Laplace Transform Integration of the Shallow Water Equations

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Outline

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Basic Theory

Residue Theorem

Numerical Inversion

Ordinary Differential Equations

Application to NWP

Kelvin Waves & Phase Errors

Lagrangian Formulation

Integral Transforms in General

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We define the **transform function** $\hat{f}(s)$ as:

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The Hilbert transform is another . . . and many more.

The Laplace Transform: Definition

For a function of time $f(t),\ t \geq 0$, the LT is defined as

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- The domain of the LT $\hat{f}(s)$ is the complex s-plane.

Recovering the Original Function

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Recall that, for the Fourier transform, we have

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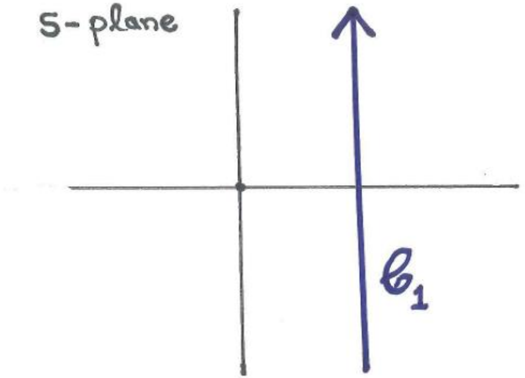
Analogously, for the LT, the inversion is an integral of $\hat{f}(s)$ multiplied by a kernel function ...

... but now the integral is taken over a contour in the complex s-plane.

Contour for inversion of Laplace Transform



For the LT, the inversion formula is

$$f(t) = \frac{1}{2\pi i} \int_{\mathcal{C}_1} e^{st} \hat{f}(s) \, \mathrm{d}s \, .$$

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- **$\mathcal{C}_1$ is parallel to the imaginary axis.**
- **$\mathcal{C}_1$ is to the right of all singularities of $\hat{f}(s)$.**

For the functions that we consider, the singularities are poles on the imaginary axis.

Thus, the contour $\mathcal{C}_1$ must be in the right half-plane.

$\star \quad \star \quad \star$

The LT is a linear operator

$$\mathcal{L}\{f(t)\} = \hat{f}(s) \equiv \int_0^\infty e^{-st} f(t)\, \mathrm{d}t\,.$$

Therefore

$$\mathcal{L}\{\alpha f(t)\} = \int_0^\infty e^{-st}\,[\alpha f(t)]\,\mathrm{d}t = \alpha \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \alpha\mathcal{L}\{f(t)\}\,.$$

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Also

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More generally,

$$\mathcal{L}\left\{\sum_{n=1}^{N} w_n f_n(t)\right\} = \sum_{n=1}^{N} w_n\mathcal{L}\{f_n(t)\}\,.$$

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Exercise: Prove these results, using the definition of the Laplace transform $\mathcal{L}\{f(t)\}$.

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where $g(z)$ is analytic inside $\mathcal{C}$.

The **residue** of $f(z)$ at $z = a$ is computed as

$$\lim_{z \longrightarrow a} (z - a) f(z) = \varrho$$

By Cauchy's Integral Formula,

$$\oint_{\mathcal{C}} g(z)\,\mathrm{d}z = 0 \qquad \textbf{and} \qquad \oint_{\mathcal{C}} \frac{\varrho}{z-a}\,\mathrm{d}z = 2\pi i\,\varrho\,.$$

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Therefore,

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More generally, if there are several poles within $\mathcal{C}$,

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = 2\pi i\,[\,\textbf{Sum of residues of } f(z) \textbf{ within } \mathcal{C}\,]\,.$$

