Boxes & Loops in Circles & Ovals, Billiards & Ballyards, Squircles & Squovals

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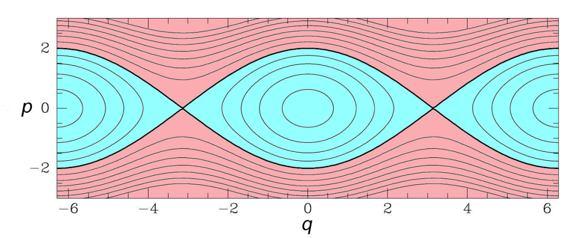
New Trends in Applied Geometric Mechanics
Celebrating Darryl Holm's 70th birthday
ICMAT, Madrid, 3–7 July 2017

Outline

Outline

Boxes & Loops

The familiar phase portrait of a simple pendulum shows how a separatrix divides the phase plane into two regions:

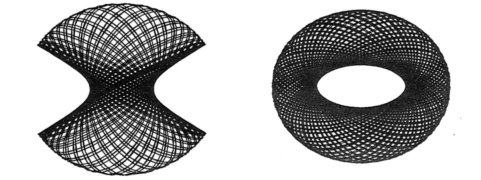


The two regions correspond to libration and rotation.

In many dynamical systems there is a similar separation of the phase plane into orbits known as boxes and loops.

Boxes & Loops

In many dynamical systems there is a similar separation
Into two types of orbits, known as boxes and loops.



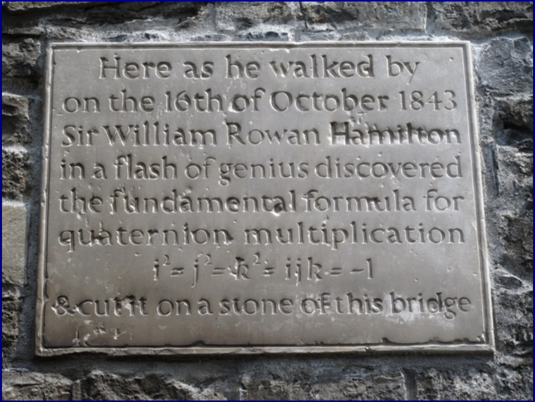
This is seen in elliptical billiards, astrodynamics,
rigid body mechanics and many other systems.

We will discuss this phenomenon and
illustrate it with a variety of examples.

Meeting Darryl: My Good Fortune

- Met Darryl at INI (AOD Programme) in 1996.
- Darryl and family in Dublin, July 1999.
  - We worked together on Swinging Spring.
- I Visited Los Alamos in Sep/Oct 2000.
  - Darryl found the 3-wave Equations.
- IMA Workshop, Minnesota, February 2002.
- Rock-n-roller. Innumerable emails.
- Recently: Numerous visits to Imperial College.

Quaternion Plaque on Hamilton's Bridge



Hamilton’s Bridge in Dublin

Figure : Darryl and Justine in Dublin, 1999?

Sand Sculpture of Hamilton’s Bridge

Figure : Hamilton’s Graffito: $i^2 = j^2 = k^2 = ijk = -1$.

Outline

The Swinging Spring

Two distinct oscillatory modes with distinct restoring forces:

- Elastic or springy modes
- Pendular or swingy modes

Two distinct oscillatory modes with distinct restoring forces:

- Elastic or springy modes
- Pendular or swingy modes

Take a peek at the Java Applet

In a paper in 1981, *Breitenberger and Mueller* made the following comment:

“This simple system looks like a toy at best, but its behaviour is astonishingly complex, with many facets of more than academic lustre.”

I hope to convince you of the validity of this remark.

The Exact Equations of Motion

In Cartesian coordinates the Lagrangian is

$$L = T - V = \underbrace{\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{Z}^2\right)}_{K.E} - \underbrace{\tfrac{1}{2}k(r-\ell_0)^2}_{E.P.E} - \underbrace{mgZ}_{G.P.E}$$

The equations of motion are (with $\omega_Z^2 \equiv k/m$):

$$\begin{aligned}
\ddot{x} &= -\omega_Z^2 \left(\frac{r-\ell_0}{r}\right) x \\
\ddot{y} &= -\omega_Z^2 \left(\frac{r-\ell_0}{r}\right) y \\
\ddot{Z} &= -\omega_Z^2 \left(\frac{r-\ell_0}{r}\right) Z - g
\end{aligned}$$

Two constants, energy and angular momentum:

$$E = T + V \qquad h = x\dot{y} - y\dot{x}\,.$$

The system is <u>not integrable</u> (two invariants, three D.O.F.).

The Canonical Equations

We consider the case of planar motion. The canonical equations of motion (in polar coordinates) are:

$$\begin{aligned}
\dot{\theta} &= p_\theta / m r^2 \\
\dot{p}_\theta &= -mgr\sin\theta \\
\dot{r} &= p_r / m \\
\dot{p}_r &= p_\theta^2 / m r^3 - k(r - \ell_0) + mg\cos\theta
\end{aligned}$$

These equations may also be written symbolically as

$$\dot{\mathbf{X}} + \mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0$$

State vector X is in 4-dimensional phase space:

$$\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}} .$$

Linear Normal Modes

Suppose that amplitude of motion is small:

$$\frac{d}{dt}\begin{pmatrix}\theta\\ p_\theta\\ r'\\ p_r\end{pmatrix} = \begin{pmatrix}0 & 1/m\ell^2 & 0 & 0\\ -mg\ell & 0 & 0 & 0\\ 0 & 0 & 0 & 1/m\\ 0 & 0 & -k & 0\end{pmatrix}\begin{pmatrix}\theta\\ p_\theta\\ r'\\ p_r\end{pmatrix}$$

The matrix is block-diagonal:

$$\mathbf{X} = \begin{pmatrix}\mathbf{Y}\\ \mathbf{Z}\end{pmatrix} : \qquad \mathbf{Y} = \begin{pmatrix}\theta\\ p_\theta\end{pmatrix} , \qquad \mathbf{Z} = \begin{pmatrix}r'\\ p_r\ .\end{pmatrix}$$

Linear dynamics evolve independently:

$$\dot{\mathbf{Y}} = \begin{pmatrix}0 & 1/m\ell^2\\ -mg\ell & 0\end{pmatrix}\mathbf{Y} , \qquad \dot{\mathbf{Z}} = \begin{pmatrix}0 & 1/m\\ -k & 0\end{pmatrix}\mathbf{Z} .$$

SLOW **FAST**

Perturbation Theory

Ratio of rotational and elastic frequencies:

$$\epsilon \equiv \left(\frac{\omega_R}{\omega_Z}\right) = \sqrt{\frac{mg}{k\ell}} .$$

For $\epsilon = 0$, there is no coupling between the modes.

For $\epsilon \ll 1$ the coupling is weak. We can apply classical *Hamiltonian perturbation theory.*

Regular and Chaotic Motion

We wish to discuss the phenomenon of **Resonance** for the spring, and its *Pulsation* and *Precession*.

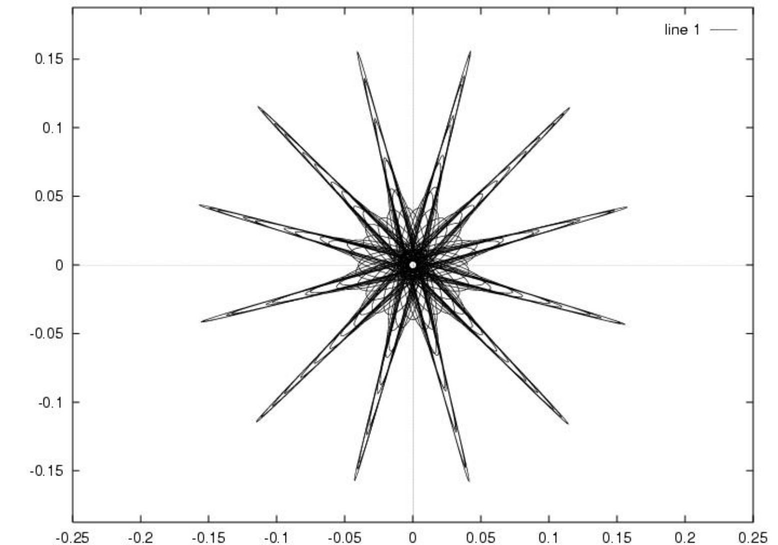
Resonance occurs for

$$\epsilon \approx \frac{1}{2} \, .$$

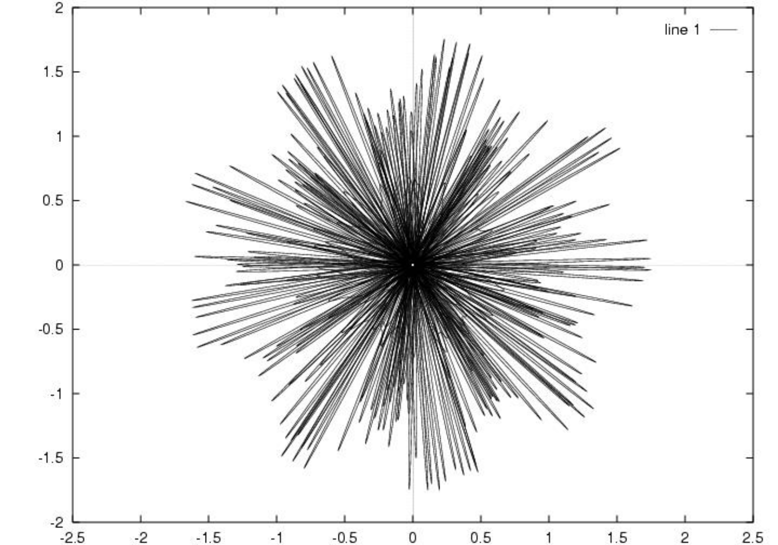
This is far from the quasi-integrable case (small ϵ).

