Recent Developments in Digital Filter Initialization

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The Idea of Filtering

A primitive filter model:

$$Good/Bad/Ugly \Longrightarrow \boxed{\textbf{Filter}} \longrightarrow Good$$

Suppose the input consists of a low-frequency (LF) signal contaminated by high-frequency (HF) noise. We use a low-pass filter which rejects the noise.

Low Frequency
High Frequency $\Longrightarrow$ **Low-Pass Filter** $\longrightarrow$ **Low Frequency**

Some Applications of Digital Filters

- **Telecommunications**
  - **Digital Switching / Multiplexing / Touch-tone Dialing**
- **Audio Equipment**
  - **Compact Disc Recording / Hi-Fi Reproduction**
- **Speech Processing**
  - **Voice Recognition / Speech Synthesis**
- **Image Processing**
  - **Image Enhancement / Data Compression**
- **Remote Sensing**
  - **Doppler Radar / Sonar Signal Processing**
- **Geophysics**
  - **Seismology / Initialization for NWP.**

Non-recursive Digital Filter

Consider a discrete time signal,

$$\{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\}$$

For example, x_n could be the value of surface pressure at time $n\Delta t$ at a specific location, say, Sapporo.

Nonrecursive Digital Filter:

A *nonrecursive* digital filter is defined by

$$y_n = \sum_{k=-N}^{+N} h_k x_{n-k}$$

The inputs are $\{x_n\}$. The outputs are $\{y_n\}$.
The outputs are <u>weighted sums of the inputs</u>.

Application to Initialization

- ***Model integrated forward for N steps:***

$$y_{\mathrm{FOR}} = \frac{1}{2}h_0 x_0 + \sum_{n=1}^{N} h_{-n} x_n$$

- ***N-step 'hindcast' is made:***

$$y_{\mathrm{BAK}} = \frac{1}{2}h_0 x_0 + \sum_{n=-1}^{-N} h_{-n} x_n$$

- ***The two sums are combined:***

$$y_0 = y_{\mathrm{FOR}} + y_{\mathrm{BAK}}$$

Digital Filters as Convolutions

Consider the nonrecursive digital filter

$$y_n = \sum_{k=-N}^{+N} h_k x_{n-k} \,.$$

The indices of x and a run in opposite directions:

$$h_{-N}, \cdots, h_{-1}, h_0, h_1, \cdots, h_N$$
$$x_{n+N}, \cdots, x_{n+1}, x_n, x_{n-1}, \cdots, x_{n-N}$$

so that the sum is in the form of a finite convolution:

$$y_n = \{h_n\} \star \{x_n\}.$$

By a careful choice of the coefficients h_n, we can design a filter with the desired selection properties.

Frequency Response of FIR Filter

Let x_n be the input and y_n the output.
Assume $x_n = \exp(in\theta)$ and $y_n = H(\theta)\exp(in\theta)$.

The transfer function $H(\theta)$ is then

$$H(\theta) = \sum_{k=-N}^{N} h_k e^{-ik\theta}\,.$$

This gives H once the coefficients h_k have been specified.

However, what is really required is the opposite: to derive coefficients which will yield the desired response.

This *inverse problem* has no unique solution, and numerous techniques have been developed.

Design of Nonrecursive Filters

We consider the simplest possible design technique, using a truncated Fourier series modified by a window function.

Consider a sequence

$$\{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\}$$

with low and high frequency components.

To filter out the high frequencies one may proceed According to the following Three-step method:

1. **Calculate the Fourier transform $X(\theta)$ of x_n;**
2. **Set the coefficients of the high frequencies to zero;**
3. **Calculate the inverse transform.**

Three-Step Procedure

1. Calculate the Fourier transform $X(\theta)$ of x_n;

2. Set the coefficients of the high frequencies to zero;

3. Calculate the inverse transform.

Step [1] is a forward Fourier transform:

$$X(\theta) = \sum_{n=-\infty}^{\infty} x_n e^{-in\theta},$$

where $\theta = \omega\Delta t$ is the *digital frequency*. $X(\theta)$ is 2π-periodic.

Step [2] may be performed by multiplying $X(\theta)$ by an appropriate weighting function $H(\theta)$.

Step [3] is an inverse Fourier transform:

Filtering as Convolution

Step [3] is an inverse Fourier transform. The product $H(\theta) \cdot F(\theta)$ is the transform of the convolution of $\{h_n\}$ with $\{x_n\}$:

$$y_n = (h * x)_n = \sum_{k=-\infty}^{\infty} h_k x_{n-k}.$$

In practice, we must truncate the sum:

$$y_n = \sum_{k=-N}^{N} h_k x_{n-k}.$$

The finite approximation to the convolution is formally identical to a nonrecursive digital filter.