A Simple Oscillation

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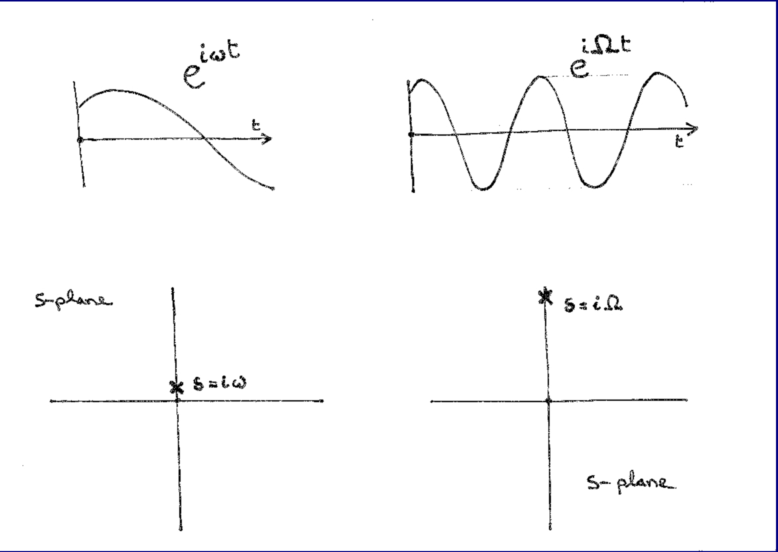
A pure oscillation in time transforms to a **holomorphic function**, with a single pole.

The frequency of the oscillation determines the position of the pole.



LF and HF oscillations and their transforms

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We augment $\mathcal{C}_1$ by a semi-circular arc $\mathcal{C}_2$ in the left half-plane. Denote the resulting closed contour by

$$\mathcal{C}_0 = \mathcal{C}_1 \cup \mathcal{C}_2 .$$

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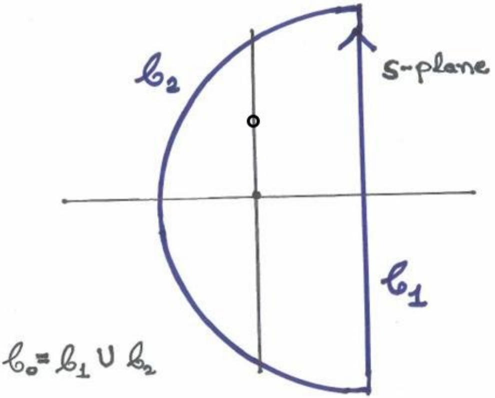
Then $f(t)$ is an integral around a closed contour $\mathcal{C}_0$.

Closed Contour



Contribution from $\mathcal{C}_2$ vanishes in limit of infinite radius

For an integral around a closed contour,

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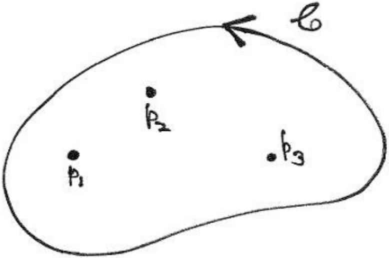
we can apply the residue theorem:

$$f(t) = \sum_{\mathcal{C}_0}\left[\textbf{Residues of }\left(\frac{\alpha\,\exp(st)}{s - i\omega}\right)\right]$$

so $f(t)$ is the sum of the residues of the integrand within the contour $\mathcal{C}_0$.

Residue Theorem



$$\frac{1}{2\pi i}\oint_{\mathcal{C}} f(z)\,dz = \left[\text{Sum of Residues of } f(z) \text{ at poles within } \mathcal{C}\right]$$

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$$f(t) = \sum_{\mathcal{C}_0} \left[\textbf{Residues of } \left(\frac{\alpha \exp(st)}{s - i\omega} \right) \right]$$

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There is just one pole, at $s = i\omega$. The residue is

$$\lim_{s \to i\omega} (s - i\omega) \left(\frac{\alpha \exp(st)}{s - i\omega} \right) = \alpha \exp(i\omega t)$$

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So we recover the input function:

$$f(t) = \alpha \exp(i\omega t)$$

A Two-Component Oscillation

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which has two simple poles, at $s = i\omega$ and $s = i\Omega$.