However, for *small amplitudes*, the motion is also quasi-integrable. We look at two numerical solutions, one with small amplitude, one with large.

Horizontal plan: Low energy case



Horizontal plan: High energy case



The Resonant Case

The Lagrangian (to cubic order) is

$$L = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \tfrac{1}{2}\left(\omega_R^2(x^2+y^2) + \omega_Z^2 z^2\right) + \tfrac{1}{2}\lambda(x^2+y^2)z\,,$$

We study the resonant case:

$$\omega_Z = 2\omega_R\,.$$

The equations of motion are

$$\begin{aligned}
\ddot{x} + \omega_R^2 x &= \lambda x z \\
\ddot{y} + \omega_R^2 y &= \lambda y z \\
\ddot{x} + \omega_Z^2 x &= \tfrac{1}{2}\lambda(x^2+y^2)\,.
\end{aligned}$$

The system is <u>not integrable.</u>

Averaged Lagrangian technique

We seek a solution of the form:

$$
\begin{aligned}
x &= \Re[a(t)\exp(i\omega_R t)]\,,\\
y &= \Re[b(t)\exp(i\omega_R t)]\,,\\
z &= \Re[c(t)\exp(2i\omega_R t)]
\end{aligned}
$$

The coefficients $a(t)$, $b(t)$ and $c(t)$ vary slowly.

The Lagrangian is averaged over fast time:

$$\langle L\rangle = \left(\frac{\omega_R}{2}\right)\left[\Im(a\dot{a}^* + b\dot{b}^* + 2c\dot{c}^*) + \kappa\,\Re(a^2+b^2)c^*)\right]$$

where $\kappa = \lambda/(4\omega_R)$ (we absorb κ in t).

The Euler-Lagrange Equations

We derive the Euler-Lagrange equations resulting from this averaged Lagrangian:

$$\begin{aligned} i\dot{a} &= a^* c\,, \\ i\dot{b} &= b^* c\,, \\ i\dot{c} &= \tfrac{1}{4}(a^2 + b^2) \end{aligned}$$

The Euler-Lagrange Equations

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i\dot{a} &= a^*c\,, \\
i\dot{b} &= b^*c\,, \\
i\dot{c} &= \tfrac{1}{4}(a^2+b^2)
\end{aligned}
$$

We transform to new dependent variables:

$$A = \tfrac{1}{2}(a+ib)\,, \qquad B = \tfrac{1}{2}(a-ib)\,, \qquad C = c\,.$$

The Three-wave Equations

The equations for the transformed amplitudes are:

$$
\begin{aligned}
i\dot{A} &= B^* C \\
i\dot{B} &= C A^* \\
i\dot{C} &= A B
\end{aligned}
$$

These are the **three-wave equations**.

Invariants

The three-wave equations conserve

$$\begin{aligned} H &= \tfrac{1}{2}(ABC^* + A^*B^*C) \\ N &= |A|^2 + |B|^2 + 2|C|^2 \\ J &= |A|^2 - |B|^2 \,. \end{aligned}$$

The three-wave equations are **completely integrable.**

Manley-Rowe Relations

Physically significant combinations of N and J:

$$N_+ \equiv \tfrac{1}{2}(N + J) = |A|^2 + |C|^2 \,,$$

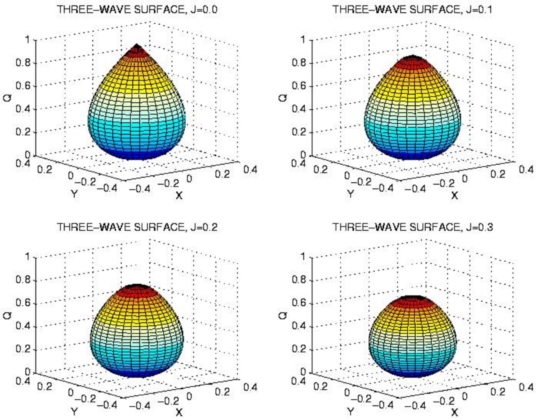
$$N_- \equiv \tfrac{1}{2}(N - J) = |B|^2 + |C|^2 \,.$$

These are the **Manley-Rowe relations.**

The quantities H, N_+ and N_- provide three independent constants of the motion.

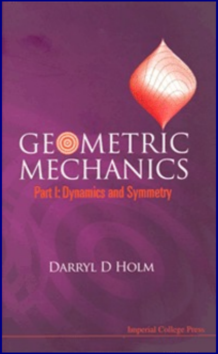
Constant N_+ and constant N_- correspond to orthogonal circular cylinders in phase-space.

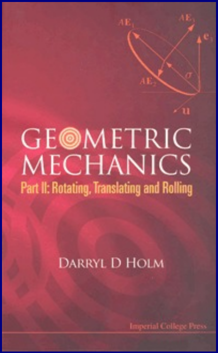
Surfaces of Revolution



Motion is on the intersection with plane of constant X.

Darryl's Books on Geometric Mechanics





Ubiquity of the Three-Wave Equations

- Modulation equations for wave interactions in fluids and plasmas.
- Three-wave equations govern envelop dynamics of light waves in an inhomogeneous material; and phonons in solids.
- Maxwell-Schrödinger envelop equations for radiation in a two-level resonant medium in a microwave cavity.
- Euler's equations for a freely rotating rigid body (when $H = 0$).

Analytical Solution of the 3WE

We can derive complete analytical expressions for the amplitudes and phases.

The amplitudes are expressed as elliptic functions.
The phases are expressed as elliptic integrals.

The complete details are given in:

Lynch, Peter, and Conor Houghton, 2004:
Pulsation and Precession of the
Resonant Swinging Spring.
Physica D, 190,1-2, 38-62

Original Reference

First comprehensive analysis of elastic pendulum:

"Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems"

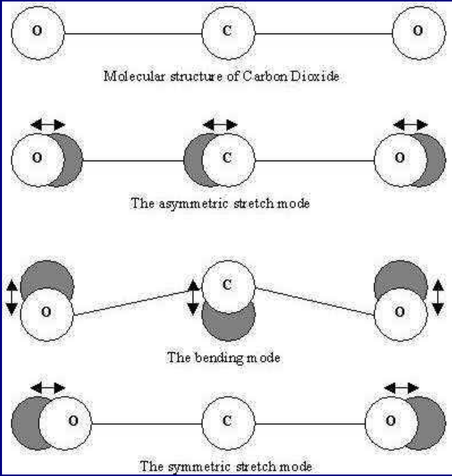
Vitt and Gorelik (1933).

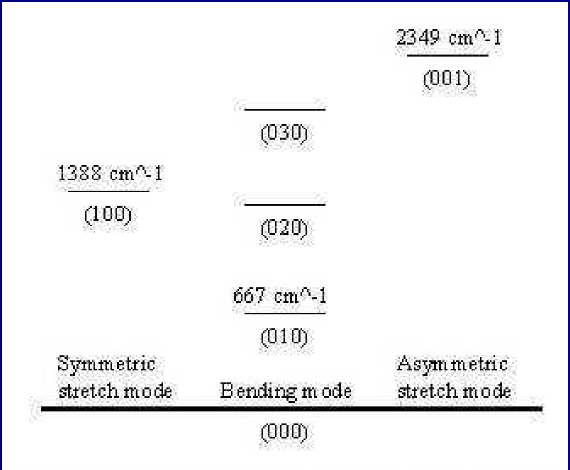
Inspired by analogy with Fermi resonance of CO_2.

Translation of this paper available as

Historical Note #3 (1999), Met Éireann, Dublin.

Vibrations of CO_2 Molecule





The first few vibrational energy levels of the CO2 molecule

$$\frac{1388}{667} \approx 2$$

Monodromy in Quantum Systems

It is 80 years since the work of Vitt and Gorelik.

" Remarkably, the swinging spring still has something interesting to offer to the quantum study of the Fermi resonance."

The CO_2 molecule as a quantum realization of the 1:1:2 resonant swing–spring with monodromy

Richard Cushman, Holger Dullin, Andrea Giacobbe, Darryl Holm, Marc Joyeux, Peter Lynch, Dmitrií Sadovskií, and Boris Zhilinskií

Published in *Phys. Rev. Lett.* (2004)

"It is now tempting to think of experimental quantum dynamical manifestations of monodromy."

