Laplace Transform Integration of the Shallow Water Equations

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Outline

Basic Theory

Residue Theorem

Numerical Inversion

Ordinary Differential Equations

Application to NWP

Kelvin Waves & Phase Errors

Lagrangian Formulation
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Lagrangian Formulation
Integral Transforms in General

The LT is one of a large family of integral transforms.

Suppose we have a function \( f(t) \) for \( t \in D \).

We define the transform function \( \hat{f}(s) \) as:

\[
\hat{f}(s) = \int_{D} K(s,t) f(t) \, dt
\]

where \( K(s,t) \) is called the kernel of the transform.

For example, the Fourier transform is:

\[
\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \, dt
\]

The Hilbert transform is another . . . and many more.
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The Laplace Transform: Definition

For a function of time $f(t)$, $t \geq 0$, the LT is defined as

$$\hat{f}(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt.$$

Here, $s$ is complex and $\hat{f}(s)$ is a complex function of $s$. 
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- The domain of the function $f(t)$ is $\mathcal{D} = [0, +\infty)$.
- The kernel of the transform is $K(s, t) = \exp(-st)$.
- The domain of the LT $\hat{f}(s)$ is the complex $s$-plane.
Recovering the Original Function

The recovery of the original function $f(t)$ from the transformed function $\hat{f}(s)$ is called inversion.
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Recall that, for the Fourier transform, we have

\[
\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \, dt \\
\hat{f}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+i\omega t} \tilde{f}(\omega) \, d\omega
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Recovering the Original Function

The recovery of the original function $f(t)$ from the transformed function $\hat{f}(s)$ is called inversion.

Recall that, for the Fourier transform, we have

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \, dt \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+i\omega t} \tilde{f}(\omega) \, d\omega$$

Analogously, for the LT, the inversion is an integral of $\hat{f}(s)$ multiplied by a kernel function . . .

. . . but now the integral is taken over a contour in the complex $s$-plane.
Contour for inversion of Laplace Transform
For the LT, the inversion formula is

\[ f(t) = \frac{1}{2\pi i} \int_{C_1} e^{st} \hat{f}(s) \, ds. \]

where \( C_1 \) is a contour in the \( s \)-plane:
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\[ f(t) = \frac{1}{2\pi i} \oint_{C_1} e^{st} \hat{f}(s) \, ds. \]

where \( C_1 \) is a contour in the \( s \)-plane:

- \( C_1 \) is parallel to the imaginary axis.
- \( C_1 \) is to the right of all singularities of \( \hat{f}(s) \).

For the functions that we consider, the singularities are poles on the imaginary axis. Thus, the contour \( C_1 \) must be in the right half-plane.
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For the functions that we consider, the singularities are poles on the imaginary axis.

Thus, the contour \( C_1 \) must be in the right half-plane.
The LT is a **linear operator**

\[ \mathcal{L}\{f(t)\} = \hat{f}(s) \equiv \int_0^\infty e^{-st} f(t) \, dt. \]

Therefore

\[ \mathcal{L}\{\alpha f(t)\} = \int_0^\infty e^{-st} \left[ \alpha f(t) \right] \, dt = \alpha \int_0^\infty e^{-st} f(t) \, dt = \alpha \mathcal{L}\{f(t)\}. \]
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Also

\[ \mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st} [f(t)+g(t)] \, dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}. \]
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\]

More generally,

\[
\mathcal{L} \left\{ \sum_{n=1}^N w_n f_n(t) \right\} = \sum_{n=1}^N w_n \mathcal{L}\{f_n(t)\}.
\]
Basic Properties of the LT

- \( \mathcal{L}\{a\} = \frac{a}{s} \) (a constant)
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- $\mathcal{L}\{\exp(i\omega t)\} = \frac{1}{s - i\omega}$ (pole on imaginary axis)

Exercise: Prove these results, using the definition of the Laplace transform $\mathcal{L}\{f(t)\}$. 

Basic Theory  Residues  N-gon  ODEs  NWP  Phase Errors  Lagrange
Basic Properties of the LT

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\[ \mathcal{L}\{ \sin at \} = \frac{a}{s^2 + a^2} \]

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Residue Theorem: Refresher

Suppose $f(z)$ is analytic inside a circle $C$ except for a simple pole at the centre $a$ of $C$. 
Residue Theorem: Refresher

Suppose \( f(z) \) is analytic inside a circle \( C \) except for a simple pole at the centre \( a \) of \( C \).