Filter Coefficients

The function $H(\theta)$ is called the

- **System Function**
- **Transfer Function**
- **Response Function.**

Typically, $H(\theta)$ is a step function:

$$\begin{aligned} H(\theta) &= 1, \qquad |\theta| \le |\theta_c|\,; \\ H(\theta) &= 0, \qquad |\theta| > |\theta_c|\,. \end{aligned}$$

$$h_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta) e^{in\theta} d\theta \quad ; \quad H(\theta) = \sum_{n=-\infty}^{\infty} h_n e^{-in\theta}$$

$$h_n = \frac{\sin n\theta_c}{n\pi}.$$

Windowing

Truncation gives rise to Gibbs oscillations.

The response of the filter is improved if h_n is multiplied by the Lanczos window

$$w_n = \frac{\sin\big(n\pi/(N+1)\big)}{n\pi/(N+1)}.$$

$$\hat{h}_n = w_n \left(\frac{\sin(n\theta_c)}{n\pi} \right).$$

$$H(\theta) = \sum_{k=-N}^{N} \hat{h}_k e^{-ik\theta} = \left[\hat{h}_0 + 2 \sum_{k=1}^{N} \hat{h}_k \cos k\theta \right].$$

Optimal Filter Design

This method uses the Chebyshev alternation theorem to obtain a filter whose maximum error in the pass- and stop-bands is minimized. Such filters are called Optimal Filters.

References:

- **Hamming (1989)**
- **Oppenheim and Schafer (1989)**

Optimal Filters require solution of complex nonlinear systems of equations. The algorithm for calculation of the coefficients involves about one thousand lines of code.

The Dolph Filter is a special optimal filter, which is much easier to calculate.

The Dolph-Chebyshev Filter

This filter is constructed using Chebyshev polynomials:

$$\begin{aligned} T_n(x) &= \cos(n \cos^{-1} x), \qquad |x| \le 1 \\ T_n(x) &= \cosh(n \cosh^{-1} x), \qquad |x| > 1\,. \end{aligned}$$

Clearly, $T_0(x) = 1$ **and** $T_1(x) = x$**. Also:**

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n \ge 2\,.$$

Now define a function in the frequency domain:

$$H(\theta) = \frac{T_{2M}\left(x_0 \cos\left(\theta/2\right)\right)}{T_{2M}(x_0)}$$

where $x_0 > 1$**. Let** θ_s **be such that** $x_0 \cos(\theta_s/2) = 1$**. The form of** $H(\theta)$ **is that of a low-pass filter with a cut-off at** $\theta = \theta_s$**.**

$H(\theta)$ **can be written as a** *finite expansion*

$$H(\theta) = \sum_{n=-M}^{+M} h_n \exp(-in\theta).$$

The coefficients $\{h_n\}$ **may be evaluated:**

$$h_n = \frac{1}{N}\left[1 + 2r\sum_{m=1}^{M} T_{2M}\left(x_0 \cos\frac{\theta_m}{2}\right)\cos m\theta_n\right],$$

where $|n| \le M$, $N = 2M+1$ **and** $\theta_m = 2\pi m/N$.
The coefficients h_n **are real and** $h_{-n} = h_n$.

The weights $\{h_n : -M \le n \le +M\}$ **define the Dolph-Chebyshev or, for short, Dolph filter.**