Outline

In a Nutshell

A mathematical equivalence with

The Swinging Spring

sheds light on the dynamics of

Resonant Rossby Waves

in the atmosphere.

Potential Vorticity Conservation

$$\begin{aligned} \zeta &= \text{Relative Vorticity,} \\ f &= \text{Planetary Vorticity,} \\ h &= \text{Fluid Depth.} \end{aligned}$$

From the **Shallow Water Equations**, we derive the principle of conservation of potential vorticity:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

Under the assumptions of quasi-geostrophic theory, the dynamics reduce to an equation for ψ alone:

$$\frac{\partial}{\partial t}[\nabla^2\psi - F\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0$$

This is the **barotropic QG potential vorticity equation** (BQGPVE) aka the Charney-Hasegawa-Mima Equation.

Rossby Waves

Wave-like solutions of the vorticity equation:

$$\psi = A\cos(kx + \ell y - \sigma t)$$

satisfies the equation provided

$$\sigma = -\frac{k\beta}{k^2 + \ell^2 + F}\,.$$

This is the celebrated Rossby wave formula

Nonlinear term vanishes for single Rossby wave:
A pure Rossby wave is solution of nonlinear equation.

When there is more than one wave present, this is no longer true: the *components interact* with each other through the nonlinear terms.

Resonant Rossby Wave Triads

Case of special interest: Two wave components produce a third such that its interaction with each generates the other.

By a multiple time-scale analysis we derive the modulation equations for the wave amplitudes:

$$\begin{aligned} i\dot{A} &= B^* C \,, \\ i\dot{B} &= C A^* \,, \\ i\dot{C} &= A B \,, \end{aligned}$$

[Canonical form of the *three-wave equations*].

The Spring Equations

and the

Triad Equations are

are

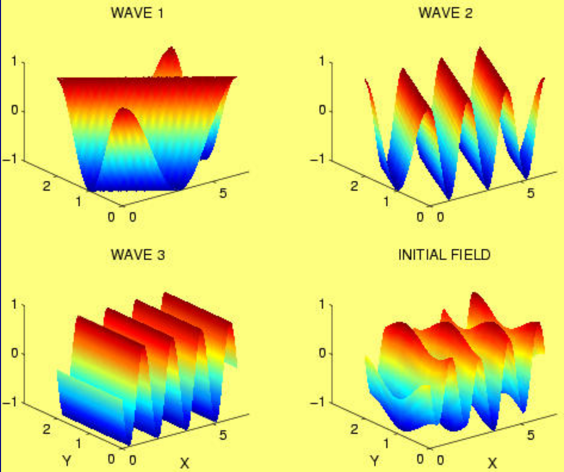
Mathematically Identical!

Numerical Example of Resonance

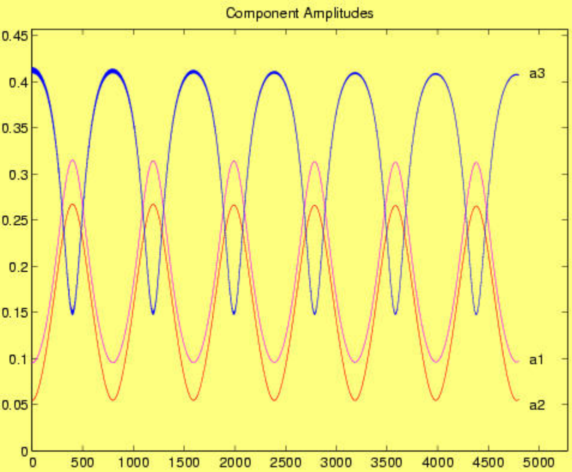
Method of numerical solution of the PDE:

$$\frac{\partial}{\partial t}[\nabla^2\psi - F\psi] + \left\{ \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} \right\} + \beta\frac{\partial\psi}{\partial x} = 0$$

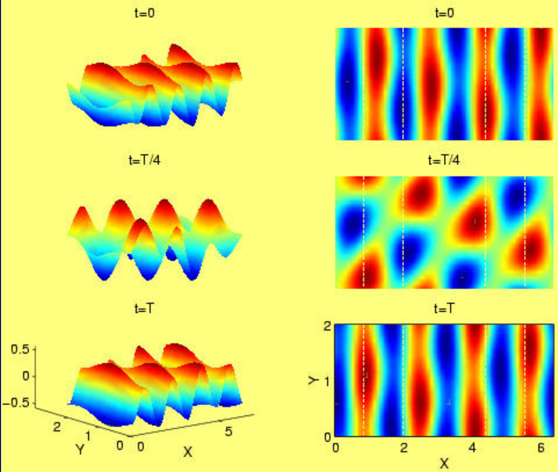
- **Potential vorticity,** $q = [\nabla^2\psi - F\psi]$ **is stepped forward (with leap-frog method)**
- ψ **is obtained by solving a Helmholtz equation with periodic boundary conditions**
- **The Jacobian term is discretized following Arakawa (to conserve energy and enstrophy)**
- **Amplitude is chosen very small. Therefore, interaction time is very long.**



Components of a resonant Rossby wave triad
All fields are scaled to have unit amplitude.



Variation with time of the amplitudes of three components of the stream function.



Stream function at three times during an integration of duration $T = 4800$ days.

Precession of Triads

- **Analogies: Interesting — Equivalences: Useful!**

Since the same equations apply to both the spring and triad systems, the stepwise precession of the spring must have a counterpart for triad interactions.

Precession of Triads

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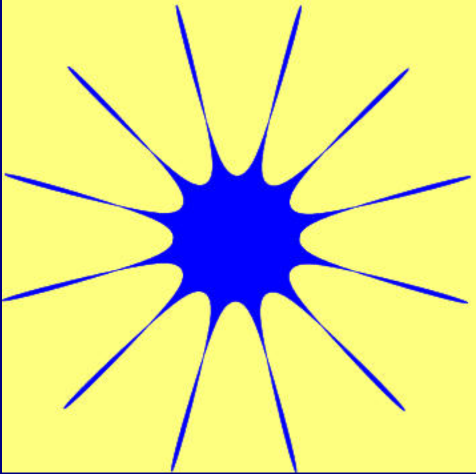
Since the same equations apply to both the spring and triad systems, the stepwise precession of the spring must have a counterpart for triad interactions.

In terms of the variables of the three-wave equations, the semi-axis major and azimuthal angle θ are

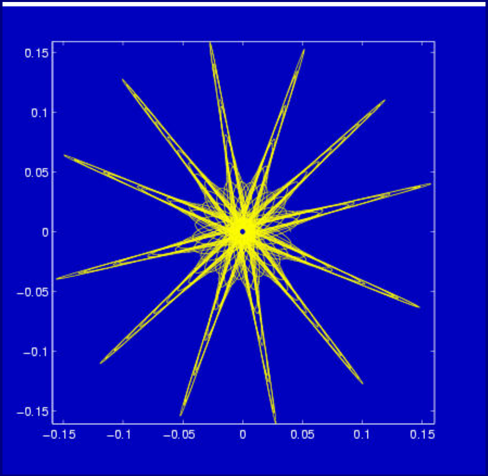
$$A_{\mathrm{maj}} = |A_1| + |A_2|\,, \qquad \theta = \frac{1}{2}(\varphi_1 - \varphi_2)\,.$$

Initial conditions chosen as for the spring (by means of the transformation relations).

Initial field scaled to ensure that small amplitude approximation is accurate.



Polar plot of A_{maj} versus θ for resonant triad.



Horizontal projection of spring solution, y vs. x.

Polar plots of A_{maj} versus θ.

(These are the quantities for the Triad, which correspond to the horizontal projection of the swinging spring.)

- The Star-like pattern is immediately evident.
- Precession angle again about 30°.

This is **remarkable**, and illustrates the value of the equivalence:

Phase precession for Rossby wave triads had not been noted before.

Resonant interactions are important for energy distribution in the atmosphere. They play a central rôle in *Wave Turbulence Theory.*

Outline

Candle-holders from Copenhagen

Fireballs (designer: Pernille Vea)

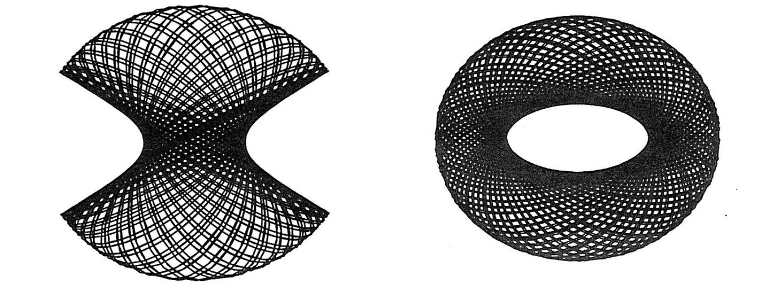
The RnR: a Topless Bowling-ball

Recession

See animated gif of RnR on website.