- **The LF pole, at $s = i\omega$, is close to the origin.**
- **The HF pole, at $s = i\Omega$, is far from the origin.**

Again

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The inverse transform of $\hat{f}(s)$ is

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\oint_{\mathcal{C}_0} \frac{a\exp(st)}{s - i\omega}\,\mathrm{d}s + \frac{1}{2\pi i}\oint_{\mathcal{C}_0} \frac{A\exp(st)}{s - i\Omega}\,\mathrm{d}s \\ &= a\exp(i\omega t) + A\exp(i\Omega t) . \end{aligned}$$

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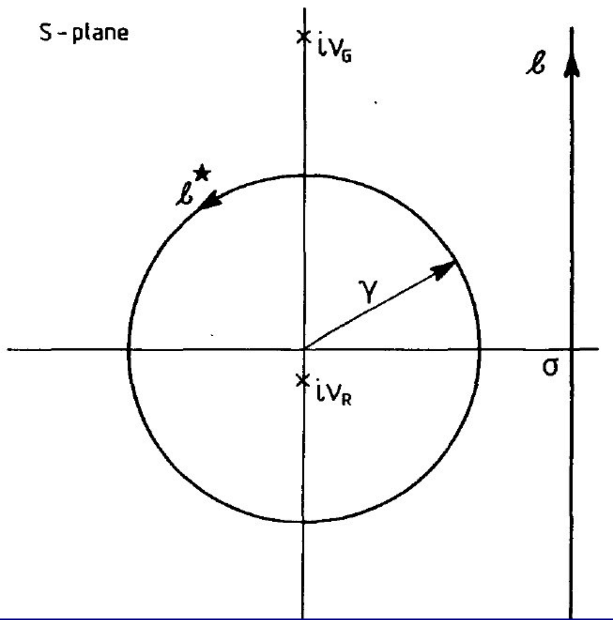
We now replace $\mathcal{C}_0$ by a circular contour $\mathcal{C}^\star$ centred at the origin, with radius γ such that $|\omega| < \gamma < |\Omega|$.


S - plane
iV_G
ℓ
ℓ*
γ
σ
iV_R

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We have filtered $f(t)$: the function $f^\star(t)$ is the LF component of $f(t)$. **The HF component is gone.**

Exercise

Consider the test function

$$f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \qquad |\omega_1| < |\omega_2|$$

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Show that the LT is

$$\hat{f}(s) = \frac{\alpha_1}{2}\left[\frac{e^{-i\psi_1}}{s - i\omega_1} + \frac{e^{i\psi_1}}{s + i\omega_1}\right] + \frac{\alpha_2}{2}\left[\frac{e^{-i\psi_2}}{s - i\omega_2} + \frac{e^{i\psi_2}}{s + i\omega_2}\right]$$

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Show how, by choosing $\mathcal{C}^\star$ with $|\omega_1| < \gamma < |\omega_2|$, the HF component can be eliminated.

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Ordinary Differential Equations

Application to NWP

Kelvin Waves & Phase Errors

Lagrangian Formulation

Approximating the Contour $\mathcal{C}^\star$

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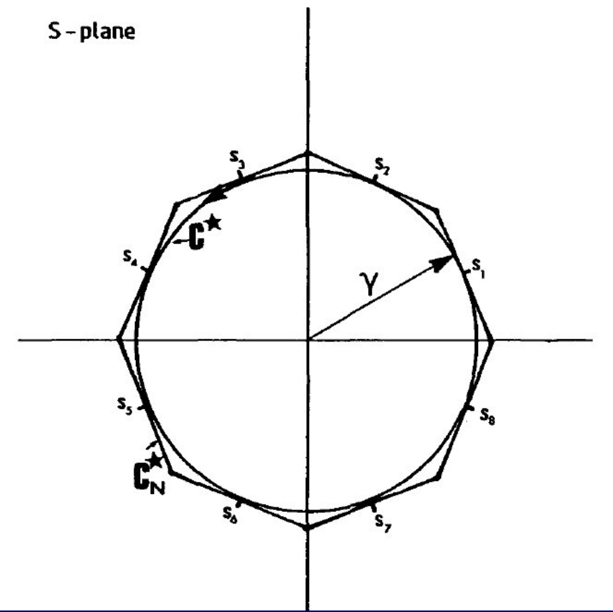
For $n = 1, 2, \ldots, N$:

- The lengths of the edges are Δs_n
- the midpoints are labelled s_n

The integrand is evaluated at the centre of each edge, and the integral is computed numerically.


