Integrable Elliptic Billiards & Ballyards

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24 July 2019.

Outline

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Billiards & Ballyards

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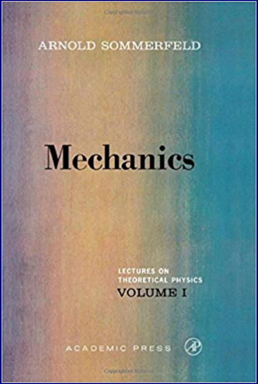
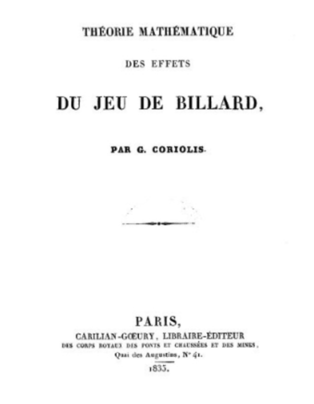
"The Beautiful Game"

"The beautiful game of billiards opens up a rich field for applications of the dynamics of rigid bodies."



***Lectures on Theoretical Physics*, Arnold Sommerfeld 1937.**

Gaspard-Gustave de Coriolis



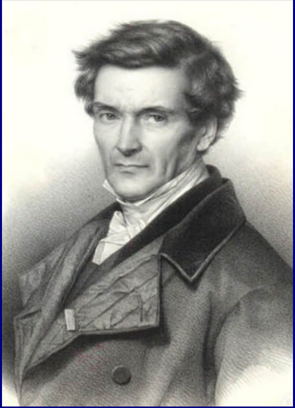
THÉORIE MATHÉMATIQUE

DES EFFETS

DU JEU DE BILLARD,

PAR G. CORIOLIS.

PARIS,
CARILIAN-GŒURY, LIBRAIRE-ÉDITEUR
DES CORPS ROYAUX DES PONTS ET CHAUSSÉES ET DES MINES,
Quai des Augustins, N° 41.
1835.



"Théorie mathématique des effets du jeu de billiard".

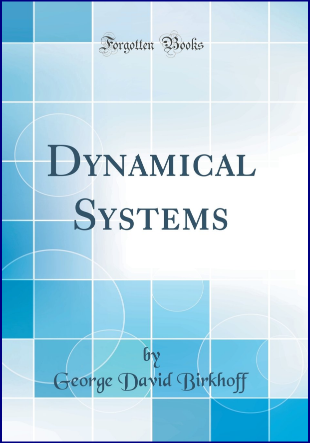
Ergodic Theory

Billiards has been used to examine questions of ergodic theory.

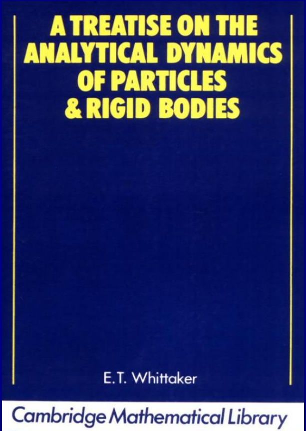
In ergodic systems, all configurations and momenta compatible with the total energy are eventually explored.

Such questions lie at the heart of statistical mechanics.

George D. Birkhoff



Edmund Taylor Whittaker



Idealizations

The ball is a point mass moving at constant velocity.

Elastic impacts with the boundary, or cushion, of the billiard table.

The energy is taken to be constant.

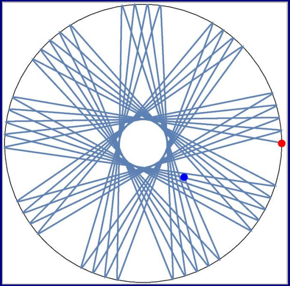
The path traced out by the moving ball may form a closed periodic loop . . .

. . . or it may cover the table (or part) densely, never returning to the starting conditions.

Circular Billiards

The simplest billiard is circular.
Every trajectory makes a constant angle with the boundary and tangent to a circle within it.