An Example of the Dolph Filter

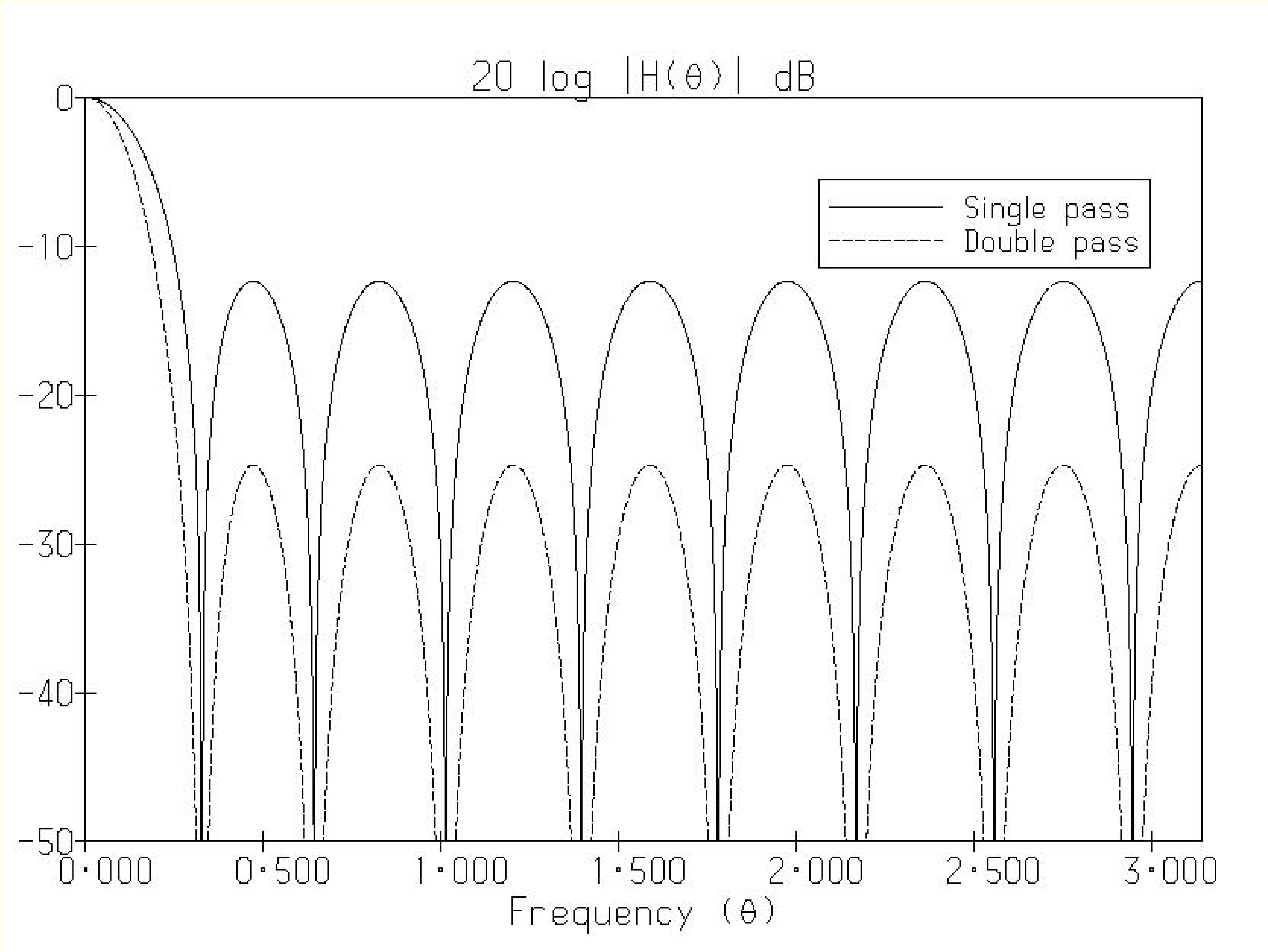
We choose the following parameters:

- **Cut-off period:** $\tau_s = 3\,\mathrm{h}$
- **Time-step:** $\Delta t = \frac{1}{8}\,\mathrm{h} = 7\frac{1}{2}\,\mathrm{min}$.
- **Filter span:** $T_{\mathrm{S}} = 2\,\mathrm{h}$.
- **Filter order:** $N = 17$.

Then the digital cut-off frequency is