S - plane
s_1
s_2
s_3
s_4
s_5
s_6
s_7
s_8
γ
C^*
C_N^*

We compute a numerical approximation: the inverse

$$\mathcal{L}^{\star}\{\hat{f}(s)\} = \frac{1}{2\pi i}\oint_{\mathcal{C}^{\star}} \exp(st)\,\hat{f}(s)\,\mathrm{d}s$$

is approximated by the summation

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We introduce a correction factor, and arrive at:

$$\mathcal{L}^{\star}_{N}\{\hat{f}(s)\} = \frac{1}{N}\sum_{n=1}^{N} \exp_N(s_n t)\,\hat{f}(s_n)\,s_n$$

Here $\exp_N(z)$ is the N-term Taylor expansion of $\exp(z)$

(For details, see Clancy and Lynch, 2011a)

Outline

Applying LT to an ODE

We consider a nonlinear ordinary differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} + i\omega w + n(w) = 0 \qquad w(0) = w_0$$

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We can immediately solve for the transform solution:

$$\hat{w}(s) = \frac{1}{s + i\omega}\left[w_0 - \frac{n_0}{s}\right]$$

Using partial fractions, we write the transform as

$$\hat{w}(s) = \left(\frac{w_0}{s + i\omega}\right) + \frac{n_0}{i\omega}\left(\frac{1}{s + i\omega} - \frac{1}{s}\right)$$

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Thus, the solution is

$$w^\star(t) = \begin{cases} \left(w_0 + \dfrac{n_0}{i\omega}\right)\exp(-i\omega t) - \dfrac{n_0}{i\omega} & : \quad |\omega| < \gamma \\ -\dfrac{n_0}{i\omega} & : \quad |\omega| > \gamma \end{cases}$$

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Thus, high frequencies are filtered out.

Again, for a HF oscillation ($|\omega| > \gamma$), the solution is

$$w^\star(t) = -\frac{n_0}{i\omega}$$

or

$$i\omega\, w^\star(t) + n_0 = 0$$

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Clearly, this corresponds to the criterion for nonlinear normal mode initialization:

Set the tendency of the HF terms to zero at $t = 0$.

Outline

A General NWP Equation

We write the general NWP equations symbolically as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

where $\mathbf{X}(t)$ is the state vector at time t.

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We apply the Laplace transform to get

$$(s\,\hat{\mathbf{X}} - \mathbf{X}_0) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_0 = 0$$

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The frequencies are entangled. How do we proceed?

Eigenanalysis

$$\dot{\mathbf{X}} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

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Assume the eigenanalysis of L is

$$\mathbf{L}\mathbf{E} = \mathbf{E}\mathbf{\Lambda}$$

where $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \ldots, \lambda_N)$ **and** $\mathbf{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_N)$.

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where $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_N)$ and $\mathbf{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_N)$.

More explicitly, assume that the eigenfrequencies split in two:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_Y & 0 \\ 0 & \boldsymbol{\Lambda}_Z \end{bmatrix}$$

$\boldsymbol{\Lambda}_Y$: Frequencies of rotational modes (LF)

$\boldsymbol{\Lambda}_Z$: Frequencies of gravity-inertia modes (HF)

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This is just

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This equation separates into two sub-systems:

$$\begin{aligned} \dot{\mathbf{Y}} + i\,\mathbf{\Lambda}_Y\mathbf{Y} + \mathbf{N}_Y(\mathbf{Y},\mathbf{Z}) &= \mathbf{0} \\ \dot{\mathbf{Z}} + i\,\mathbf{\Lambda}_Z\mathbf{Z} + \mathbf{N}_Z(\mathbf{Y},\mathbf{Z}) &= \mathbf{0} \end{aligned}$$

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The variables $\mathbf{Y}$ and $\mathbf{Z}$ are all coupled through the nonlinear terms $\mathbf{N}_Y(\mathbf{Y},\mathbf{Z})$ and $\mathbf{N}_Z(\mathbf{Y},\mathbf{Z})$.