Every trajectory is either a polygon or is everywhere dense in an annular region.



Elliptic Billiards

The elliptical billiard problem is completely resolved, thanks to Poncelet's theorem.

There are periodic trajectories, or ones that are dense in regions of two distinct topological types.

Elliptical Billiards

We examine the orbits for an elliptical table. The boundary is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In parametric form

$$x = a\cos\theta\,, \qquad y = b\sin\theta$$

The foci are at $(f, 0)$ and $(-f, 0)$.

Eccentricity e defined by $e^2 = 1 - (b/a)^2$.

Initial Conditions

We assume that the ball moves at unit speed.

Suppose a trajectory starts at a boundary point θ_0 and moves at an angle ψ_0 to the *x*-axis.

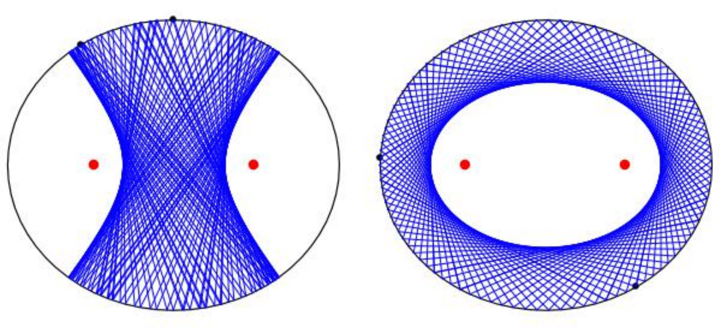
The initial values $\{\theta_0, \psi_0\}$ determine the motion.

The tangential component of velocity is unchanged at impact, while the normal component reverses sign.

Each segment of the trajectory is tangent to a conic confocal with the boundary.

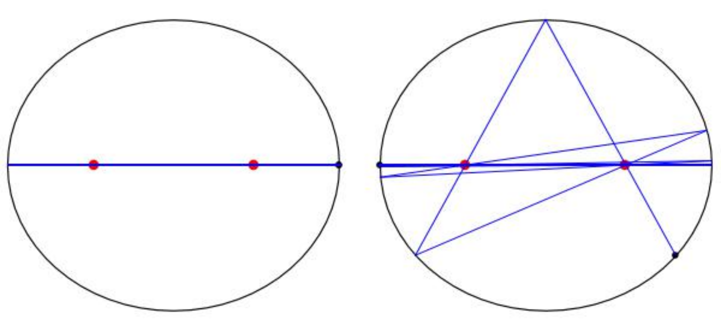
This caustic may be an ellipse or hyperbola.

Generic Motion: Box Orbits & Loop Orbits



Generic orbits. Left: Box orbit, Right: Loop orbit.

The Homoclinic Orbit



Homoclinic orbits

Discrete Mapping

The billiard problem is a Hamiltonian system.
Between impacts the equations are:

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{p}\,, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{0}\,.$$

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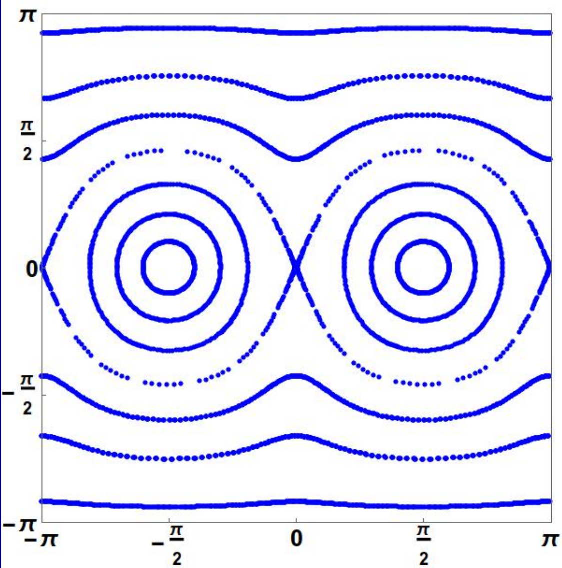
The dynamics are specified by a discrete mapping.
Given the values $(x, y; m)_n$ we can get $(x, y; m)_{n+1}$