$$\theta_s = 2\pi\Delta t/\tau_s \approx 0.26\,.$$

This filter attenuates high frequency components by more than 12 dB. Double application gives 25 dB attenuation.



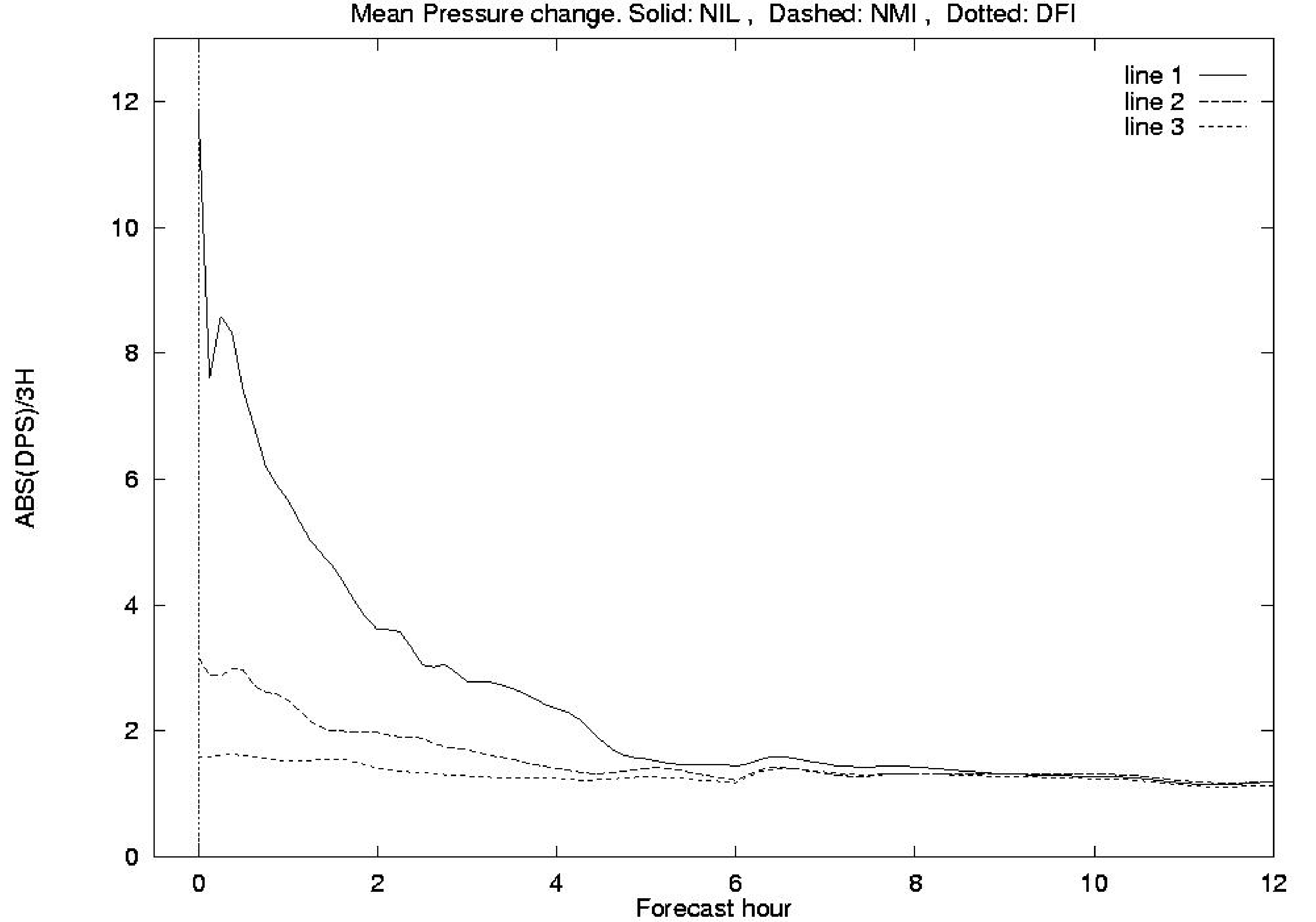
Frequency response for Dolph filter with span $T_S = 2\text{h}$, order $N = 2M + 1 = 17$ and cut-off $\tau_s = 3\text{h}$. Results for single and double application are shown.

