

# On resonant Rossby–Haurwitz triads

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## ABSTRACT

The dynamics of non-divergent flow on a rotating sphere are described by the conservation of absolute vorticity. The analytical study of the non-linear barotropic vorticity equation is greatly facilitated by the expansion of the solution in spherical harmonics and truncation at low order. The normal modes are the well-known Rossby–Haurwitz (RH) waves, which represent the natural oscillations of the system. Triads of RH waves, which satisfy conditions for resonance, are of critical importance for the distribution of energy in the atmosphere.

We show how non-linear interactions of resonant RH triads may result in dynamic instability of large-scale components. We also demonstrate a mathematical equivalence between the equations for an orographically forced triad and a simple mechanical system, the forced-damped swinging spring. This equivalence yields insight concerning the bounded response to a constant forcing in the absence of damping. An examination of triad interactions in atmospheric reanalysis data would be of great interest.

## 1. Introduction

It is well known that the dynamical behaviour of planetary waves in the atmosphere is modelled by the barotropic vorticity equation (BVE; Rossby et al., 1939; Haurwitz, 1940). Charney et al. (1950) integrated the BVE to produce the earliest numerical weather predictions. They used a finite difference approximation to the equation. Silberman (1954) devised a numerical solution method in which the streamfunction is expanded in spherical surface harmonics. The non-linear terms introduced interaction coefficients between the components. A more efficient spectral technique, the transform method, was later devised by Eliassen et al. (1970) and Orszag (1970).

Highly truncated versions of the spectral BVE have been analysed to gain understanding of atmospheric phenomena. Lorenz (1960) introduced what he called the ‘maximum simplification’ of the system, reducing it to three non-linear ODEs. This enabled him to study the energy exchanges between the zonal mean flow and wave-like disturbances and to advance conjectures about the mechanism of the index cycle. He recommended using such ‘systematically imperfect’ systems of equations as a means of gaining understanding of atmospheric phenomena.

In a series of papers, Platzman undertook a systematic study of the truncated spectral vorticity equation (Platzman, 1960; Platzman, 1962). From the work of Fjørtoft (1953), it is known

that a three-component system is the lowest order system capable of exhibiting energy exchanges. (For background theory of three-wave resonance, see Craik, 1985). Platzman (1962) showed that a three-component system has periodic solutions: the equations are integrable, and omitting singular cases, the solutions are expressible in terms of Jacobi elliptic functions. Interactions are particularly effective when the component parameters are related by ‘resonance conditions’. These ensure that the phase relationships between the components are constant over a large number of periods, so that energy exchanges are facilitated. Platzman (1962) did not explicitly consider this resonant case.

The non-linear interactions between different scales play a critical role in establishing the statistical energy spectrum of the atmosphere (Newell et al., 2001; Chen et al., 2005). The role of non-linear interactions of RH modes was considered by Reznik et al. (1993). They concluded that interactions for which the resonance conditions are approximately satisfied, can generate an intensive redistribution of energy amongst the scales much smaller than the Earth’s radius.

Resonant triads are also crucial in determining the dynamical stability of planetary waves (Lorenz, 1972; Hoskins, 1973). Gill (1974) showed the role of resonant triads in initiating instabilities. Baines (1976) showed that planetary waves with total wavenumber greater than two are unstable for sufficiently large amplitude. He also explicitly identified a number of resonant Rossby–Haurwitz (RH) triads (see his Table 3). Hoskins (1973) found that RH mode (4,5) was stable. However, his integrations were on a quadrant and only modes whose zonal wavenumbers are multiples of four were admissible. A numerical study of

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mode RH(4,5) by Thuburn and Li (2000) showed it to be unstable, with a near-resonant triad interaction as the likely mechanism. We provide numerical evidence that strongly supports this hypothesis.

Burzclaff et al. (2008) showed that at the second order of a perturbation approximation, zonal flow can be generated by triad interactions. This mechanism provides a channel for energy exchange between mean and eddy motions in the atmosphere. The phenomenon of vacillation in the stratospheric flow was first examined by Holton and Mass (1976). They found that for wave forcing beyond a critical amplitude, the response to a steady forcing is not steady, but the mean zonal flow and eddy components oscillate quasi-periodically. Such oscillatory response to steady forcing is consistent with forced resonant triads, as we will see in Section 4.

In this paper, we first review the spectral analysis of the BVE and the normal mode solutions of the equation (Section 2). We then derive the equations for resonant RH triad solutions (Section 3). We present numerical evidence that the instability of RH mode (4,5) can be accounted for in terms of a resonant triad interaction. Miles (1985) studied resonantly forced Rossby waves using the quasi-geostrophic potential vorticity equation, including Ekman damping, and found resonant triad solutions. In Section 4, we revisit the problem of a forced planetary wave in the context of the BVE and find a periodic response involving a resonant triad. We introduce forcing and damping terms that represent the interaction of orography with the flow and the effects of energy dissipation by various mechanisms. We show by a numerical example that a constant forcing by orography can result in a periodic response in the absence of damping. To understand the mechanism of this periodic response to constant forcing, we digress in Section 5 to consider a simple mechanical system, the forced-damped swinging spring. The envelope equations for this system are found to be mathematically identical to the equations derived in Section 4 for the orographically forced system. We see how the forcing of the spring increases or decreases the energy of the system depending on the relative phase. The pulsation from the vertical to horizontal modes and back again results in a reversal of the phase, so that the overall solution exhibits periodicity. Thus, the isomorphism between the systems yields insight into the atmospheric dynamics. We conclude (in Section 6) with some remarks on possible extensions of this work.