$$\begin{aligned} x_{n+1} &= -x_n - \frac{2a^2 m_n(y_n - m_n x_n)}{m_n^2 a^2 + b^2}\,, \\ y_{n+1} &= y_n + m_n(x_{n+1} - x_n)\,, \\ m_{n+1} &= \frac{2\nu_{n+1} - (1 - \nu_{n+1}^2)m_n}{(1 - \nu_{n+1}^2) + 2\nu_{n+1} m_n}\,, \end{aligned}$$

where $\nu_{n+1} = (a^2 y_{n+1})/(b^2 x_{n+1})$ is known when needed.

Phase Portrait



Constants of Motion

The kinetic energy

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

is a constant of the motion.

For a circular table the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r)\,.$$

Since ϑ is ignorable, $p_\vartheta = \partial\mathcal{L}/\partial\dot{\vartheta} = r^2\dot{\vartheta}$ is conserved.

For an elliptical table, the angular momentum about the centre is no longer conserved.

Constants of Motion

We use elliptic coordinates (ξ, η):

$$x = f\cosh\xi\cos\eta\,, \qquad y = f\sinh\xi\sin\eta\,.$$

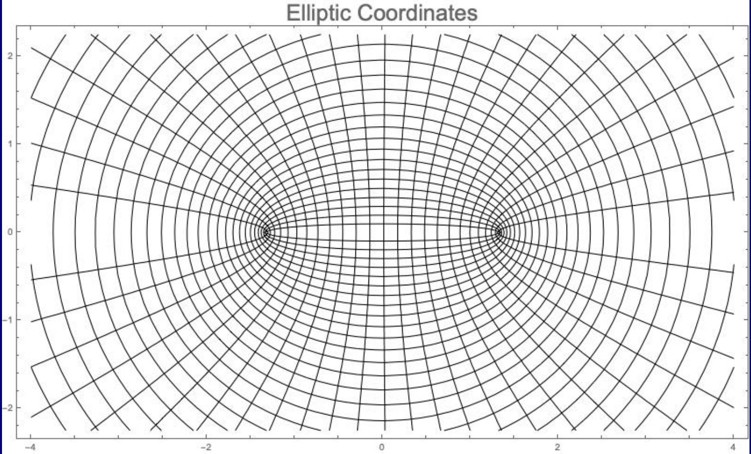
The components of the velocity $\mathbf{v} = (u, v)$ are:

$$\begin{aligned} \dot{x} &= u &= f\sinh\xi\cos\eta\,\dot{\xi} - f\cosh\xi\sin\eta\,\dot{\eta} \\ \dot{y} &= v &= f\cosh\xi\sin\eta\,\dot{\xi} + f\sinh\xi\cos\eta\,\dot{\eta} \end{aligned}$$

The radii from the center and foci are

$$\begin{aligned} \mathbf{r_0} = (x, y) &= f(\cosh\xi\cos\eta, \sinh\xi\sin\eta) \\ \mathbf{r_1} = (x - f, y) &= f(\cosh\xi\cos\eta - 1, \sinh\xi\sin\eta) \\ \mathbf{r_2} = (x + f, y) &= f(\cosh\xi\cos\eta + 1, \sinh\xi\sin\eta) \end{aligned}$$

Elliptical Coordinates



Constants of Motion

The angular momenta about the foci are

$$\mathbf{L_1} = \mathbf{r_1} \times \mathbf{v} \qquad \text{and} \qquad \mathbf{L_2} = \mathbf{r_2} \times \mathbf{v}$$

Then we have

$$\mathbf{L_1} \cdot \mathbf{L_2} = L_1 L_2 = f^4 (\cosh^2 \xi - \cos^2 \eta) \left[(-\sin^2 \eta)\dot{\xi}^2 + (\sinh^2 \xi)\dot{\eta}^2 \right]$$

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The quantity $L_1 L_2$ does not change at an impact.