For example, \( f(z) \) might be of the form

\[
f(z) = \frac{\varrho}{z - a} + g(z)
\]

where \( g(z) \) is analytic inside \( C \).
Residue Theorem: Refresher

Suppose $f(z)$ is analytic inside a circle $C$ except for a simple pole at the centre $a$ of $C$.

For example, $f(z)$ might be of the form

$$f(z) = \frac{\varrho}{z - a} + g(z)$$

where $g(z)$ is analytic inside $C$.

The residue of $f(z)$ at $z = a$ is computed as

$$\lim_{z \to a} (z - a) f(z) = \varrho$$
By Cauchy’s Integral Formula,

\[ \oint_C g(z) \, dz = 0 \quad \text{and} \quad \oint_C \frac{\varrho}{z - a} \, dz = 2\pi i \varrho. \]
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\oint_{C} g(z) \, dz = 0 \quad \text{and} \quad \oint_{C} \frac{\varrho}{z - a} \, dz = 2\pi i \varrho.
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Therefore,

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\oint_{C} f(z) \, dz = 2\pi i \varrho = 2\pi i \left[ \text{Residue of } f(z) \text{ at } a \right]
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By Cauchy’s Integral Formula,
\[ \oint_C g(z) \, dz = 0 \quad \text{and} \quad \oint_C \frac{\varrho}{z - a} \, dz = 2\pi i \varrho. \]

Therefore,
\[ \oint_C f(z) \, dz = 2\pi i \varrho = 2\pi i [ \text{Residue of } f(z) \text{ at } a ] \]

More generally, if there are several poles within \( C \),
\[ \oint_C f(z) \, dz = 2\pi i [ \text{Sum of residues of } f(z) \text{ within } C ] . \]
A Simple Oscillation

Let $f(t)$ have a single harmonic component

$$f(t) = \alpha \exp(i\omega t)$$
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The LT of $f(t)$ has a simple pole at $s = i \omega$:

$$\hat{f}(s) = \frac{\alpha}{s - i \omega},$$

A pure oscillation in time transforms to a holomorphic function, with a single pole. The frequency of the oscillation determines the position of the pole.
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The frequency of the oscillation determines the position of the pole.
LF and HF oscillations and their transforms
Again

\[ \hat{f}(s) = \mathcal{L}\{\alpha \exp(i\omega t)\} = \frac{\alpha}{s - i\omega}. \]
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The inverse transform of \( \hat{f}(s) \) is

\[ f(t) = \frac{1}{2\pi i} \int_{C_1} e^{st} \hat{f}(s) \, ds = \frac{1}{2\pi i} \int_{C_1} \frac{\alpha \exp(st)}{s - i\omega} \, ds. \]
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We augment \( C_1 \) by a semi-circular arc \( C_2 \) in the left half-plane. Denote the resulting closed contour by

\[ C_0 = C_1 \cup C_2. \]
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In cases of interest, we can show that this leaves the value of the integral unchanged (see Doetsch, 1977).
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In cases of interest, we can show that this leaves the value of the integral unchanged (see Doetsch, 1977).

Then \( f(t) \) is an integral around a closed contour \( C_0 \).
Contribution from $C_2$ vanishes in limit of infinite radius

Closed Contour

$C_0 = C_1 \cup C_2$

$s$-plane
For an integral around a closed contour,

\[ f(t) = \frac{1}{2\pi i} \oint_{C_0} \frac{\alpha \exp(st)}{s - i\omega} \, ds , \]

we can apply the residue theorem:
For an integral around a closed contour,

\[ f(t) = \frac{1}{2\pi i} \oint_{C_0} \frac{\alpha \exp(st)}{s - i\omega} \, ds , \]

we can apply the **residue theorem**:

\[ f(t) = \sum_{C_0} \text{Residues of} \left( \frac{\alpha \exp(st)}{s - i\omega} \right) \]

so \( f(t) \) is the sum of the residues of the integrand within the contour \( C_0 \).
Residue Theorem

