Laplace Transforms and Exponential Integrators

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The accuracy and efficiency of weather and climate models has been greatly enhanced by the introduction of better numerical algorithms for the solution of the equations of motion. Two of the most notable schemes are the semi-implicit (SI) scheme for treating the gravity-wave terms and the semi-Lagrangian scheme for advection processes.

Many operational NWP models use a semi-implicit scheme for time integration, increasing efficiency by enabling the use of a large time step. But this comes at a price: stabilization is achieved by slowing down the high-frequency gravity waves. However, the meteorologically significant components of the flow are also distorted by the time averaging of the SI scheme.

It was pointed out in Lynch & Clancy (2016) that the LT method with analytic inversion gives an exact treatment of the linear modes. This is due to the fact that the LT scheme does not involve time-averaging of the linear terms. Harney & Lynch (2019) describe a Laplace transform integration scheme in a baroclinic model and show that it yields more accurate forecasts than SI. A version of the LT scheme for use with semi-Lagrangian advection is under development.

An alternative way of achieving accuracy is to use an exponential integrator (e.g., Pudykiewicz and Clancy, 2019; Peixoto and Schreiber, 2019). In this note we demonstrate the close relationship between Laplace transform integration and exponential integrators.

LTI and Exponential Integrators

We may write the model equations in the form

$$\frac{\partial \mathbf{X}}{\partial t} = \mathsf{L}\mathbf{X} + \mathbf{N}(\mathbf{X})\,. \tag{1}$$

Assuming that the matrix L has an orthogonal eigenvector matrix E with $\mathsf{LE} = \mathsf{E}\Lambda$, we have

$$\mathsf{L} = \mathsf{E}\Lambda\mathsf{E}^{\mathrm{T}}\,, \quad \Lambda = \mathsf{E}^{\mathrm{T}}\mathsf{L}\mathsf{E} \qquad \text{and} \qquad e^{\mathsf{L}t} = \mathsf{E}e^{\Lambda t}\mathsf{E}^{\mathrm{T}}$$

$$\mathsf{L}^{-1} = \mathsf{E}\Lambda^{-1}\mathsf{E}^{\mathrm{T}}\,, \quad \Lambda^{-1} = \mathsf{E}^{\mathrm{T}}\mathsf{L}^{-1}\mathsf{E} \qquad \text{and} \qquad e^{\Lambda t} = \mathsf{E}^{\mathrm{T}}e^{\mathsf{L}t}\mathsf{E}$$

If the solution of (1) at time $t_n = n\Delta t$ is known, the Laplace transform with this initial time is

$$s\widehat{\mathbf{X}} - \mathbf{X}^n = \mathsf{L}\widehat{\mathbf{X}} + \widehat{\mathbf{N}}$$

where $\mathcal{L}\{\mathbf{X}\} = \widehat{\mathbf{X}}$ is the Laplace transform of the state vector. Solving for this, we get

$$\widehat{\mathbf{X}} = (s\mathsf{I} - \mathsf{L})^{-1}[\mathbf{X}^n + \widehat{\mathbf{N}}]\,. \tag{2}$$

We note that

$$(s\mathsf{I} - \mathsf{L})^{-1} = \mathsf{E}(s\mathsf{I} - \Lambda)^{-1}\mathsf{E}^{\mathrm{T}}$$

and also note the transforms

$$(s\mathsf{I} - \Lambda)^{-1} = \mathcal{L}\{\exp(\Lambda t)\} \qquad \text{and} \qquad (s\mathsf{I} - \mathsf{L})^{-1} = \mathcal{L}\{\exp(\mathsf{L}t)\}\,.$$

We can write the nonlinear term as

$$(s\mathsf{I} - \mathsf{L})^{-1}\widehat{\mathbf{N}} = \mathcal{L}\{\exp(\mathsf{L}t)\} \cdot \mathcal{L}\{\mathbf{N}\}\,.$$

The convolution theorem allows this to be written

$$(s\mathsf{I} - \mathsf{L})^{-1}\widehat{\mathbf{N}} = \mathcal{L}\left\{\int_{t_n}^{t} \exp(\mathsf{L}(t-\tau))\mathbf{N}(\tau)\,\mathrm{d}\tau\right\}\,.$$

The transformed equation (2) now becomes

$$\widehat{\mathbf{X}} = \mathcal{L}\{\exp(\mathsf{L}t)\}\mathbf{X}^n + \mathcal{L}\left\{\int_{t_n}^{t} \exp(\mathsf{L}(t-\tau))\mathbf{N}(\tau)\,\mathrm{d}\tau\right\}$$

We invert this at time $t_{n+1} = t_n + \Delta t$ to get

$$\mathbf{X}^{n+1} = e^{\mathsf{L}t_{n+1}}\mathbf{X}^n + e^{\mathsf{L}t_{n+1}} \int_{t_n}^{t_{n+1}} e^{-\mathsf{L}\tau}\mathbf{N}(\tau)\,\mathrm{d}\tau \qquad (3)$$

We note that (3) is formally identical to Equation (8) of Peixoto and Schreiber (2019) which they call the variation-of-constants formula.

We have thus established a close relationship between the Laplace transform scheme and exponential integrators.

Approximating the Nonlinear Term

The convolution term must be evaluated by approximate means, since it involves unknown quantities. Suppose we evaluate the nonlinear term at time t_n and assume that it is constant throughout the timestep (t_n, t_{n+1}). Then the convolution integral can be evaluated, giving

$$\begin{aligned}\mathbf{X}^{n+1} &= e^{\mathsf{L}t_{n+1}}\mathbf{X}^n + e^{\mathsf{L}t_{n+1}} \left(\int_{t_n}^{t_{n+1}} e^{-\mathsf{L}\tau}\,\mathrm{d}\tau\right)\mathbf{N}^n \\ &= e^{\mathsf{L}t_{n+1}}\mathbf{X}^n + (-\mathsf{L})^{-1}[\mathsf{I} - \exp(\mathsf{L}\Delta t)]\mathbf{N}^n\,.\end{aligned}$$

Assuming a small time-step, this reduces to

$$\mathbf{X}^{n+1} = e^{\mathsf{L}t_{n+1}}\mathbf{X}^n + \Delta t\mathbf{N}^n\,.$$

This is perhaps the simplest version of an exponential integrator. There is a wide range of more sophisticated and accurate approximations of the convolution integral. For example, we might estimate N at the centre of the time step by extrapolation $N^{n+1/2} = (3N^n - N^{n-1})/2$. Many other possibilities exist.

The time-averaging of the SI scheme also results in an error in the nonlinear term, even when this term is constant (see Harney & Lynch, 2019, Eq. 3). In this ideal case, the LT scheme has no error in the nonlinear term (*loc. cit.*, Eq. 4).

References

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