Globular Cluster: Messier 54, NGC 6715
Class III Extragalactic Globular Cluster.

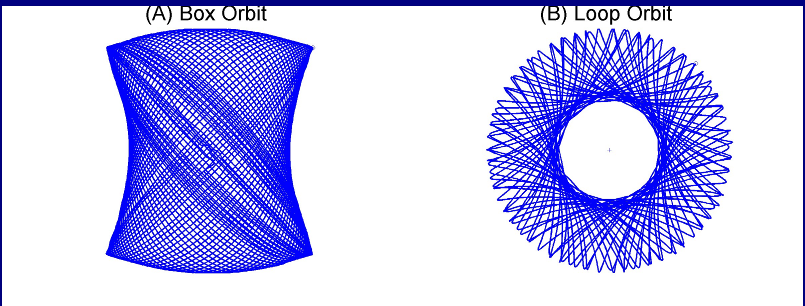
Box and Loop Orbits: Globular Cluster



Two orbits in a logarithmic gravitational potential.
Left: a box orbit. Right: a loop orbit.

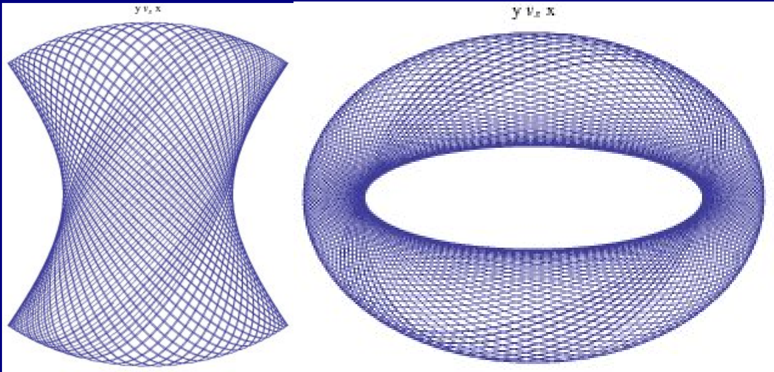
Galactic Dynamics. Binney and Tremaine (2008) [pg. 174]

Box and Loop Orbits: Rock'n'roller



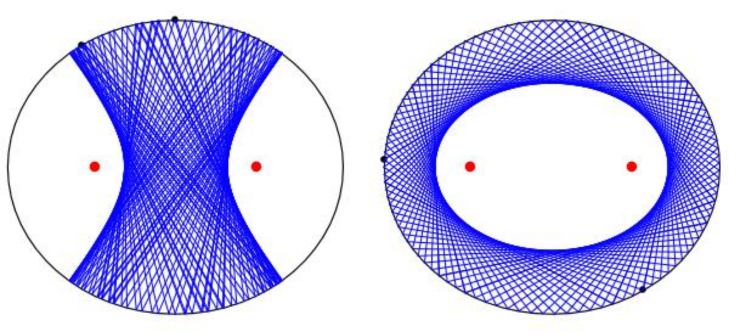
Trajectory of the Rock'n'roller in θ–ϕ-plane (θ radial, ϕ azimuthal) with $\epsilon = 0.1$.

Box and Loop Orbits: Perturbed SHO



Box and Loop orbits for the perturbed SHO.

Box and Loop Orbits: Billiards



Box and Loop orbits on a billiard table.

The RnR: Main Topics

- Two types of trajectories: boxes and loops.
- Simple model: Perturbed 2D harmonic oscillator.
- Small-amplitude motion of rock'n'roller.
- Equations of motion in quaternionic form.
- Recession is associated with box orbits.

Motivation

One of the motivations for studying the Rock'n'roller is the hope of finding an invariant of the motion in addition to the energy. This expectation arises from the symmetry of the body.

For the general Chaplygin Sphere, there is a finite angle δ between the principal axis corresponding to I_3 and the line joining the centres of gravity and symmetry. For the Rock'n'roller, this angle is zero and the Lagrangian is independent of the azimuthal angle ϕ.

However, we have not found a second invariant and, considering the non-holonomic nature of the problem, its existence remains an open question.

Outline

The Perturbed Harmonic Oscillator

Unperturbed system: 2D SHO with equal frequencies:

$$L_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}\omega_0^2(x^2 + y^2)$$

The perturbed system has Lagrangian:

$$L = L_0 - \delta y^2 - \epsilon r^4 \,,$$

where $\delta \ll \omega_0^2$ and $\epsilon \ll 1$.

The δ-term breaks the $1:1$ resonance.

The ϵ-term is a radially symmetric stiffening.

To analyse the system, we assume a solution

$$x(t) = \Re\{A(t)\exp(i\omega_0 t)\} \qquad y(t) = \Re\{B(t)\exp(i\omega_0 t)\}$$

and average the Lagrangian over the fast motion.

We let $A = |A|\exp(i\alpha)$ and $B = |B|\exp(i\beta)$.

Defining $W = |A|^2 - |B|^2$ and $\phi = \alpha - \beta$, we have

$$\frac{dW}{d\tau} = \lambda(1 - W^2)\sin\phi\cos\phi$$

$$\frac{d\phi}{d\tau} = \lambda W \sin^2\phi - 1$$

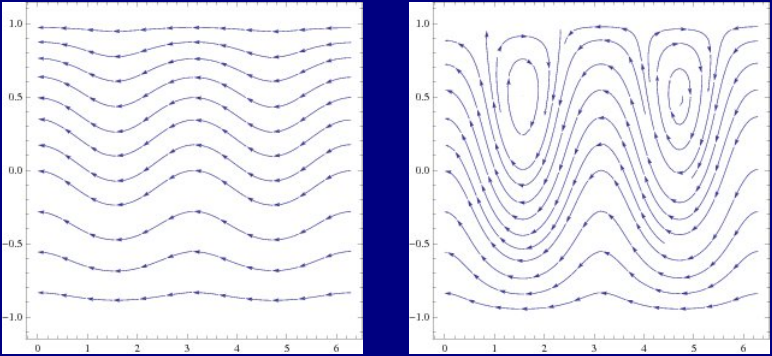
where $\lambda = 2\epsilon U/\delta$ is a non-dimensional parameter.

Again,

$$\frac{dW}{d\tau} = \lambda(1 - W^2)\sin\phi\cos\phi$$
$$\frac{d\phi}{d\tau} = \lambda W \sin^2\phi - 1$$

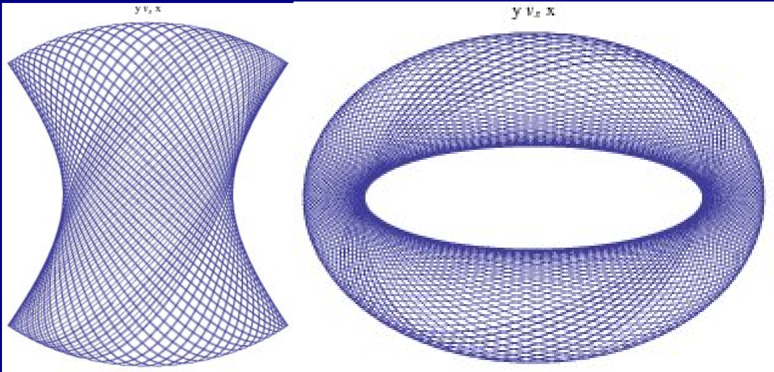
These are the canonical equations for the Hamiltonian

$$H = \tfrac{1}{2}\lambda(1 - W^2)\sin^2\phi + W \,.$$



Phase portraits (W–ϕ plane) for the perturbed SHO. Left panel: $\lambda = 0.5$. Right panel: $\lambda = 2.0$.

Box and Loop Orbits: Perturbed SHO



Box and Loop orbits for the perturbed SHO.

Outline

Sergey Alexeyevich Chaplygin

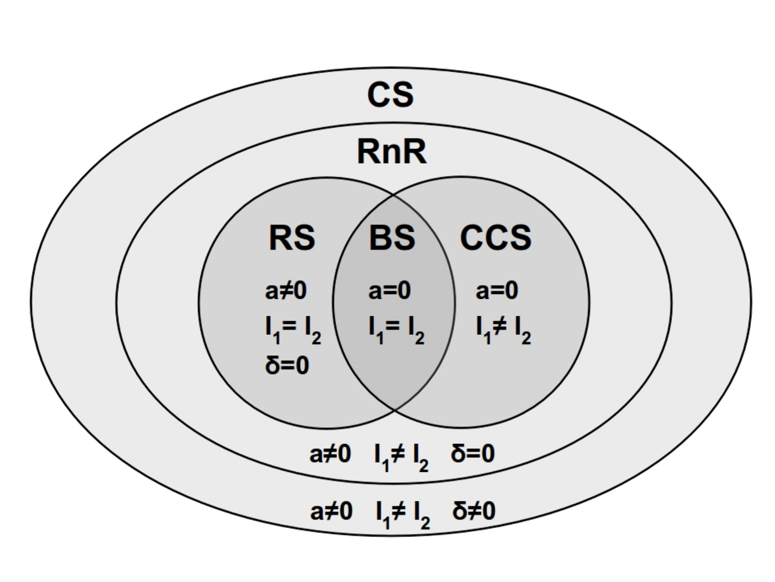
Sergey Alexeyevich Chaplygin

Sergey Alexeyevich Chaplygin (1869–1942) was a Russian physicist, mathematician, and mechanical engineer. He is known for mathematical formulas such as Chaplygin's equation.