\[ \frac{1}{2\pi i} \oint_{c} f(z) \, dz = \left[ \text{Sum of residues of } f(z) \text{ at poles within } \mathcal{C} \right] \]
Again

\[ f(t) = \sum_{c_0} \left[ \text{Residues of} \left( \frac{\alpha \exp(st)}{s - i\omega} \right) \right] \]
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There is just one pole, at \( s = i\omega \). The residue is

\[ \lim_{s \to i\omega} (s - i\omega) \left( \frac{\alpha \exp(st)}{s - i\omega} \right) = \alpha \exp(i\omega t) \]
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There is just one pole, at \( s = i\omega \). The residue is

\[ \lim_{s \to i\omega} (s - i\omega) \left( \frac{\alpha \exp(st)}{s - i\omega} \right) = \alpha \exp(i\omega t) \]

So we recover the input function:

\[ f(t) = \alpha \exp(i\omega t) \]
A Two-Component Oscillation

Let $f(t)$ have two harmonic components

$$f(t) = a \exp(i\omega t) + A \exp(i\Omega t) \quad |\omega| \ll |\Omega|$$
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The LT is a linear operator, so the transform of \( f(t) \) is

\[
\hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega},
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which has two simple poles, at \( s = i\omega \) and \( s = i\Omega \).
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The LT is a linear operator, so the transform of $f(t)$ is

$$\hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega},$$

which has two simple poles, at $s = i\omega$ and $s = i\Omega$.

- The LF pole, at $s = i\omega$, is close to the origin.
- The HF pole, at $s = i\Omega$, is far from the origin.
Again

\[ \hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}. \]
Again

\[ \hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}. \]

The inverse transform of \( \hat{f}(s) \) is

\[ f(t) = \frac{1}{2\pi i} \oint_{c_0} \frac{a \exp(st)}{s - i\omega} \, ds + \frac{1}{2\pi i} \oint_{c_0} \frac{A \exp(st)}{s - i\Omega} \, ds \]

\[ = a \exp(i\omega t) + A \exp(i\Omega t). \]
Again

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The inverse transform of $\hat{f}(s)$ is

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$$= a \exp(i\omega t) + A \exp(i\Omega t).$$

We now replace $C_0$ by a circular contour $C^*$ centred at the origin, with radius $\gamma$ such that $|\omega| < \gamma < |\Omega|$.
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$. We denote the modified operator by $L^*$. Since the pole $s = i\omega$ falls within the contour $C^*$, it contributes to the integral. Since the pole $s = i\Omega$ falls outside the contour $C^*$, it makes no contribution. Therefore, $f^*(t) \equiv L^*\{\hat{f}(s)\} = \frac{1}{2\pi i} \oint_{C^*} a \exp(st) s^{-i\omega} ds = a \exp(i\omega t)$.

We have filtered $f(t)$: the function $f^*(t)$ is the LF component of $f(t)$. The HF component is gone.
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$.

We denote the modified operator by $\mathcal{L}^\star$. 

Since the pole $s = i\omega$ falls within the contour $C^\star$, it contributes to the integral.

Since the pole $s = i\Omega$ falls outside the contour $C^\star$, it makes no contribution.

Therefore, $f^\star(t) \equiv \mathcal{L}^\star\{\hat{f}(s)\} = \frac{1}{2\pi i} \oint_{C^\star} a \exp(st) s^{-i\omega} ds = a \exp(i\omega t)$. 

We have filtered $f(t)$: the function $f^\star(t)$ is the LF component of $f(t)$. The HF component is gone.
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$.

We denote the modified operator by $L^*$.

Since the pole $s = i\omega$ falls **within** the contour $C^*$, it contributes to the integral.

Since the pole $s = i\Omega$ falls **outside** the contour $C^*$, it makes **no contribution**.
Again: We replace $C_0$ by $C^\ast$ with $|\omega| < \gamma < |\Omega|$.

We denote the modified operator by $L^\ast$.

Since the pole $s = i\omega$ falls within the contour $C^\ast$, it contributes to the integral.

Since the pole $s = i\Omega$ falls outside the contour $C^\ast$, it makes no contribution.