General Solution Method

We recall that the Laplace transform of the equation is

$$(s\hat{\mathbf{X}} - \mathbf{X}_0) + i\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_0 = \mathbf{0}$$

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But now we take $n\Delta t$ to be the initial time:

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Laplace transform integration of the shallow water equations. Part 1: Eulerian formulation and Kelvin waves

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Laplace transform integration of the shallow water equations. Part 2: Lagrangian formulation and orographic resonance

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Outline

Phase Errors of SI and LT Schemes

Consider the phase error of the oscillation equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + i\omega\, u = 0 \qquad R = \frac{\text{Numerical frequency}}{\text{Physical frequency}}$$

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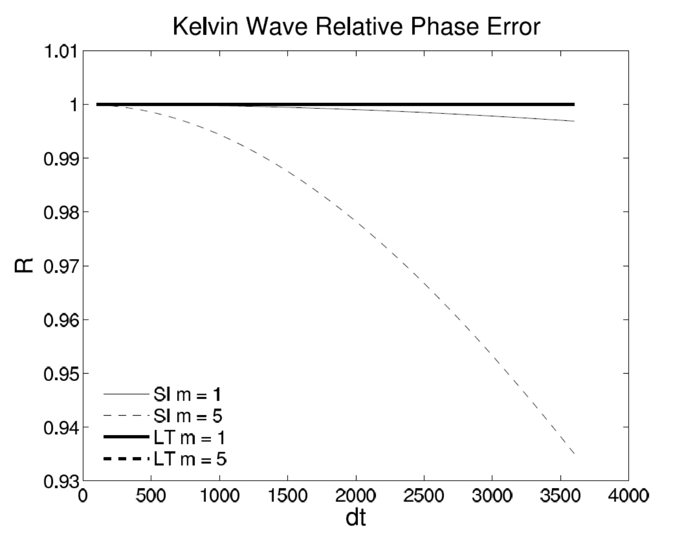
$$R_{\mathrm{SI}} = 1 - \frac{1}{12}(\omega\Delta t)^2$$

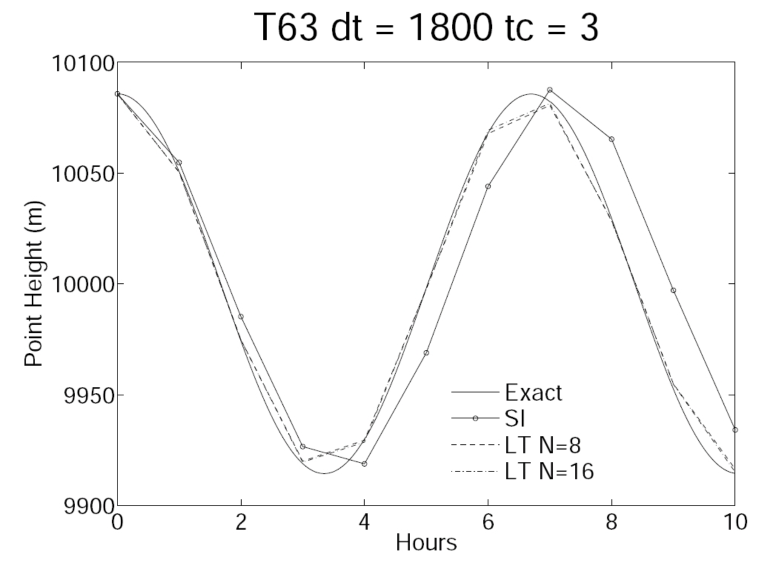
For the LT scheme, the corresponding error is

$$R_{\mathrm{LT}} = 1 - \frac{1}{N!}\,(\omega\Delta t)^N$$

Even for modest values of N, this is negligible.