Moreover, $\mathbf{L_1} = \mathbf{r_1} \times \mathbf{v}$ and $\mathbf{L_2} = \mathbf{r_2} \times \mathbf{v}$
are constant along each segment.

Therefore, $L_1 L_2$ is a constant of the motion.

Constant $L_1 L_2$

For loop orbits, L_1 and L_2 are either both positive or both negative, so $L_1 L_2$ is positive.

For box orbits, which pass between the foci, L_1 and L_2 are of opposite signs.

For the homoclinic orbit, passing through the foci, one or other of these components vanishes.

Thus, $L_1 L_2$ acts as a discriminant for the motion:

$$\text{Orbit is} \begin{cases} \text{Box type} & \text{if } L_1 L_2 < 0 \\ \text{Homoclinic} & \text{if } L_1 L_2 = 0 \\ \text{Loop type} & \text{if } L_1 L_2 > 0. \end{cases}$$

Outline

Circular Ballyard Table

For billiards, the potential well has a step discontinuity at the boundary.

We can approximate this behaviour by a high-order polynomial. But can we integrate this system?

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We can approximate this behaviour by a high-order polynomial. But can we integrate this system?

For a circular table of radius a we take the potential energy to be $V(r) = V_0(r/a)^N$ where N is large.

The Lagrangian may be written

$$\mathcal{L} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r)$$

Since this is independent of ϑ, p_ϑ is a constant of the motion.

Elliptical Billiard Table

The kinetic energy in elliptic coordinates is:

$$T = \tfrac{1}{2} f^2 (\cosh^2 \xi - \cos^2 \eta)(\dot{\xi}^2 + \dot{\eta}^2)$$

The Lagrangian then becomes

$$\mathcal{L} = \tfrac{1}{2} f^2 (\cosh^2 \xi - \cos^2 \eta)(\dot{\xi}^2 + \dot{\eta}^2) - V(\xi, \eta)\,.$$

We note the form of the kinetic energy:

$$T = [\mathcal{U}_1(q_1) + \mathcal{U}_2(q_2)](\dot{q}_1^2 + \dot{q}_2^2)$$

where (q_1, q_2) are the generalized coordinates.

Liouville Integrable Systems

In 1848 Joseph Liouville identified a broad class of integrable systems.

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If the kinetic and potential energies take the form

$$T = [\mathcal{U}_1(q_1) + \mathcal{U}_2(q_2)] \cdot [\mathcal{V}_1(q_1)\dot{q}_1^2 + \mathcal{V}_2(q_2)\dot{q}_2^2]$$

$$V = \frac{\mathcal{W}_1(q_1) + \mathcal{W}_2(q_2)}{\mathcal{U}_1(q_1) + \mathcal{U}_2(q_2)}$$

the solution can be solved in quadratures (see Whittaker, 1937).

The Ballyard Potential

"Our" kinetic energy is

$$T = \tfrac{1}{2} f^2 (\cosh^2 \xi - \cos^2 \eta)(\dot{\xi}^2 + \dot{\eta}^2)\,.$$

This is of Liouville form with

$$\mathcal{U}_1(\xi) = f^2 \cosh^2 \xi \qquad \mathcal{U}_2(\eta) = -f^2 \cos^2 \eta \qquad \mathcal{V}_1 \equiv \mathcal{V}_2 \equiv 1$$

If the potential energy function is of the form

$$V(\xi, \eta) = \frac{\mathcal{W}_1(\xi) + \mathcal{W}_2(\eta)}{\mathcal{U}_1(\xi) + \mathcal{U}_2(\eta)}$$

the ballyard problem is of Liouville type.

The Ballyard Potential

We seek a potential surface close to constant within the ellipse and rising rapidly near the boundary.

We define the potential surfaces by setting

$$\mathcal{W}_1(\xi) = V_N f^2 \cosh^N \xi \qquad \mathcal{W}_2(\eta) = -V_N f^2 \cos^N \eta$$

where N is an even integer.