Therefore,

$$f^\ast(t) \equiv L^\ast\{\hat{f}(s)\} = \frac{1}{2\pi i} \oint_{C^\ast} \frac{a \exp(st)}{s - i\omega} \, ds = a \exp(i\omega t).$$
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$.

We denote the modified operator by $L^*$.

Since the pole $s = i\omega$ falls within the contour $C^*$, it contributes to the integral.

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$$f^*(t) \equiv L^*\{\hat{f}(s)\} = \frac{1}{2\pi i} \oint_{C^*} \frac{a \exp(st)}{s - i\omega} \, ds = a \exp(i\omega t).$$

We have filtered $f(t)$: the function $f^*(t)$ is the LF component of $f(t)$. The HF component is gone.
Exercise

Consider the test function

\[ f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \quad |\omega_1| < |\omega_2| \]
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Consider the test function

\[ f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \quad |\omega_1| < |\omega_2| \]

Show that the LT is

\[ \hat{f}(s) = \frac{\alpha_1}{2} \left[ \frac{e^{-i\psi_1}}{s - i\omega_1} + \frac{e^{i\psi_1}}{s + i\omega_1} \right] + \frac{\alpha_2}{2} \left[ \frac{e^{-i\psi_2}}{s - i\omega_2} + \frac{e^{i\psi_2}}{s + i\omega_2} \right] \]
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\[ f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \quad |\omega_1| < |\omega_2| \]

Show that the LT is

\[ \hat{f}(s) = \frac{\alpha_1}{2} \left[ \frac{e^{-i\psi_1}}{s - i\omega_1} + \frac{e^{i\psi_1}}{s + i\omega_1} \right] + \frac{\alpha_2}{2} \left[ \frac{e^{-i\psi_2}}{s - i\omega_2} + \frac{e^{i\psi_2}}{s + i\omega_2} \right] \]

Show how, by choosing \( \mathcal{C}^* \) with \( |\omega_1| < \gamma < |\omega_2| \), the HF component can be eliminated.
Outline

Basic Theory

Residue Theorem

Numerical Inversion

Ordinary Differential Equations

Application to NWP

Kelvin Waves & Phase Errors

Lagrangian Formulation
We have to compute a contour integral around the circular contour $C^*$ in the $s$-plane.
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This is done numerically, by replacing the circle $C^*$ by an $N$-sided polygon or N-gon $C^*_N$. 
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This is done numerically, by replacing the circle $C^*$ by an $N$-sided polygon or N-gon $C_N^*$.

For $n = 1, 2, \ldots, N$:

- The lengths of the edges are $\Delta s_n$
- The midpoints are labelled $s_n$

The integrand is evaluated at the centre of each edge, and the integral is computed numerically.
S-plane

Diagram showing a polygon with labeled vertices and paths labeled $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$.
We compute a numerical approximation: the inverse

\[ \mathcal{L}^*\{\hat{f}(s)\} = \frac{1}{2\pi i} \oint_{C^*} \exp(st) \hat{f}(s) \, ds \]

is approximated by the summation

\[ \mathcal{L}_N^*\{\hat{f}(s)\} = \frac{1}{2\pi i} \sum_{n=1}^{N} \exp(s_n t) \hat{f}(s_n) \Delta s_n \]
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\[ \mathcal{L}^*_N\{\hat{f}(s)\} = \frac{1}{2\pi i} \sum_{n=1}^{N} \exp(s_n t) \hat{f}(s_n) \Delta s_n \]

We introduce a correction factor, and arrive at:

\[ \mathcal{L}^*_N\{\hat{f}(s)\} = \frac{1}{N} \sum_{n=1}^{N} \exp_N(s_n t) \hat{f}(s_n) s_n \]

Here \( \exp_N(z) \) is the \( N \)-term Taylor expansion of \( \exp(z) \)

(For details, see Clancy and Lynch, 2011a)
Outline

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Lagrangian Formulation
Applying LT to an ODE

We consider a nonlinear ordinary differential equation

\[ \frac{dw}{dt} + i\omega w + n(w) = 0 \quad w(0) = w_0 \]

The LT of the equation is

\[ (s\hat{w} - w_0) + i\omega \hat{w} + n_0 s = 0. \]

We have frozen \( n(w) \) at its initial value \( n_0 = n(w_0) \).