Relative phase errors for semi-implicit (SI) and Laplace transform (LT) schemes for Kelvin waves $m = 1$ and $m = 5$.



Hourly height at $0.0°$E, $0.9°$N over 10 hours, with $\tau_c = 3$ h.

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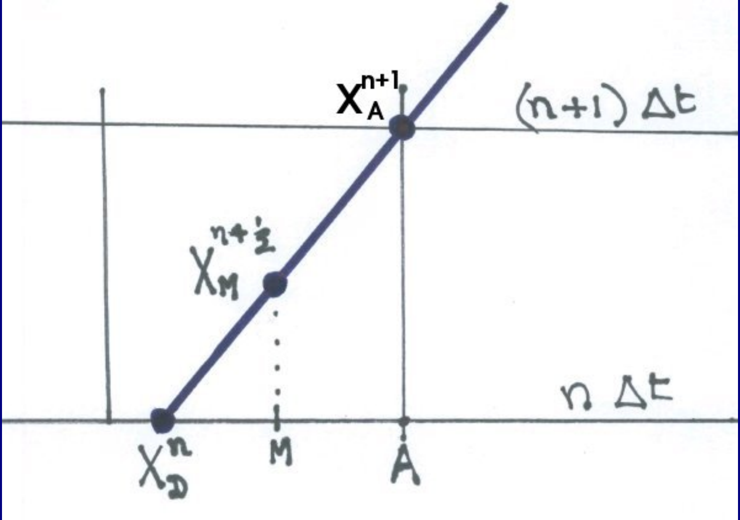
We *re-define* the Laplace transform to be the integral in time *along the trajectory of a fluid parcel*:

$$\hat{\mathbf{X}}(s) \equiv \int_{\mathcal{T}} e^{-st}\,\mathbf{X}(t)\,\mathrm{d}t$$



We compute $\mathcal{L}$ along a fluid trajectory $\mathcal{T}$.

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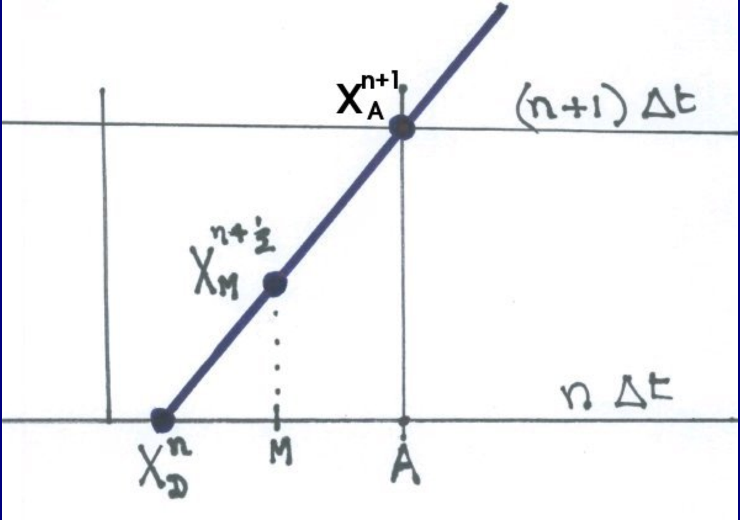
The **initial values** when transforming the Lagrangian time derivatives are $\mathbf{X}_{\mathrm{D}}^{n}$.

The equations thus transform to

$$(s\,\hat{\mathbf{X}} - \mathbf{X}_{\mathrm{D}}^{n}) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_{\mathrm{M}}^{n+\frac{1}{2}} = \mathbf{0}$$

where we evaluate nonlinear terms at a **mid-point**, interpolated in space and extrapolated in time.



Departure point, arrival point and mid-point.