The potential energy function is then

$$V(\xi, \eta) = \frac{\mathcal{W}_1(\xi) + \mathcal{W}_2(\eta)}{\mathcal{U}_1(\xi) + \mathcal{U}_2(\eta)} = V_N \left[\frac{\cosh^N \xi - \cos^N \eta}{\cosh^2 \xi - \cos^2 \eta} \right]$$

The Ballyard Potential: Special Cases

For $N = 2$ we have potential energy constant:

$$\mathcal{W} = V_2 f^2(\cosh^2\xi - \cos^2\eta)\,, \qquad V \equiv V_2$$

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For $N = 4$, we have

$$\mathcal{W} = V_4 f^2(\cosh^4\xi - \cos^4\eta) \qquad V = V_4(\cosh^2\xi + \cos^2\eta)$$

The potential energy is proportional to $x^2 + y^2$ (the orbits are closed ellipses).

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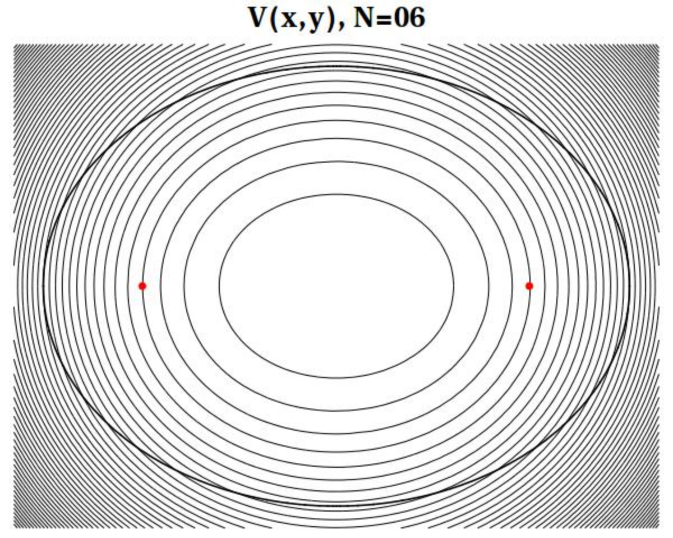
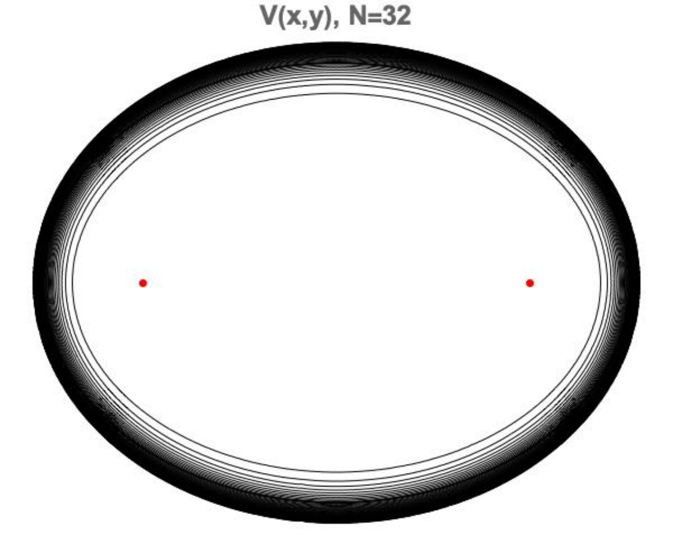
For $N = 6$, we have

$$\begin{aligned} \mathcal{W} &= V_6 f^2(\cosh^6\xi - \cos^6\eta) \\ V &= V_6(\cosh^4\xi + \cosh^2\xi\cos^2\eta + \cos^4\eta) \end{aligned}$$