We can immediately solve for the transform solution:

\[ \hat{w}(s) = \frac{1}{s + i\omega} \left[ w_0 - n_0 s \right] \]
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Using partial fractions, we write the transform as

\[ \hat{w}(s) = \left( \frac{w_0}{s + i\omega} \right) + \frac{n_0}{i\omega} \left( \frac{1}{s + i\omega} - \frac{1}{s} \right) \]

There are two poles, at \( s = -i\omega \) and at \( s = 0 \).
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The pole at \( s = 0 \) always falls within the contour \( \mathcal{C}^\star \).
The pole at \( s = -i\omega \) may or may not fall within \( \mathcal{C}^\star \).

Thus, the solution is

\[ w^\star(t) = \begin{cases} 
(w_0 + \frac{n_0}{i\omega}) \exp(-i\omega t) - \frac{n_0}{i\omega} & : |\omega| < \gamma \\
-\frac{n_0}{i\omega} & : |\omega| > \gamma 
\end{cases} \]
Again,

\[ w^*(t) = \begin{cases} 
(w_0 + \frac{n_0}{i\omega}) \exp(-i\omega t) - \frac{n_0}{i\omega} & : |\omega| < \gamma \\
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So we see that, for a LF oscillation (|\omega| < \gamma), the solution \( w^*(t) \) is the full solution \( w(t) \) of the ODE.
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For a HF oscillation (|\omega| > \gamma), the solution contains only a constant term.
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\[ w^*(t) = \begin{cases} 
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So we see that, for a LF oscillation ($|\omega| < \gamma$), the solution $w^*(t)$ is the full solution $w(t)$ of the ODE.

For a HF oscillation ($|\omega| > \gamma$), the solution contains only a constant term.

Thus, high frequencies are filtered out.
Again, for a HF oscillation ($|\omega| > \gamma$), the solution is

$$w^*(t) = -\frac{n_0}{i\omega}$$

or

$$i\omega w^*(t) + n_0 = 0$$
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Clearly, this corresponds to the criterion for nonlinear normal mode initialization:

Set the tendency of the HF terms to zero at $t = 0$. 
A General NWP Equation

We write the general NWP equations symbolically as

\[
\frac{dX}{dt} + iLX + N(X) = 0
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where \(X(t)\) is the state vector at time \(t\).
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We apply the Laplace transform to get

\[ (s\hat{X} - X_0) + iL\hat{X} + \frac{1}{s}N_0 = 0 \]

where \( X_0 \) is the initial value of \( X \) and \( N_0 = N(X_0) \) is held constant at its initial value.
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The frequencies are entangled. How do we proceed?
Eigenanalysis

\[ \dot{X} + iLX + N(X) = 0 \]

Assume the eigenanalysis of \( L \) is \( LE = E\Lambda \)

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and \( E = (e_1, \ldots, e_N) \).

More explicitly, assume that the eigenfrequencies split in two:

\[ \Lambda = \begin{bmatrix} \Lambda_Y & 0 \\ 0 & \Lambda_Z \end{bmatrix} \]

\( \Lambda_Y \): Frequencies of rotational modes (LF)
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This is just

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\dot{W} + i \Lambda W + E^{-1} N(X) = 0
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Basic Theory Residues N-gon ODEs NWP Phase Errors Lagrange
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This equation separates into two sub-systems:

$$\dot{Y} + i \Lambda_Y Y + N_Y(Y, Z) = 0$$
$$\dot{Z} + i \Lambda_Z Z + N_Z(Y, Z) = 0$$

where $W = (Y, Z)^T$. 

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\dot{Z} + i\Lambda_ZZ + N_Z(Y,Z) = 0
\]

where \( W = (Y,Z)^T \).