The solution can be written formally:

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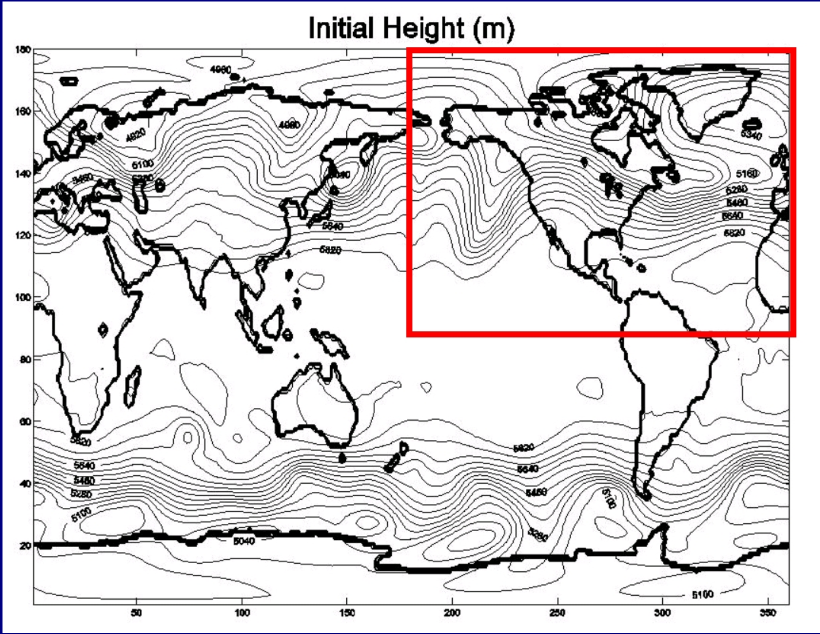
- Running at T119 resolution

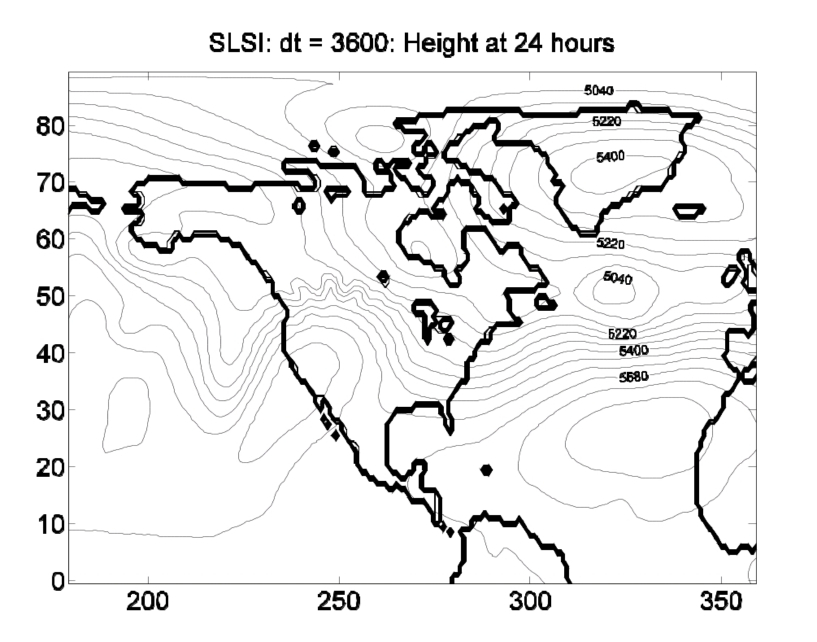
Orographic Resonance

- Spurious resonance arises from coupling the semi-Lagrangian and semi-implicit methods
- Linear analysis of orographically forced stationary waves confirms this
- This motivates an investigating of orographic resonance in a full model.