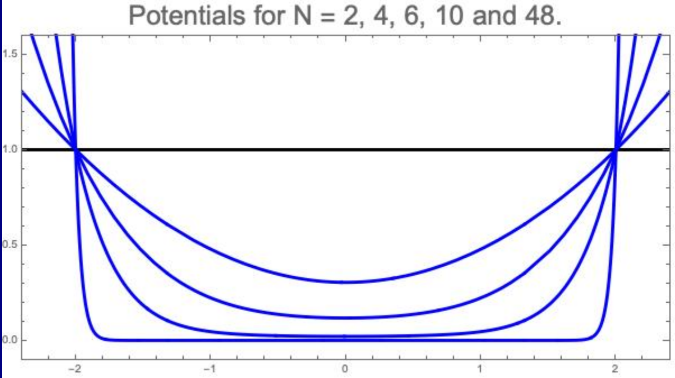
The Ballyard Potential for $N = 6$



The Ballyard Potential for $N = 32$



Ballyard Potential Cross-sections



Ballyard approaches square potential-well as $N \to \infty$.

Integrals

From the theory of Liouville systems, we have

$$\tfrac{1}{2}\mathcal{U}^2\dot{\xi}^2 = E\mathcal{U}_1(\xi) - \mathcal{W}_1(\xi) + \gamma_1$$
$$\tfrac{1}{2}\mathcal{U}^2\dot{\eta}^2 = E\mathcal{U}_2(\eta) - \mathcal{W}_2(\eta) + \gamma_2$$

where γ_1 and γ_2 are constants of integration, and $\gamma_1 + \gamma_2 = 0$. We write $\gamma = \gamma_1$.

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where γ_1 and γ_2 are constants of integration, and $\gamma_1 + \gamma_2 = 0$. We write $\gamma = \gamma_1$.

We partition the energy as $E = E_1 + E_2$, where

$$E_1 = \tfrac{1}{2}\mathcal{U}(\xi,\eta)\dot{\xi}^2 + \frac{\mathcal{W}_1(\xi)}{\mathcal{U}(\xi,\eta)} \quad \text{and} \quad E_2 = \tfrac{1}{2}\mathcal{U}(\xi,\eta)\dot{\eta}^2 + \frac{\mathcal{W}_2(\eta)}{\mathcal{U}(\xi,\eta)}$$

Then the constants of motion can be written

$$\gamma_1 = \mathcal{U}E_1 - E\mathcal{U}_1 \qquad \gamma_2 = \mathcal{U}E_2 - E\mathcal{U}_2\,.$$

Note that E_1 and E_2 are not constants.

The equations for $\dot{\xi}$ and $\dot{\eta}$ can be integrated:

$$\int^{\xi} \frac{\mathcal{U}_1(\xi)\mathrm{d}\xi}{\sqrt{2[E\mathcal{U}_1(\xi) - \mathcal{W}_1(\xi) + \gamma_1]}} = \int^{t} \mathrm{d}t$$

$$\int^{\eta} \frac{\mathcal{U}_2(\eta)\mathrm{d}\eta}{\sqrt{2[E\mathcal{U}_2(\eta) - \mathcal{W}_2(\eta) + \gamma_2]}} = \int^{t} \mathrm{d}t$$

Analytical evaluation may or may not be possible.

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Analytical evaluation may or may not be possible.

For the case $N = 6$, we get:

$$\int_{\xi_0}^{\xi} \frac{f^2 \cosh^2 \xi \, \mathrm{d}\xi}{\sqrt{2[Ef^2 \cosh^2 \xi - V_6 f^2 \cosh^6 \xi + \gamma]}} = t - t_0$$

$$\int_{\eta_0}^{\eta} \frac{-f^2 \cos^2 \eta \, \mathrm{d}\eta}{\sqrt{2[-Ef^2 \cos^2(\eta) + V_6 f^2 \cos^6 \eta - \gamma]}} = t - t_0$$

The Angular Momentum Integral

For the billiard dynamics, $L_1 L_2$ is a constant.
We seek a corresponding integral for the ballyard.