The variables \( Y \) and \( Z \) are all coupled through the nonlinear terms \( N_Y(Y,Z) \) and \( N_Z(Y,Z) \).
General Solution Method

We recall that the Laplace transform of the equation is

\[(s \hat{X} - X_0) + i L \hat{X} + \frac{1}{s} N_0 = 0\]

where \(X_0\) is the initial value of \(X\) and \(N_0 = N(X_0)\) is held constant at its initial value.
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The solution can be written formally:

\[\hat{X}(s) = (s I + i L)^{-1} \left[ X^n - \frac{1}{s} N^n \right] \]
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We recover the filtered solution at time \((n + 1)\Delta t\) by applying \( L^* \) at time \( \Delta t \) beyond the initial time:

\[ X^*((n + 1)\Delta t) = L^*\{\hat{X}(s)\} \bigg|_{t=\Delta t} \]
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Further details are given in Clancy and Lynch, 2011a,b
Laplace transform integration of the shallow water equations.  
Part 1: Eulerian formulation and Kelvin waves

Colm Clancy* and Peter Lynch  
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Laplace transform integration of the shallow water equations.  
Part 2: Lagrangian formulation and orographic resonance

Colm Clancy* and Peter Lynch  
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Outline

Basic Theory
Residue Theorem
Numerical Inversion
Ordinary Differential Equations
Application to NWP

Kelvin Waves & Phase Errors

Lagrangian Formulation
Phase Errors of SI and LT Schemes

Consider the phase error of the oscillation equation

\[
\frac{du}{dt} + i\omega u = 0 \quad R = \frac{\text{Numerical frequency}}{\text{Physical frequency}}
\]

For the semi-implicit (SI) scheme, the error is

\[
R_{SI} = 1 - \frac{1}{12} (\omega \Delta t)^2
\]

For the LT scheme, the corresponding error is

\[
R_{LT} = 1 - \frac{1}{N!} (\omega \Delta t)^N
\]

Even for modest values of \(N\), this is negligible.
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Relative phase errors for semi-implicit (SI) and Laplace transform (LT) schemes for Kelvin waves $m = 1$ and $m = 5$. 
Hourly height at $0.0^\circ\text{E}, 0.9^\circ\text{N}$ over 10 hours, with $\tau_c = 3\text{ h}$.
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Lagrangian Formulation
We now consider how to combine the Laplace transform approach with Lagrangian advection.
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where advection is now included in the time derivative.
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The general form of the equation is

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\frac{DX}{Dt} + iLX + N(X) = 0
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where advection is now included in the time derivative.

We re-define the Laplace transform to be the integral in time along the trajectory of a fluid parcel:

\[
\hat{X}(s) \equiv \int_{T} e^{-st} X(t) \, dt
\]
We compute $\mathcal{L}$ along a fluid trajectory $\mathcal{T}$. 
We consider parcels that arrive at the gridpoints at time \((n+1)\Delta t\). They originate at locations not corresponding to gridpoints at time \(n\Delta t\).
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- The value at the **arrival point** is \(X_{A}^{n+1}\).
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- The value at the *arrival point* is $X_{A}^{n+1}$.
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The initial values when transforming the Lagrangian time derivatives are $X_{D}^{n}$. 

The equations thus transform to:

\[(\hat{s}X - X_{D}^{n}) + iL\hat{X}^{n+1} + \frac{1}{2}M = 0\]

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Departure point, arrival point and mid-point.
The solution can be written formally:

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We recover the filtered solution by applying \( \mathcal{L}^* \) at time \( (n + 1)\Delta t \), or \( \Delta t \) after the initial time:

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Further details are given in Clancy and Lynch, 2011a,b
Orographic Resonance

- Spurious resonance arises from coupling the semi-Lagrangian and semi-implicit methods

- Linear analysis of orographically forced stationary waves confirms this

- This motivates an investigating of orographic resonance in a full model.

Test Case:
- Initial data: ERA-40 analysis of 12 UTC on 12th February 1979
- Used by Ritchie & Tanguay (1996) and by Li & Bates (1996)
- Running at T119 resolution
- Shows LT method has benefits over SI scheme.
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SLSI: $dt = 3600$: Height at 24 hours
SLLT: dt = 3600: Height at 24 hours
Conclusion

- LT scheme effectively filters HF waves
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- LT scheme more accurate than SI scheme
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- LT scheme effectively filters HF waves
- LT scheme more accurate than SI scheme
- LT scheme has no orographic resonance.

Next job:
Implement the LT scheme in a full baroclinic model.
Conclusion

- LT scheme effectively filters HF waves
- LT scheme more accurate than SI scheme
- LT scheme has no orographic resonance.

Next job: Implement the LT scheme in a full baroclinic model.
Thank you