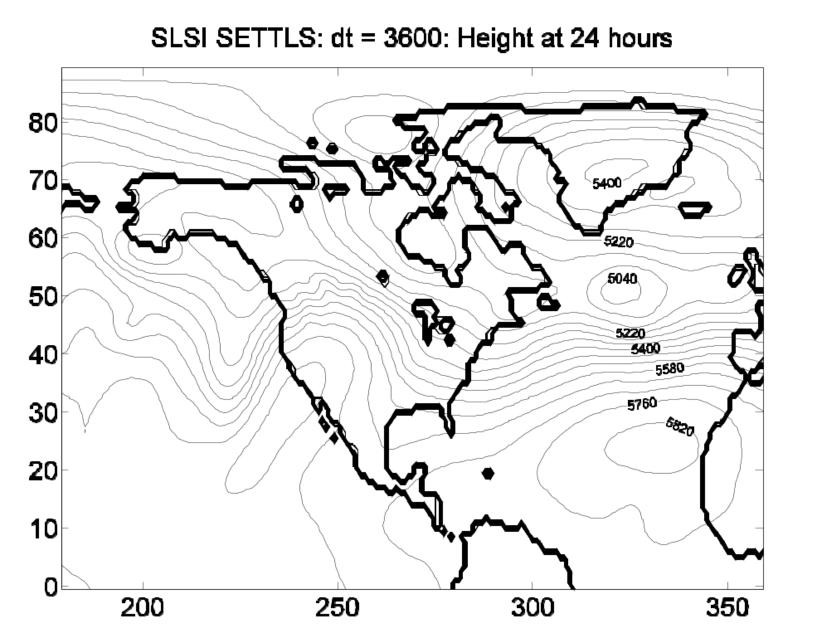
Test Case:

- Initial data: ERA-40 analysis of 12 UTC on 12th February 1979
- Used by Ritchie & Tanguay (1996) and by Li & Bates (1996)
- Running at T119 resolution
- Shows LT method has benefits over SI scheme.

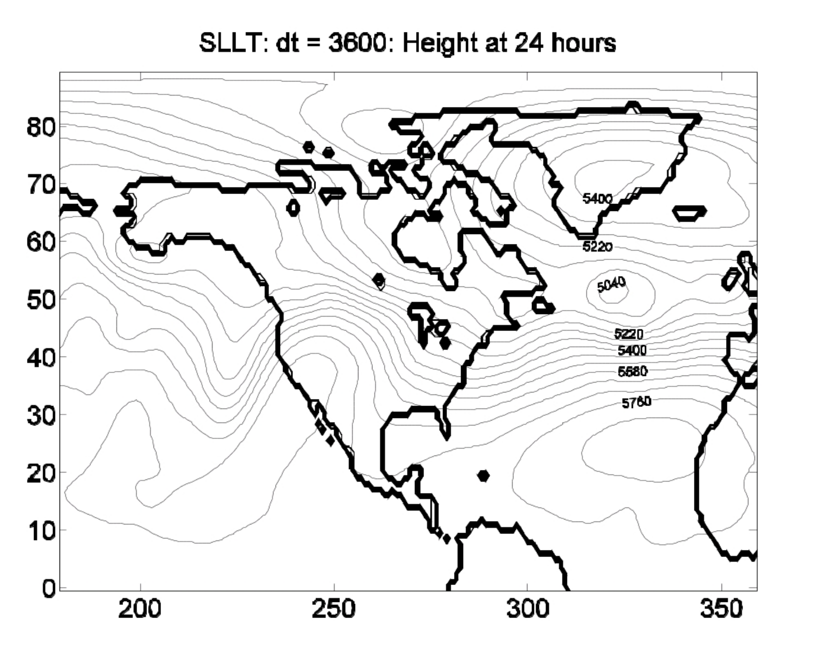

Initial Height (m)
180
160
140
120
100
80
60
40
20
50
100
150
200
250
300
350
4960
4920
5100
5820
5640
5480
5280
5040
5340
5160



SLSI: dt = 3600: Height at 24 hours
5040
5220
5400
5220
5040
5220
5400
5580
80
70
60
50
40
30
20
10
0
200
250
300
350



SLSI SETTLS: dt = 3600: Height at 24 hours
80
70
60
50
40
30
20
10
0
200
250
300
350
5400
5220
5040
5220
5400
5580
5760
5820


SLLT: dt = 3600: Height at 24 hours
80
70
60
50
40
30
20
10
0
200
250
300
350
5400
5220
5040
5220
5400
5580
5760

Conclusion

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Next job:

Implement the LT scheme in a full baroclinic model.

Thank you