Implementation in HIRLAM:

Hop, Skip and Jump

The initialization and forecast are performed in three stages:

- ***Hop:* *Adiabatic backward integration. Output filtered to give fields valid at*** $t = -\frac{1}{2}T_{\mathrm{S}}$.
- ***Skip:* *Forward diabatic run spanning range*** $[-\frac{1}{2}T_{\mathrm{S}}, +\frac{1}{2}T_{\mathrm{S}}]$. ***Output filtered to provide initialized values.***
- ***Jump:* *Normal forecast, covering desired range.***

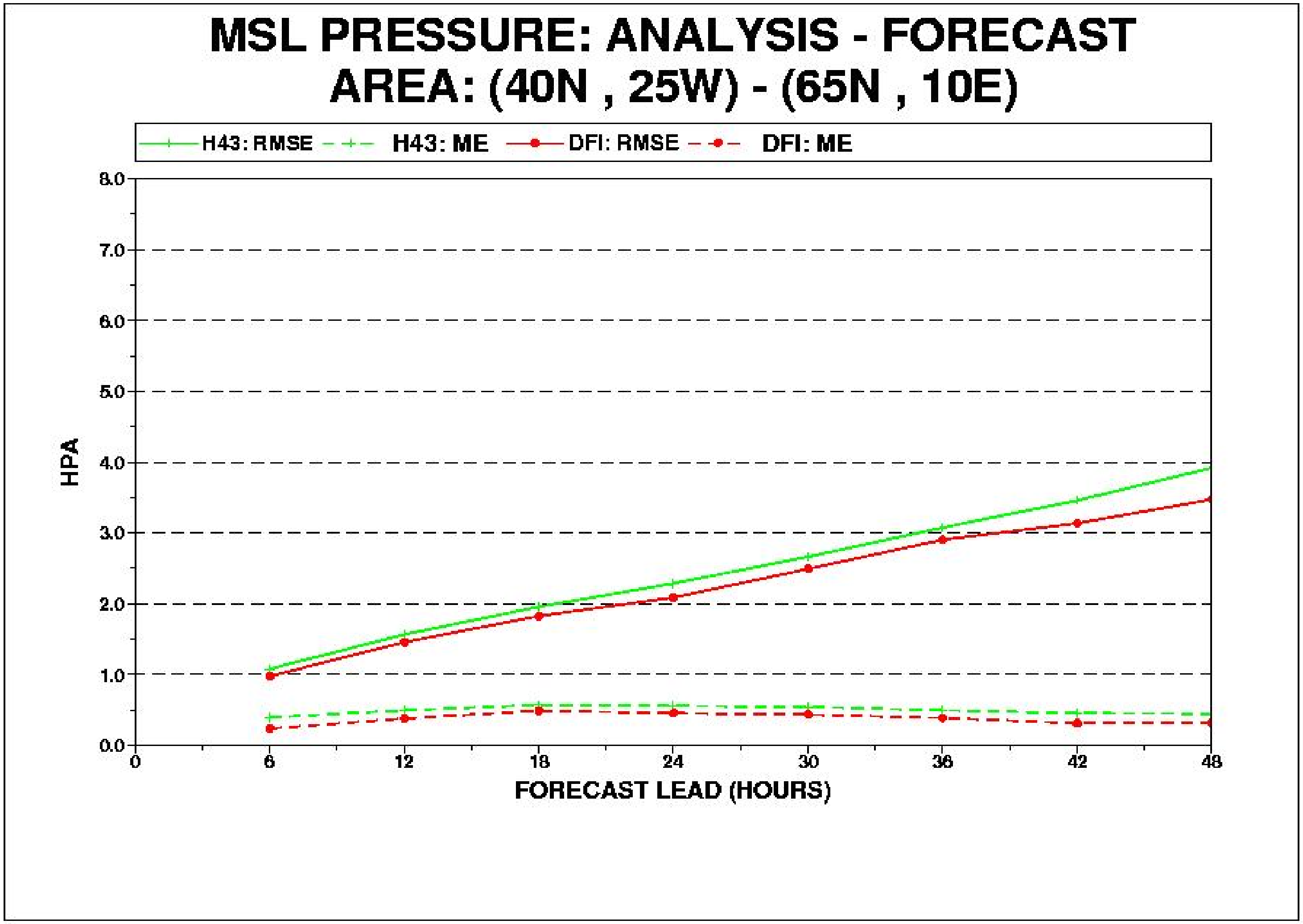


Mean absolute surface pressure tendency for three forecasts. Solid: uninitialized analysis (NIL). Dashed: Normal mode initialization (NMI). Dotted: Digital filter initialization (DFI). Units are hPa/3 hours.

Changes in Surface Pressure

Table 1: Changes in model prognostic variables at analysis time and for the 24-hour forecast, induced by DFI. Units are hPa.

Psurf	Analysis		Forecast	
	max	rms	max	rms
	2.21	.493	.924	.110



Root-mean-square (solid) and bias (dashed) errors for mean sea-level pressure. Average over thirty Fastex forecasts. Green: reference run (NMI); Red: DFI run.

Application to Richardson Forecast

- ***NIL:*** $$\frac{dp_s}{dt} = +145\,\mathrm{hPa}/6\,\mathrm{h}.$$
- ***LANCZOS:*** $$\frac{dp_s}{dt} = -2.3\,\mathrm{hPa}/6\,\mathrm{h}.$$
- ***DOLPH:*** $$\frac{dp_s}{dt} = -0.9\,\mathrm{hPa}/6\,\mathrm{h}.$$

Observations: Barometer steady!

IDFI in GME Model at DWD

A DFI scheme is used in the initialization of the GME model at the Deutscher Wetterdienst.

Incremental DFI is applied: Only the analysis increments are filtered.

$$
\begin{aligned}
X_A &= X_F + (X_A - X_F) \\
X_A \longrightarrow \bar{X}_A\,, &\quad X_F \longrightarrow \bar{X}_F \\
\bar{X}_A &= X_F + (\bar{X}_A - \bar{X}_F)
\end{aligned}
$$

If analysis increment vanishes, filter has no effect.

The scheme is applied in vertical normal mode space. The first ten vertical modes are filtered, the remaining 21 of the 31-level GME are left unchanged.

The damping of physical processes, such as precipitation and convection, by the IDFI is thus reduced to an acceptably low level.

Half-sinc Filters

An ideal low-pass filter has an impulse response

$$h_n = \frac{\sin n\theta_c}{n\pi} = \left(\frac{\theta_c}{\pi}\right) \mathrm{sinc}\left(\frac{n\theta_c}{\pi}\right), \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots .$$

For a causal filter we require $n \geq 0$. Then

$$h_n = \frac{\sin n\theta_c}{n\pi}, \qquad n = 0, 1, \ldots, N-1 .$$

We refer to this sequence as a half-sinc sequence.
The frequency response may be written

$$\sum_{n=0}^{N-1} h_n e^{in\theta} = H(\theta) = M(\theta) e^{i\varphi(\theta)} .$$

Boundary Filters

The group delay is defined as $\delta = -d\varphi/d\theta$.