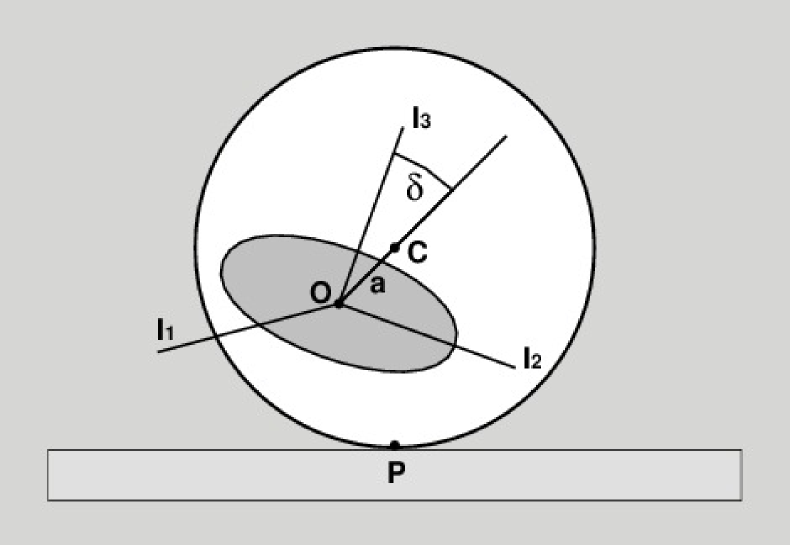
He graduated in 1890 from Moscow University, and later became a professor. He taught mechanical engineering at Moscow's Woman College in 1901, and applied mathematics at Moscow School of Technology, 1903.

Chaplygin was elected to the Russian Academy of Sciences in 1924. The lunar crater Chaplygin and town Chaplygin are named in his honor. His "Collected Works" in four volumes were published in 1948.

The Hierarchy of Spheres



Schematic Diagram of Chaplygin Sphere



RnR: The Physical System

Consider a spherical rigid body with
an asymmetric mass distribution.

Specifically, we consider a loaded sphere.

The dynamics are essentially the same as for the **tippe-top**, which has been studied extensively.

Unit radius and unit mass.

Centre of mass off-set a distance a from the centre.

Moments of inertia I_1, I_2 and I_3, with $I_1 \approx I_2 < I_3$.

The Lagrangian

The Lagrangian of the system is easily written down:

$$L = \tfrac{1}{2}(\mathbf{I_1}\omega_1^2 + \mathbf{I_2}\omega_2^2 + \mathbf{I_3}\omega_3^2) + \tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - ga(1 - \cos\theta)$$

The equations may then be written (in vector form):

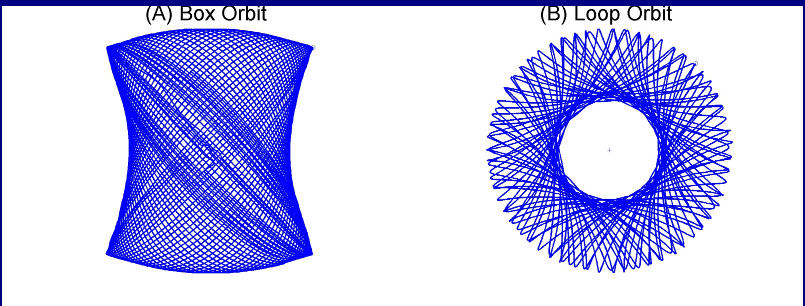
$$\boldsymbol{\Sigma}\dot{\theta} = \omega\,, \qquad \mathbf{K}\dot{\omega} = \mathbf{P}_\omega$$

where the matrices Σ and K are known and

$$\mathbf{P}_\omega = \begin{pmatrix} -(g + \omega_1^2 + \omega_2^2)as\chi + (\mathbf{I_2} - \mathbf{I_3} - af)\omega_2\omega_3 \\ (g + \omega_1^2 + \omega_2^2)as\sigma + (\mathbf{I_3} - \mathbf{I_1} + af)\omega_1\omega_3 \\ (\mathbf{I_1} - \mathbf{I_2})\omega_1\omega_2 + as(-\chi\omega_1 + \sigma\omega_2)\omega_3 \end{pmatrix}$$

Note that neither K nor $\mathbf{P}_\omega$ depends explicitly on ϕ.

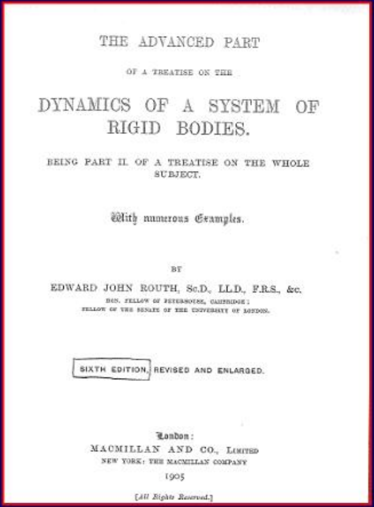
Box and Loop Orbits: Rock'n'roller



Trajectory of the Rock'n'roller in θ–ϕ-plane (θ radial, ϕ azimuthal) with $\epsilon = 0.1$.

Outline

The Routh Sphere: $I_1 = I_2$



THE ADVANCED PART

OF A TREATISE ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

BEING PART II. OF A TREATISE ON THE WHOLE SUBJECT.

With numerous Examples.

BY

EDWARD JOHN ROUTH, Sc.D., LL.D., F.R.S., &c.

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FELLOW OF THE SENATE OF THE UNIVERSITY OF LONDON.

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Cover of Routh's *Dynamics* Part II

In the Cambridge Mathematical Tripos Examination of 1854, James Clark Maxwell came second.

Edward John Routh came first (senior wrangler).

Constants of Motion for Routh Sphere

In case $\mathbf{I}_1 = \mathbf{I}_2$, there are three degrees of freedom and three constants of integration.

The kinetic energy is

$$K = \frac{1}{2}[u^2 + v^2 + w^2] + \frac{1}{2}[\mathbf{I}_1(\omega_1^2 + \omega_2^2) + \mathbf{I}_3\omega_3^2]$$

The potential energy is

$$V = mga(1 - \cos\theta)\,.$$

Since there is no dissipation,

$$E = K + V = \text{constant}\,.$$

Constants of Motion for Routh Sphere

Jellett's constant is the scalar product:

$$C_J = \mathbf{L} \cdot \mathbf{r} = \mathbf{I}_1 s(\sigma\omega_1 + \chi\omega_2) + \mathbf{I}_3 f\,\omega_3 = \text{constant}\,.$$

where $s = \sin\theta$, $f = \cos\theta - a$, $\sigma = \sin\psi$ and $\chi = \cos\psi$.
[S O'Brien & J L Synge first gave this interpretation.]

Routh's constant (difficult to interpret physically):

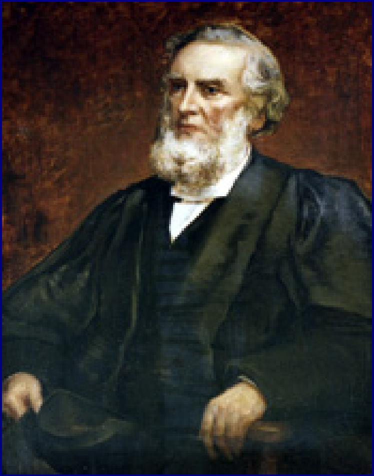
$$C_R = \left[\sqrt{\mathbf{I}_3 + s^2 + (\mathbf{I}_3/\mathbf{I}_1)f^2}\right]\omega_3 = \text{constant}\,.$$

Constant C_R implies conservation of *sign* of ω_3 . . .
. . . but this does not automatically preclude recession!

Edward J Routh

John H Jellett

1831–1907



1817–1888

Edward J Routh

Edward John Routh (20 January 1831 to 7 June 1907), an English mathematician, noted as the outstanding coach of students preparing for the Mathematical Tripos examination of the University of Cambridge.

He also did much to systematize the mathematical theory of mechanics and created several ideas critical to the development of modern control systems theory.

In 1854, Routh graduated just above James Clerk Maxwell, as Senior Wrangler, sharing the Smith's prize with him. He coached over 600 pupils between 1855 and 1888, 27 of them making Senior Wrangler.

Known for: Routh-Hurwitz theorem, Routh stability criterion, Routh array, Routhian, Routh's theorem, Routh's algorithm, Kirchhoff-Routh function.

