega_Z^2 z^2] - \tfrac{1}{2}\lambda(x^2+y^2)z\,.\end{aligned}$$

where x, y and z are Cartesian coordinates centered at the point of equilibrium.

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where x, y and z are Cartesian coordinates centered at the point of equilibrium.

$\zeta(t) = \Re\{\zeta_0 e^{i\omega_Z t}\}$ is displacement of suspension point

$\omega_R = (g/\ell)^{1/2}$ is frequency of pendular motion

$\omega_Z = (k/m)^{1/2}$ is frequency of elastic oscillations

$\lambda = \ell_0\omega_Z^2/\ell^2$ is a parameter.

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We employ the average Lagrangian technique to obtain an approximate solution.

We confine attention to the **resonant case** $\omega_Z = 2\omega_R$.
The solution is assumed to be of the form

$$
\begin{aligned}
x &= \Re\{a(t)\exp(i\omega_R t)\}\,, \\
y &= \Re\{b(t)\exp(i\omega_R t)\}\,, \\
z &= \Re\{c(t)\exp(i\omega_Z t)\}\,.
\end{aligned}
$$

The time scale of variation of a, b and c is much longer than $\tau = 2\pi/\omega_R$.

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The time scale of variation of a, b and c is much longer than $\tau = 2\pi/\omega_R$.

If the Lagrangian and the dissipation function are averaged over time τ, the amplitude equations are

$$\begin{aligned} i\dot{a} &= -\mu a^* c - i\nu a \\ i\dot{b} &= -\mu b^* c - i\nu b \\ i\dot{c} &= -\tfrac{1}{4}\mu(a^2 + b^2) - i\nu c + \tfrac{1}{2}\omega_Z \zeta_0\,. \end{aligned}$$

where $\mu = \lambda/4\omega_R$.

Defining new variables by

$$\alpha = \tfrac{1}{2}\mu(a + ib)\,, \qquad \beta = \tfrac{1}{2}\mu(a - ib)\,, \qquad \gamma = \mu c$$

the equations for the envelope dynamics become

$$\begin{aligned} i\dot\alpha &= \beta^*\gamma - i\nu\alpha \\ i\dot\beta &= \gamma\alpha^* - i\nu\beta \\ i\dot\gamma &= \alpha\beta - i\nu\gamma + iF\,, \end{aligned}$$

where $F = -\frac{1}{2}i\mu\omega_Z\zeta_0$ represents the external forcing.

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where $F = -\frac{1}{2}i\mu\omega_Z\zeta_0$ represents the external forcing.

This system is **isomorphic to the system for a forced-damped resonant Rossby triad.**

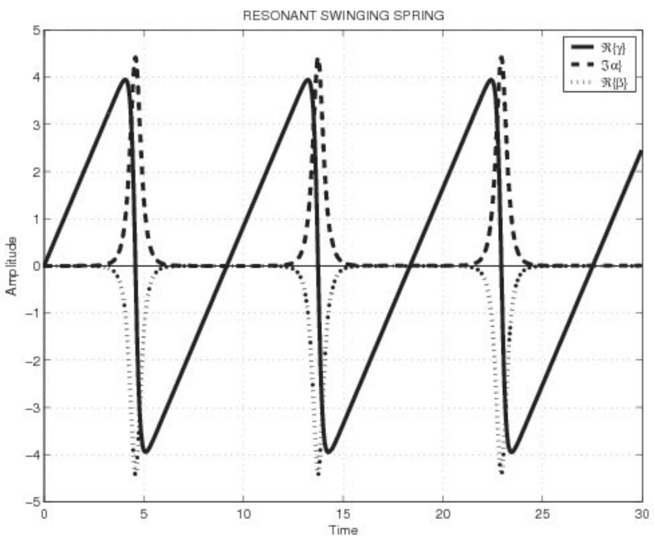
Numerical Example

We integrated the system over thirty time units, with unit forcing $F = 1$ and no damping

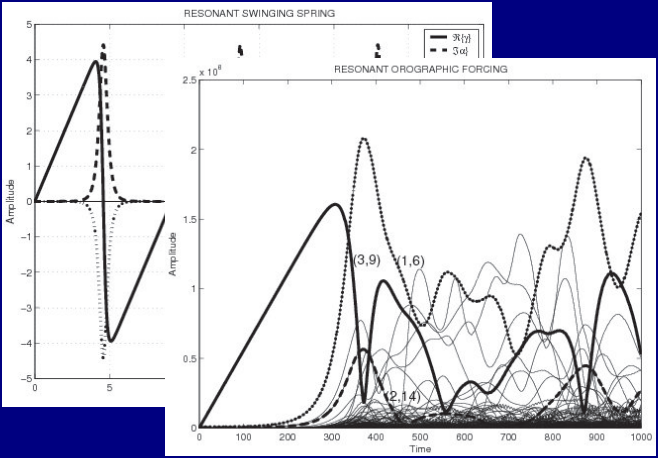
The initial conditions are

$$\begin{aligned} \alpha_0 &= (+0.0005, 0.0000)\,, \\ \beta_0 &= (-0.0005, 0.0005)\,, \\ \gamma_0 &= (+0.0000, 0.0000)\,. \end{aligned}$$

The amplitudes of the components (real and imaginary parts) are shown in the figure below. Initially, the forced component, γ, grows linearly.



Amplitudes of α, β and γ. Components $\Im\{\alpha\}$, $\Re\{\beta\}$ and $\Re\{\gamma\}$ are shown bold. Other amplitudes remain small.


RESONANT SWINGING SPRING
Amplitude
RESONANT OROGRAPHIC FORCING
x 10^8
Amplitude
Time
(3,9)
(1,6)
(2,14)

As the forced mode γ gains energy, there is a sudden surge of energy into the other two components, (α, β).

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Full account in *Tellus* paper:

Lynch, Peter, 2009: On Resonant Rossby-Haurwitz triads. *Tellus*, 61A, 438–445.

Outline

Concluding remarks

- Resonant triads can explain the instability of large-scale RH waves.
- A constant forcing can lead to a periodic response, even in the absence of damping.
- There is a mathematical equivalence between forced resonant RH triads and the forced-damped swinging spring.

Concluding remarks

- Triad interactions are important in establishing and maintaining the atmospheric enegy spectrum.
- These interactions can account for quasi-periodic variations of long time-scale.
- An examination of the spectral characteristics of ERA40 (triads) data would be of great interest.

Outline

Selection of publications

- Rossby et al. (1939); Haurwitz (1940)
- Charney et al. (1950): ENIAC integrations
- Fjørtoft (1953): energy/enstrophy cascade
- Lorenz (1960): “Maximum simplification”
- Platzman (1962): Spectral analysis
- Baines (1976): Resonant RH triads
- Reznik et al. (1993): More triads
- Newell et al. (2001), Chen et al (2005).
- Lynch (2009).

Thank You