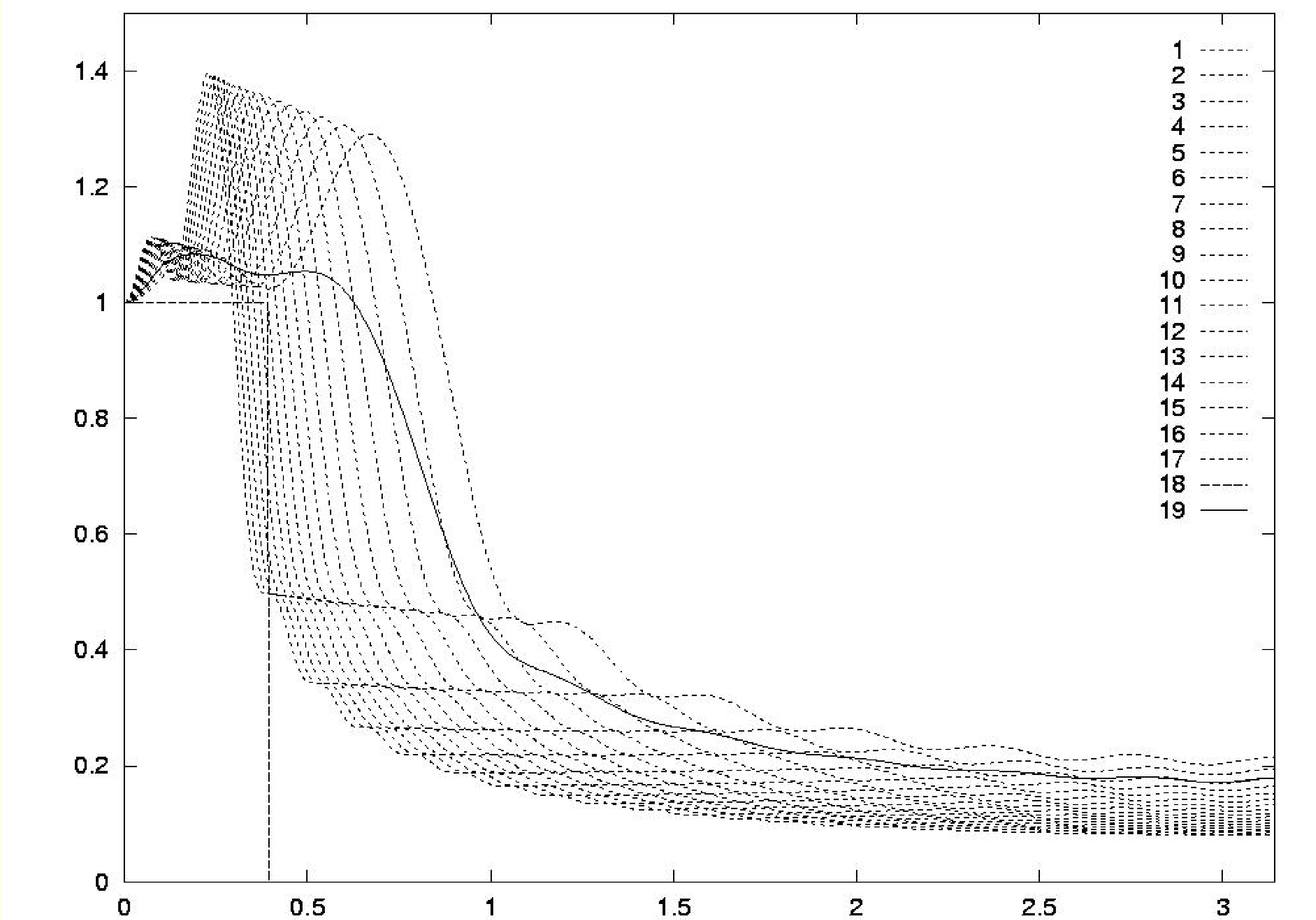
$$\delta_0 = \delta(0) = \sum nh_n$$

A boundary filter must be zero-delay with $\delta_0 = 0$.