John H Jellett

J. H. Jellett was a native of Cashel, County Tipperary, the son of a clergyman. He graduated from Trinity College with honors in mathematics in 1838, and was elected to Fellowship in 1840. In 1847 he was appointed to the newly established chair of Natural Philosophy (Applied Mathematics), which he held until 1870.

Jellett was a scholar of considerable eminence and his publications cover the fields of pure and applied mathematics, notably the theory of friction and the properties of optically active solutions, as well as sermons and lectures on religious topics.

He was President of the Royal Irish Academy for five years from 1869, received the Royal Society's Medal in 1881 and an honorary degree from Oxford in 1887.

His politics were sufficiently liberal to make him an acceptable candidate to Gladstone who appointed him Provost of Trinity College Dublin in April 1881. He died in office on 19 February 1888.

Integrability of Routh Sphere

Using Routh's constant C_R, we have $\omega_3 = \omega_3(\theta)$.

Then, using Jellett's constant C_J, we have $\omega_2 = \omega_2(\theta)$.

Using the energy equation, we can now write:

$$\dot{\theta}^2 = f(\theta)\,.$$

For a given θ, both ω_2 and ω_3 are fixed:
This confirms that recession is impossible.

Invariants of the Rock'n'roller

The only known constant of motion is total energy E.

There remains a symmetry: the system is unchanged under the transformation

$$\phi \longrightarrow \phi + \delta\phi$$

The spirit of Noether's Theorem would indicate another constant associated with this symmetry;

So far, we have not found a "missing constant".

Outline

Quaternionic Formulation

The Euler angles have a singularity when $\theta = 0$
The angles ϕ and ψ are not uniquely defined there.

We can obviate this problem by using
Euler's symmetric parameters:

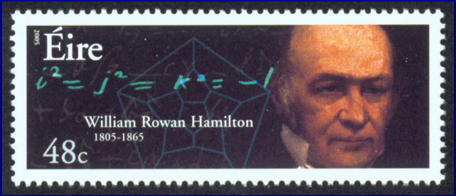
$$\gamma = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi) \qquad \xi = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi)$$
$$\zeta = \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \qquad \eta = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)$$

These are the components of a unit quaternion

$$\mathbf{q} = \gamma + \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$$

$$\gamma^2 + \xi^2 + \eta^2 + \zeta^2 = 1$$

William Rowan Hamilton (1805–1865)



Quaternion Equations

Euler's symmetric parameters,

or

Euler-Rodrigues parameters:

$$\gamma = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi) \qquad \xi = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi)$$

$$\zeta = \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \qquad \eta = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)$$

The components of angular velocity are

$$\begin{aligned}\omega_1 &= 2[\gamma\dot\xi - \xi\dot\gamma + \zeta\dot\eta - \eta\dot\zeta]\\ \omega_2 &= 2[\gamma\dot\eta - \eta\dot\gamma + \xi\dot\zeta - \zeta\dot\xi]\\ \omega_3 &= 2[\gamma\dot\zeta - \zeta\dot\gamma + \eta\dot\xi - \xi\dot\eta]\end{aligned}$$

Lagrangian and Hamiltonian

The quaternion equations arise from the Lagrangian

$$L = \tfrac{1}{2}(k_1\dot{\mu}^2 + k_2\dot{\nu}^2) - \tfrac{1}{2}(k_1\tilde{\Omega}_1^2\mu^2 + k_2\tilde{\Omega}_2^2\nu^2) + k_1k_2(\mu\dot{\nu} - \nu\dot{\mu})$$

where $(\gamma, \zeta, \xi, \eta) \rightarrow (\gamma, \zeta, \mu, \nu)$**.**

The generalized momenta are

$$p_\mu = k_1(\dot{\mu} - k_2\nu) \qquad \textbf{and} \qquad p_\nu = k_2(\dot{\nu} + k_2\mu)$$

The Hamiltonian is

$$H = \tfrac{1}{2}\left(\frac{p_\mu^2}{k_1} + \frac{p_\nu^2}{k_2}\right) - [k_1\mu p_\nu - k_2\nu p_\mu] + \tfrac{1}{2}[k_1(k_1k_2 + \tilde{\Omega}_1^2)\mu^2 + k_2(k_1k_2 + \tilde{\Omega}_2^2)\nu^2]$$

Constants of the Motion

The numerical value of the Hamiltonian (energy) is

$$E_{\mu+\nu} = \tfrac{1}{2}(k_1\dot{\mu}^2 + k_2\dot{\nu}^2) + \tfrac{1}{2}(k_1\tilde{\Omega}_1^2\mu^2 + k_2\tilde{\Omega}_2^2\nu^2)$$

An additional constant of the motion can be found:

$$K_1 \equiv \left(\frac{\lambda_2\dot{\mu}+\beta_2\nu}{\beta_1\lambda_2-\beta_2\lambda_1}\right)^2 + \left(\frac{\dot{\nu}-\beta_2\lambda_2\mu}{\beta_1\lambda_1-\beta_2\lambda_2}\right)^2 = \mu_1^2\,,$$

$$K_2 \equiv \left(\frac{\lambda_1\dot{\mu}+\beta_1\nu}{\beta_1\lambda_2-\beta_2\lambda_1}\right)^2 + \left(\frac{\dot{\nu}-\beta_1\lambda_1\mu}{\beta_1\lambda_1-\beta_2\lambda_2}\right)^2 = \mu_2^2\,.$$

Numerical tests confirm that K_1 and K_2 are constant.

Aspiration

To find an invariant of the motion of the Rock'n'roller
in addition to the energy.

This expectation arises from the symmetry of the body.

In view of the non-holonomic nature of the problem,
its existence remains an open question.

However, the box and loop orbits suggest
that a search would be worthwhile.

Aspiration

To find an invariant of the motion of the Rock'n'roller
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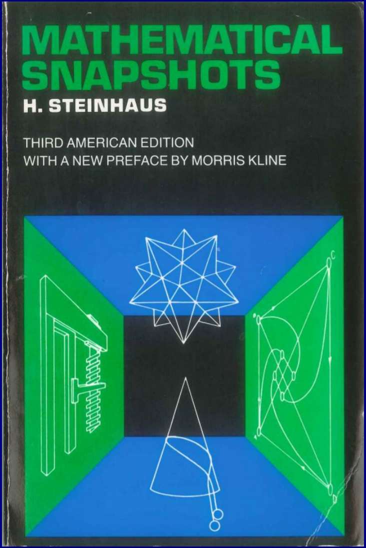
its existence remains an open question.

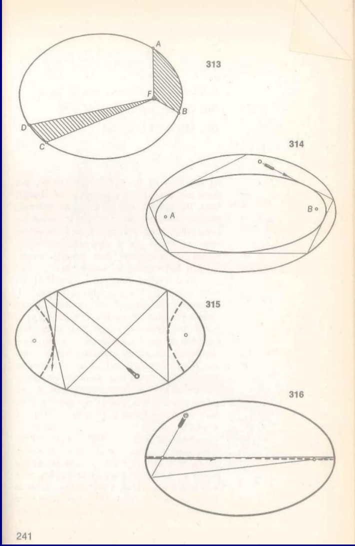
However, the box and loop orbits suggest
that a search would be worthwhile.

In elliptical billiards there is an "extra" invariant: $p_1 \times p_2$.

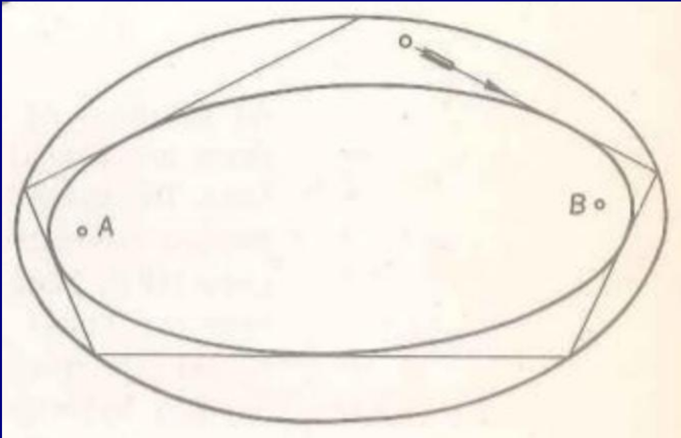
Outline

Kalejdoskop Matematyczny (1939)