In elliptical coordinates, we can write

$$L_1 L_2 = f^2 \mathcal{U}(\xi,\eta)[\sinh^2\xi\,\dot{\eta}^2 - \sin^2\eta\,\dot{\xi}^2]$$

The Angular Momentum Integral

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In elliptical coordinates, we can write

$$L_1 L_2 = f^2 \mathcal{U}(\xi, \eta)[\sinh^2 \xi \, \dot{\eta}^2 - \sin^2 \eta \, \dot{\xi}^2]$$

We use γ_1 and γ_2 to substitute for $\dot{\xi}^2$ and $\dot{\eta}^2$.
"After some manipulation", we find that

$$L_1 L_2 + \frac{2f^2(\sinh^2 \xi \, \mathcal{W}_2 - \sin^2 \eta \, \mathcal{W}_1)}{\mathcal{U}} = -2(f^2 E + \gamma) \, .$$

The right side is constant. Therefore, so is the left!

Again

$$L_1 L_2 + \frac{2f^2(\sinh^2\xi\,\mathcal{W}_2 - \sin^2\eta\,\mathcal{W}_1)}{\mathcal{U}} = -2(f^2E+\gamma)\,.$$

If we define the quantity

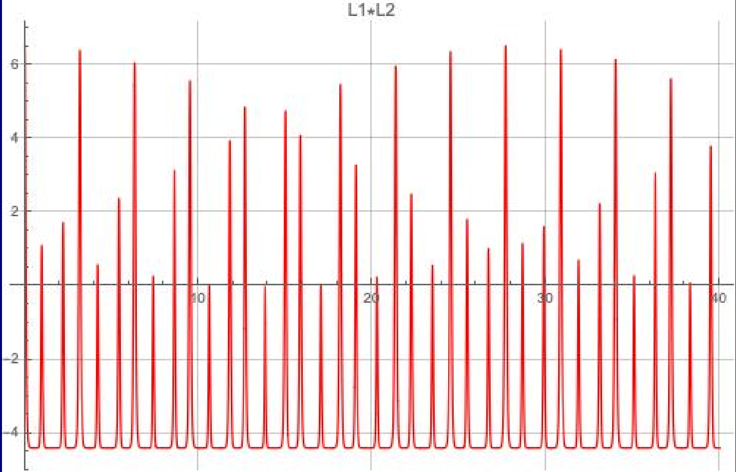
$$\Lambda(\xi,\eta) = \frac{2f^2[\sinh^2\xi\,\mathcal{W}_2(\eta) - \sin^2\eta\,\mathcal{W}_1(\xi)]}{\mathcal{U}(\xi,\eta)}$$

then the top equation becomes

$$\mathbb{L} \equiv [L_1 L_2 + \Lambda] = -2(f^2E+\gamma) = \textbf{constant}$$

and $\mathbb{L}$ is an integral of the motion.

$L_1 L_2$ for a Box Orbit



We easily show that, on the major axis ($y = 0$),

$$\Lambda(\xi, \eta) = \Lambda_0 = -2f^2 V_N$$

This means that $L_1 L_2 = \mathbb{L} - \Lambda$ is also constant there.

But $L_1 L_2 < 0$ on the inter-focal segment $-f < x < f$ and $L_1 L_2 > 0$ when $x < -f$ or $x > f$.

Therefore, the orbits fall into boxes and loops.

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Therefore, the orbits fall into boxes and loops.

If a trajectory passes through a focus
then $L_1 L_2$ must vanish there.

It can cross the axis only through the foci.

This special case ($\mathbb{L} = \Lambda_0$) separates boxes and loops.

Limiting Form of $\mathbb{L}$

We note that, as $N \to \infty$,

$$\mathcal{W}_1 = O\left(\frac{\cosh\xi}{\cosh\xi_{\mathrm{B}}}\right)^N , \qquad \mathcal{W}_2 = O\left(\frac{1}{\cosh\xi_{\mathrm{B}}}\right)^N .$$

Thus, for $|\xi| < |\xi_{\mathrm{B}}|$, we have

$$\lim_{N\to\infty} \mathbb{L} = L_1 L_2 .$$

The integral $\mathbb{L}$ corresponds in this limit to the quantity $L_1 L_2$ conserved for motion on a billiard.