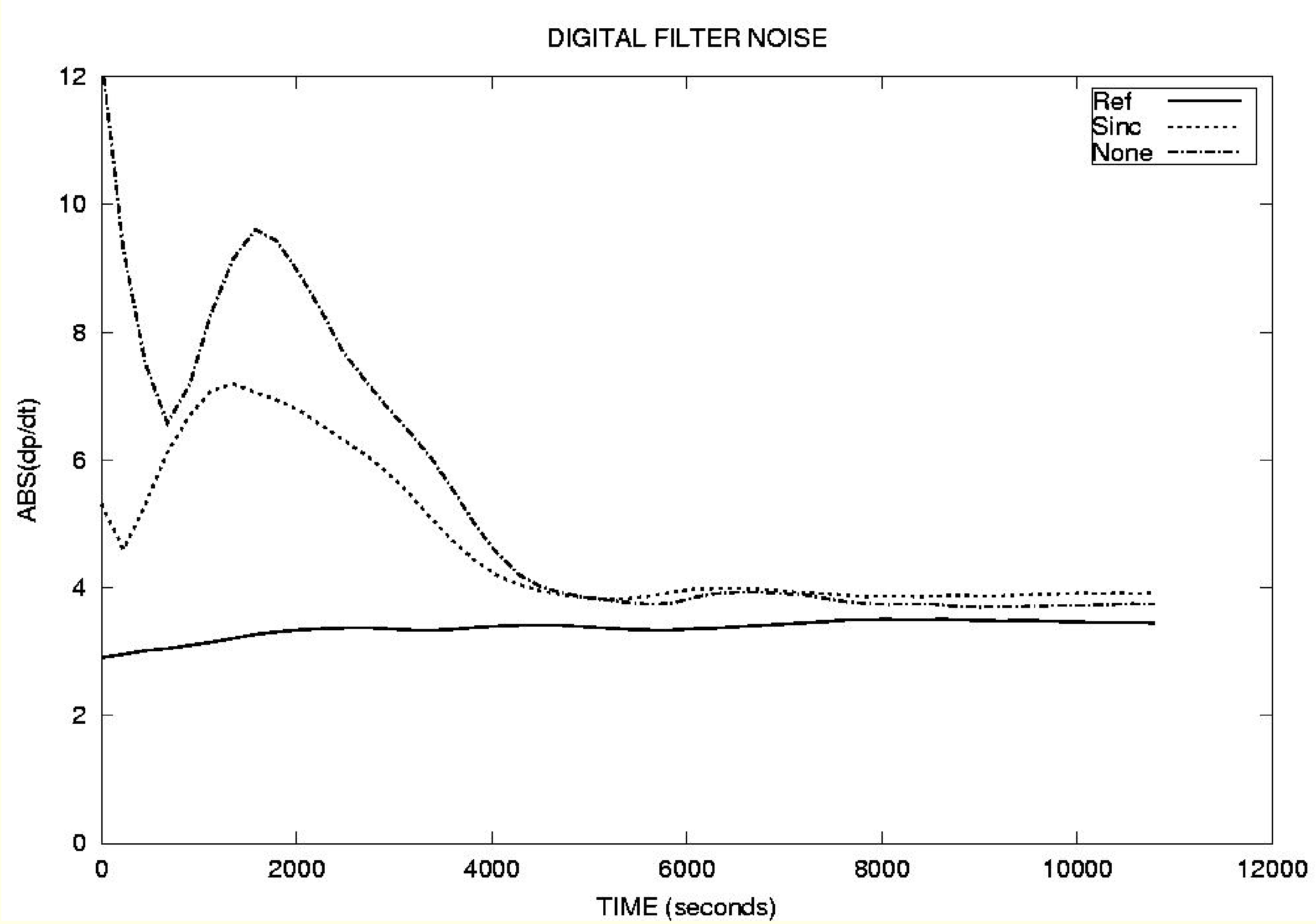
For the half-sinc sequence, this can be satisfied if we truncate after an exact number of wavelengths:

$$\sum_{n=0}^{N-1} nh_n = \frac{1}{\pi} \sum_{n=0}^{N-1} \sin n\theta_c = 0$$

provided $(N-1)\theta_c = 2\pi K$ for some integer K.



Dashed curves: **Frequency responses $H(\theta)$ for seventeen half-sincs with varying spans. Solid curve: weighted sum of seventeen half-sincs, to reduce intermediate frequency boost.**



Time evolution, during a 3-hour forecast, of the area-averaged absolute value of the surface pressure tendency (units: hPa per 3 hours) for three forecasts. Dot-dashed line:No initialization. Dotted line: BFI scheme (Sinc Filter). Solid line: Reference DFI scheme.

Padé Filtering

$\star\star\star$ **Work in Progress** $\star\star\star$

The Padé approximation represents a sequence of length N by a sum of $M = N/2$ components of complex exponential form:

$$x_n = \sum_{m=1}^{M} c_m \gamma_m^n \,.$$

The Z-transform of $\{x_n\}$ is then the sum of M terms

$$X(z) = \sum_{m=1}^{M} \left(\frac{c_m z}{z - \gamma_m} \right) \,.$$

The Z-transform has M simple poles at positions $z = \gamma_m$ with residues c_m.

We approximate the Z-transform of an arbitrary finite sequence by a function with $M = N/2$ components:

$$\Xi(z) = \sum_{m=1}^{M} \left(\frac{c_m z}{z - \gamma_m} \right) .$$

The poles are obtained by solving a Toeplitz system.

The residues are obtained from a Vandermonde system.

Filtering the Input Sequence

To filter an input signal, we select a weighting function $H(\gamma)$ such that for components corresponding to low frequency oscillations or long time-scales it is exactly or approximately equal to unity, and for components corresponding to high frequencies or short time-scales it is small.

Then we define the filtered transform to be

$$\bar{X}(z) = \sum_{m=1}^{M} \left(\frac{H(\gamma_m) c_m z}{z - \gamma_m} \right) .$$

On inverting this, we get the filtered signal

$$\bar{x}_n = \sum_{m=1}^{M} H(\gamma_m) c_m \gamma_m^n .$$

Note that the complete freedom of choice of $H(z)$ is a powerful aspect of this filtering procedure.
Warning: There are Pitfalls in the Numerical Procedure.

DF as a Constraint in 4DVAR

If the system is noise-free at a particular time, *i.e.*, is close to the slow manifold, it will remain noise-free, since the slow manifold is an invariant subset of phase-space.