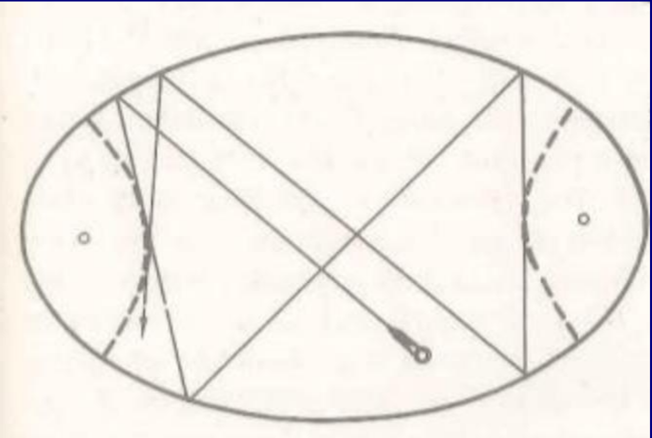




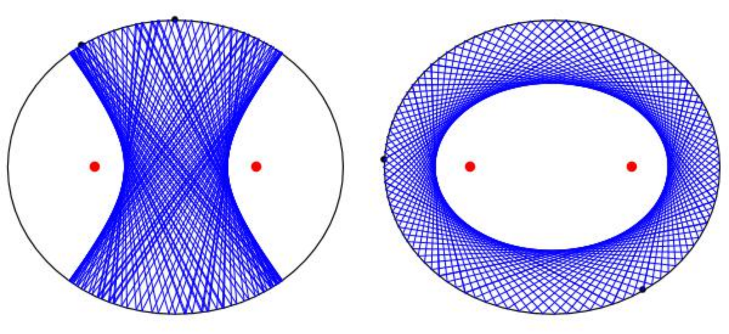
Billiard Shot leading to a Loop Orbit



Billiard Shot leading to a Box Orbit



Box and Loop Orbits: Billiards



Box and Loop orbits on an elliptical billiard table.

Extra invariant: $p_1 \times p_2$ is conserved.

Billiards and Ballyards

Main idea:

Billiard Table with Soft Cushions $\Longrightarrow$ Ballyard.

Playing surface no longer quite flat.

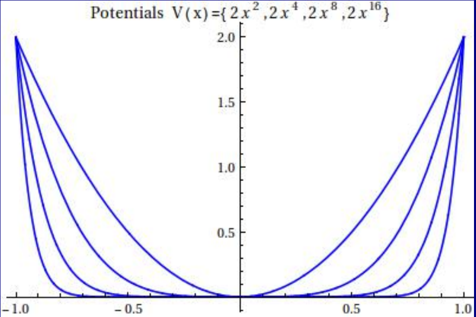


Figure : Potentials $2x^N$ for $N \in \{2, 4, 8, 16\}$.

Back to Basics: 1 Dimension

A particle in a parabolic well $z = \frac{1}{2} z_1 x^2$ has Lagrangian

$$L = \frac{1}{2}\dot{x}^2(1 + z_1^2 x^2) - \frac{1}{2}(g z_1) x^2$$

The Euler-Lagrange equations are

$$(1 + z_1^2 x^2)\ddot{x} + g z_1 x + z_1^2 x \dot{x}^2 = 0$$

Back to Basics: 1 Dimension

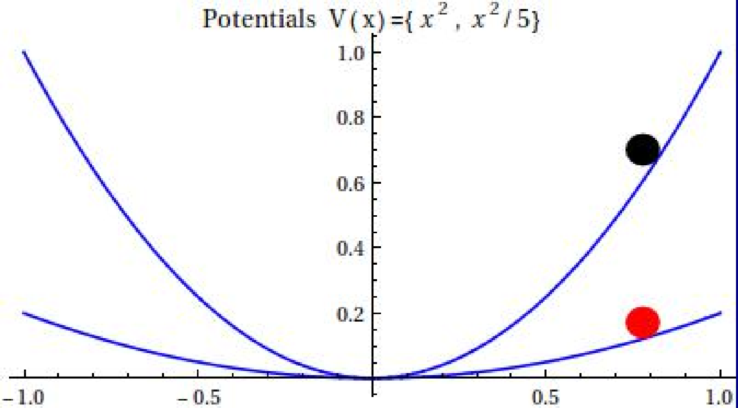


Figure : Potentials x^2 and $x^2/5$.

Back to Basics: 1 Dimension

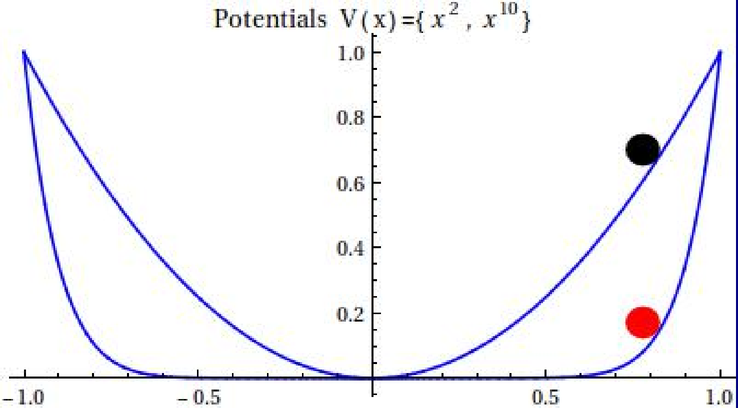


Figure : Potentials x^2 and x^{10}.

Back to Basics: 1 Dimension

To linearize, make z_1 small but keep gz_1 fixed:

$$\ddot{x} + (gz_1)x = 0\,.$$

This is a geometric/gravimetric approximation.

We are flattening the table while turning up gravity.

Moving to 2 Dimensions: Circular Table

A particle in a paraboloidal well with axial symmetry, $z = \frac{1}{2} z_1 r^2$ has Lagrangian

$$L = \tfrac{1}{2}[(1 + z_1^2 r^2)\dot{r}^2 + r^2\dot{\theta}^2] - \tfrac{1}{2}(g z_1) r^2$$

As before, we let $z_1 \to 0$ with $\frac{1}{2} g z_1 = 1$.

Since θ is a cyclic variable, $\partial L / \partial \dot{\theta} = r^2 \dot{\theta}$ is constant.

My Gaffe: Eliminate $\dot{\theta}$ from L using $h = r^2 \dot{\theta}$.

Correct: Get E-L equation, then use $h = r^2 \dot{\theta}$.

Circular Ballyards

Since the restoring force is central,
the angular momentum is conserved.

The system is integrable.

Potentials with Increasing Power

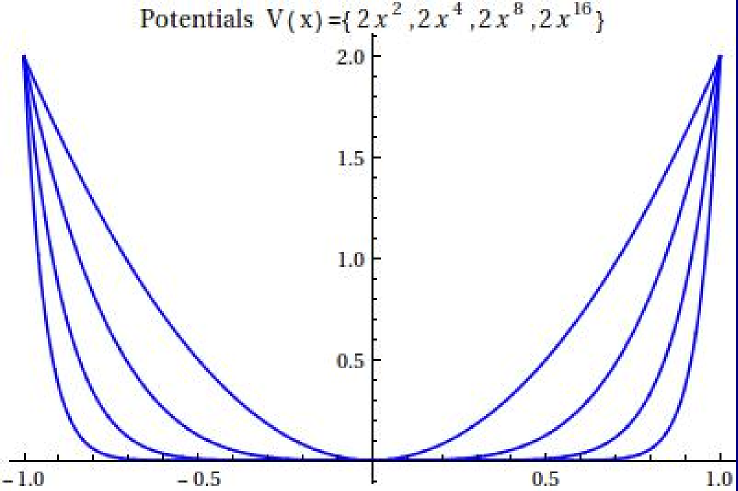


Figure : Potentials $2x^2$, $2x^4$, $2x^8$, and $2x^{16}$.

Changing the Potential

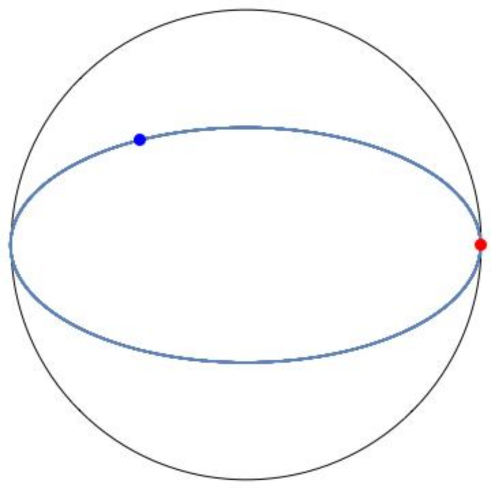


Figure : N=2.

Changing the Potential

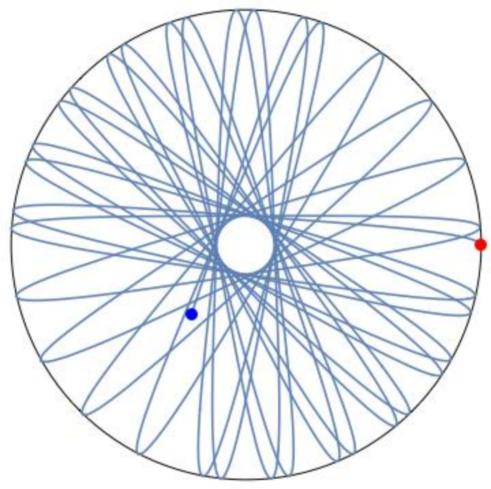


Figure : N=4.