Numerical Results

Numerical integrations confirm the dichotomy between boxes and loops for the ballyard potentials.

A large number of numerical experiments were performed with $N = 6$, and several for larger N.

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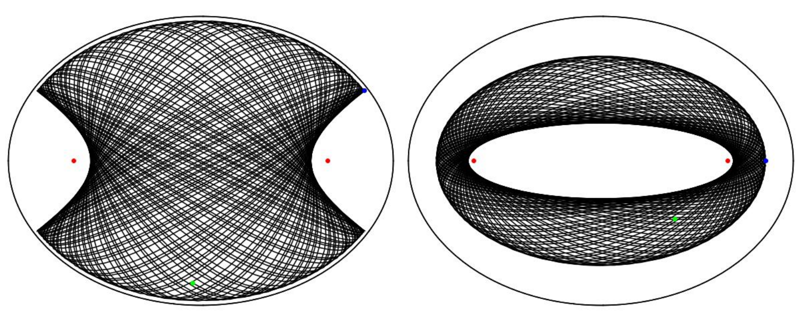
A large number of numerical experiments were performed with $N = 6$, and several for larger N.

For orbits passing through the foci, the equations in (ξ, η) coordinates are singular.

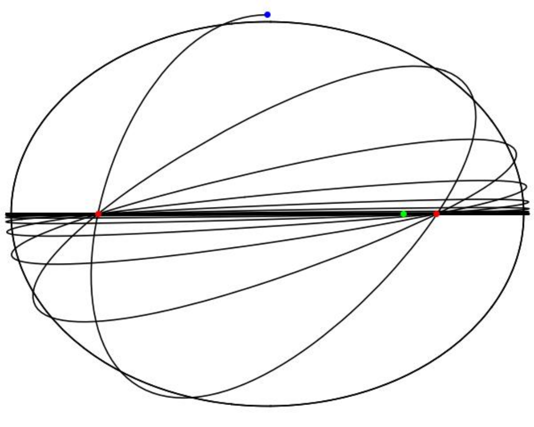
A re-coding using cartesian coordinates enabled numerical integrations along homoclinic orbits.

Boxes and Loops for $N = 6$ Ballyard



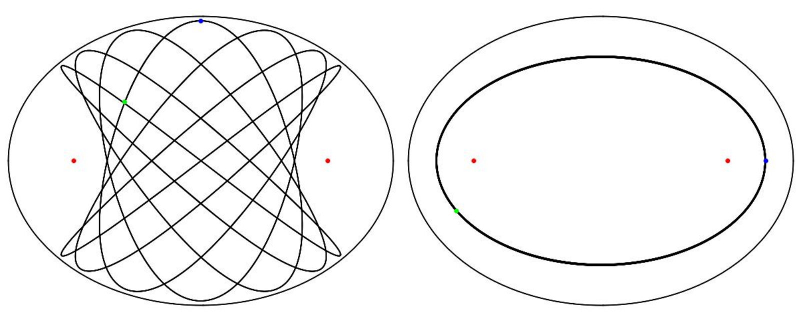
Box orbit. **Loop orbit.**

Homiclinic Orbit for $N = 6$ Ballyard



Homoclinic orbit

Special Solutions for $N = 6$ Ballyard



Periodic box orbit. **Pure elliptic orbit.**

Outline

A High-Order Ballyard

Summary

We have reviewed motion on an elliptical billiard.

We have replaced the the flat-bedded, hard-edged billiard by a smooth surface, a ballyard.

The ballyard Lagrangians are of Liouville type and so are completely integrable.

A new constant of the motion ($\mathbb{L}$) was found, showing that the orbits split into boxes and loops.

The discriminant that determines the character of the solution is the sign of $L_1 L_2$ on the major axis.

Thank you