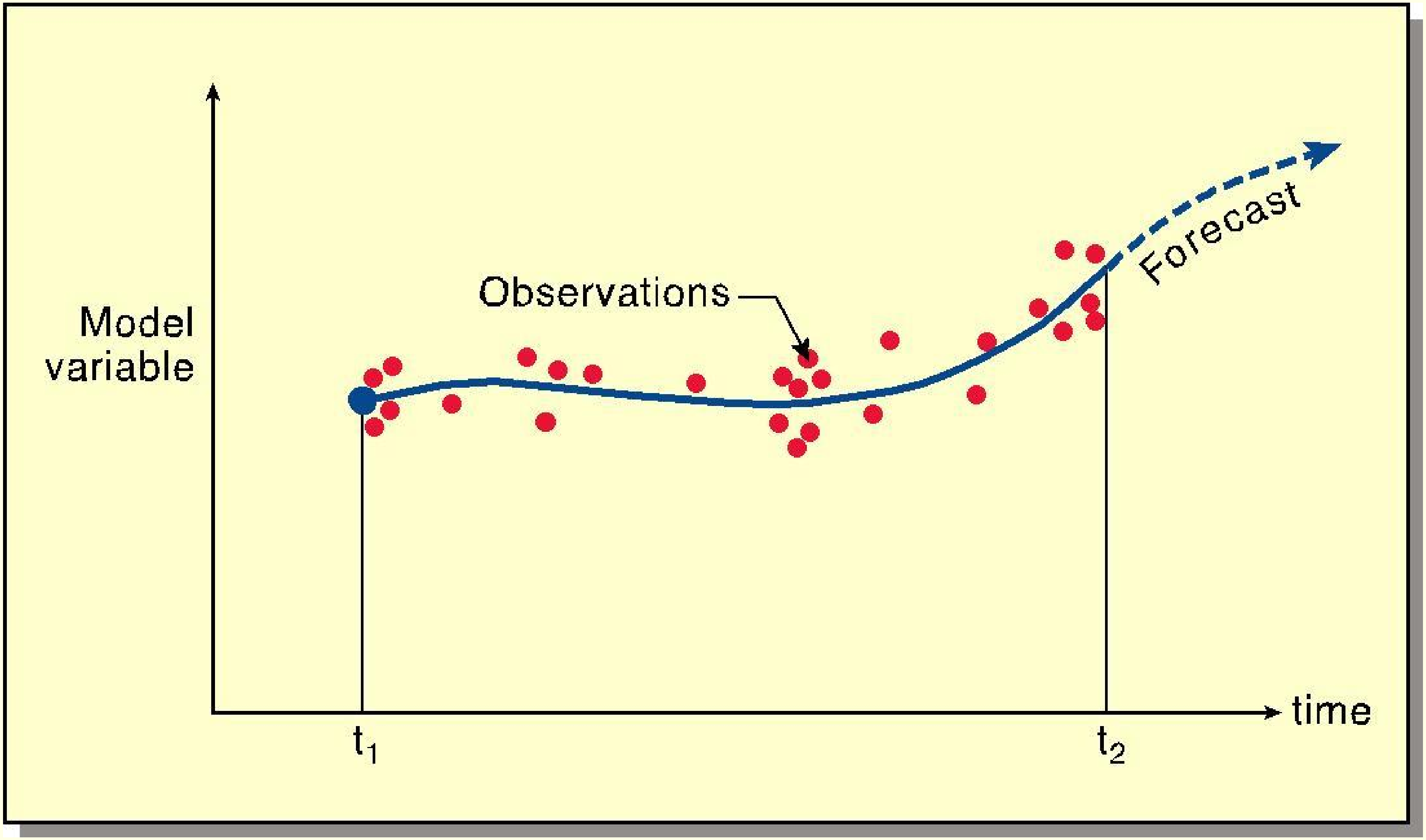
We consider a sequence of values $\{x_0, x_1, x_2, \cdots x_N\}$ and form the filtered value

$$\bar{x} = \sum_{n=0}^{N} h_n x_n. \tag{1}$$

The evolution is constrained, so that the value at the mid-point in time is close to this filtered value, by addition of a term

$$J_c = \frac{1}{2}\mu ||x_{N/2} - \bar{x}||^2$$

to the cost function to be minimized (μ is an adjustable parameter).



Schematic of smooth trajectory approximating observations.

$$J_c = \frac{1}{2}\mu||x_{N/2} - \bar{x}||^2$$

It is straightforward to derive the adjoint of the filter.

Gauthier and Thépaut (2001) found that a digital filter weak constraint imposed on the low-resolution increments of the 4DVAR system of Météo-France:

- **Efficiently controlled the emergence of fast oscillations**
- **Maintained a close fit to the observations.**

The dynamical imbalance was significantly less in 4DVAR than in 3DVAR.

Fuller details: Gauthier and Thépaut (2001).

Advantages of DFI

1. **No need to compute or store normal modes;**
2. **No need to separate vertical modes;**
3. **Complete compatibility with model discretisation;**
4. **Applicable to exotic grids on arbitrary domains;**
5. **No iterative numerical procedure which may diverge;**
6. **Ease of implementation and maintenance;**
7. **Applicable to all prognostic model variables;**
8. **Applicable to non-hydrostatic models.**
9. **Economic and effective Constraint in 4D-Var Analysis.**