Changing the Potential

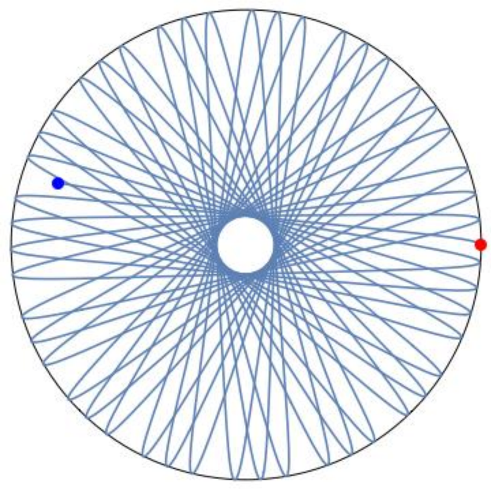


Figure : N=8.

Changing the Potential

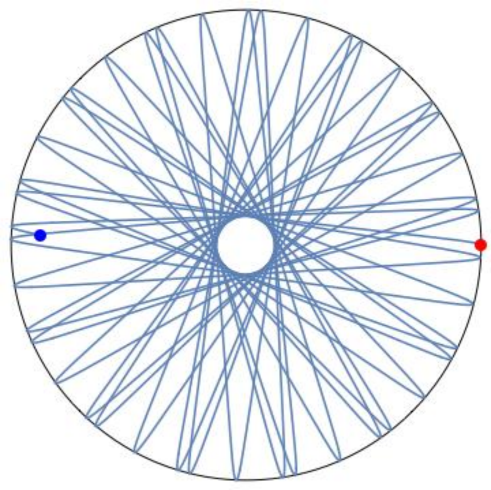


Figure : N=16.

Changing the Potential

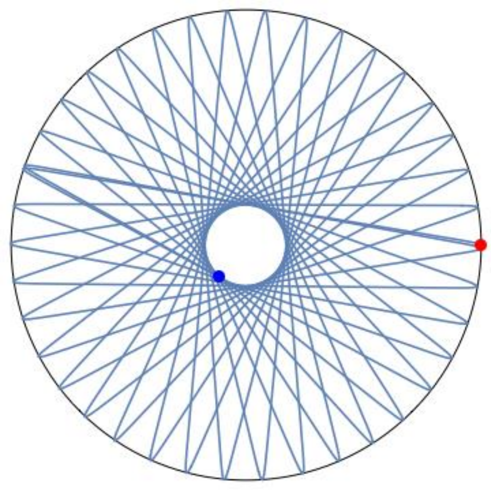


Figure : N=32.

Changing the Potential

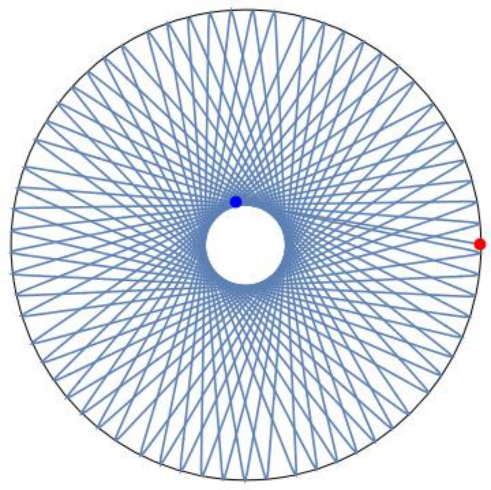
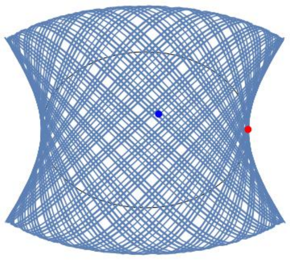
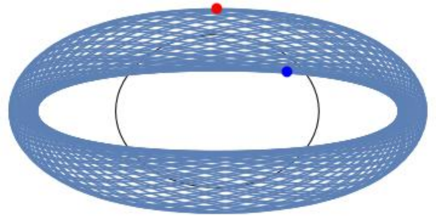


Figure : N=64.

Elliptic Ballyards: $N = 4$





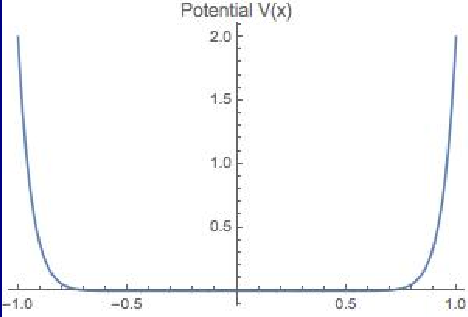
Left: Angular Momentum varies from -6 to $+6$.
Right: Angular Momentum varies from -4 to -10.

Additional Constant of Motion not found (yet!)

Outline

Squircular Ballyard Potential

$$x^4 + y^4 = 1$$



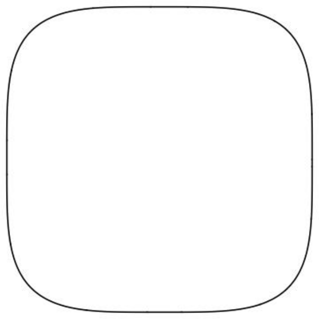


Figure : Potential, $V(x)$, $N = 8$. Left: VER-X. Right: HOR-X.

Squircular Ballyard

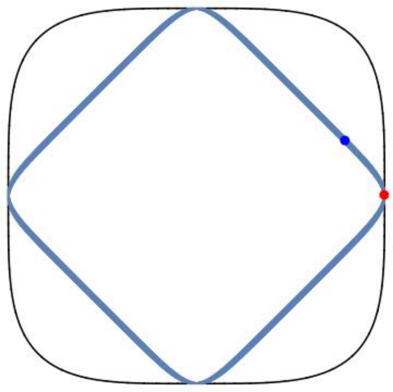


Figure : $N = 2$. ICs = 1.

Squircular Ballyard

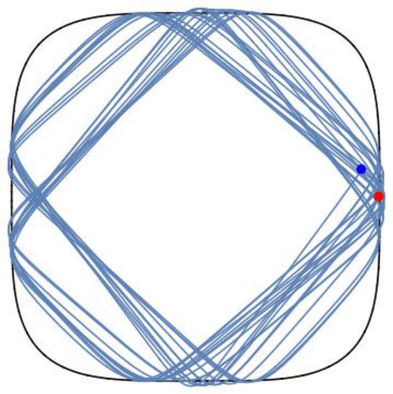
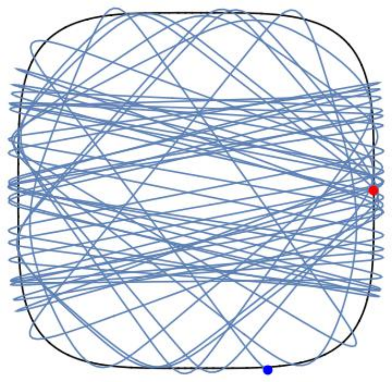


Figure : $N = 2$. ICs = 2.

Squircular Ballyard



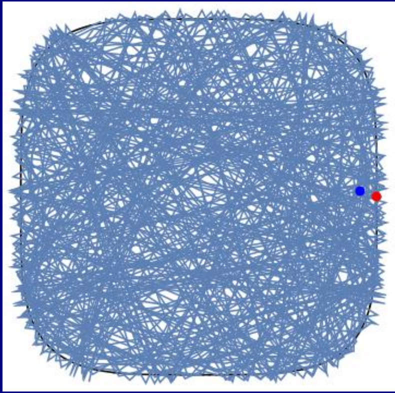


Figure : $N = 2$. ICs = 3.

Squovular Ballyard

$$\left(\frac{x}{a}\right)^8 + \left(\frac{y}{b}\right)^8 = 1$$

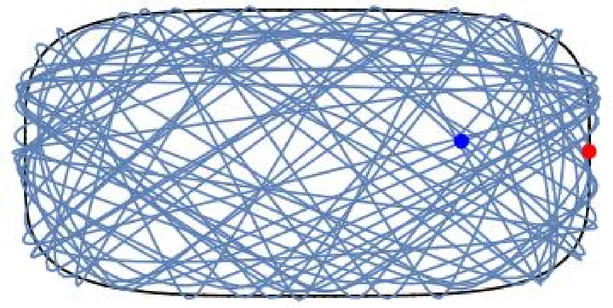


Figure : Aspect ratio 2 : 1.

Outline

Conclusion

AIM:
To find an invariant of the motion of the Rock'n'roller in addition to the energy.

This expectation arises from the symmetry of the body.

The box and loop orbits suggest that a search would be worthwhile.

The investigation of elliptical billiards may be fruitful.

But the goal is not yet reached

Conclusion

"I Still Haven't Found What I'm Looking For"

Conclusion

"I Still Haven't Found What I'm Looking For"

Let's hope for success by Darryl's 75th Birthday.

Thank You All

Thank You All

and

Happy Birthday, Darryl!