## 2. Normal modes of the BVE

We consider a shallow layer of incompressible fluid on a rotating sphere, assuming the horizontal velocity to be non-divergent. The radius of the sphere is  $a$ , the rotation rate is  $\Omega$ , and longitude/latitude coordinates  $(\lambda, \phi)$  will be used. The dynamics of the fluid are governed by conservation of absolute vorticity

$$\frac{d}{dt}(\zeta + f) = 0, \tag{1}$$

where  $f = 2\Omega \sin \phi$  is the planetary vorticity,  $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V}$  is the vorticity of the flow and the time derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \phi}.$$

Introducing a stream-function  $\psi$  such that  $\mathbf{V} = \mathbf{k} \times \nabla \psi$  and  $\zeta = \nabla^2 \psi$  and defining  $\mu = \sin \phi$ , the equation may be written

$$\frac{\partial \zeta}{\partial t} + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} + \frac{1}{a^2} \frac{\partial(\psi, \zeta)}{\partial(\lambda, \mu)} = 0 \quad \text{[BVE]} \tag{2}$$

This is the non-divergent BVE. The non-linear advection is represented by the Jacobian term. Temporarily omitting this, it is immediately found that (2) has solutions of the form

$$\psi = \psi_0 Y_n^m(\lambda, \mu) \exp(-i\sigma t) = \psi_0 P_n^m(\mu) \exp[i(m\lambda - \sigma t)], \tag{3}$$

where  $\psi_0$  is the constant amplitude and the frequency  $\sigma$  is given by the dispersion formula

$$\sigma = \sigma_n^m \equiv -\frac{2\Omega m}{n(n+1)}. \tag{4}$$

Here,  $m$  is the zonal wavenumber,  $n$  is the total wavenumber (both are integers) and  $Y_n^m(\lambda, \mu)$  are the spherical harmonics, which are eigenfunctions of the Laplacian operator on the sphere:

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m. \tag{4}$$

A comprehensive classical treatise on spherical harmonics is available (Hobson, 1931). The most important properties of the surface harmonics are presented in Machenhauer (1979). Orszag (1974) provides an excellent summary of the advantages and disadvantages of the spectral method. We assume the functions to be normalized so that

$$\frac{1}{4\pi} \iint (Y_{n_1}^{m_1})^* Y_{n_2}^{m_2} d\lambda d\mu = \delta_{m_2}^{m_1} \delta_{n_2}^{n_1}. \tag{5}$$

Solutions (3) are called Rossby–Haurwitz waves, or RH waves (Rossby et al., 1939; Haurwitz, 1940). It is remarkable that, for a single RH wave, the non-linear Jacobian term vanishes identically, so that such a wave is a solution of the non-linear eq. (2). This is not generally true for a combination of such waves: the velocity of one component will advect the vorticity of another, so that the waves interact and their amplitudes change.<sup>1</sup>

The spherical harmonics form an orthonormal basis on the sphere: any sufficiently smooth function may be expressed as a sum of such components. Thus, the stream function has an expansion

$$\psi(\lambda, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \psi_n^m(t) Y_n^m(\lambda, \mu). \tag{5}$$

<sup>1</sup> Craig (1945) and Neamtan (1946) found that combinations of RH waves with different zonal wavenumbers  $m$  but the same total wavenumber  $n$  were solutions of the non-linear equation. All components have the same angular phase-speed  $\sigma/m = -2\Omega/n(n+1)$ .

The vorticity has a similar expansion, with coefficients  $\zeta_n^m = (-n(n+1)/a^2)\psi_n^m$ . The coefficients  $\psi_n^m$  and  $\zeta_n^m$  are normally functions of time. For brevity, we define a vector wavenumber  $\gamma = (m, n)$  and denote its conjugate by  $\bar{\gamma} = (-m, n)$ . We can then write the expansions

$$\psi = \sum_{\gamma} \psi_{\gamma}(t) Y_{\gamma}(\lambda, \mu) \exp(-i\sigma_{\gamma}t) \quad (6)$$

and

$$\zeta = \sum_{\gamma} \zeta_{\gamma}(t) Y_{\gamma}(\lambda, \mu) \exp(-i\sigma_{\gamma}t), \quad (7)$$

with  $\psi_{\gamma} = -a^2 \kappa_{\gamma} \zeta_{\gamma}$ , where  $\kappa_{\gamma} = 1/(n(n+1))$ . For a pure RH wave, or a collection of non-interacting waves, the coefficients  $\psi_{\gamma}$  and  $\zeta_{\gamma}$  are constants. The variation of these factors represents the amplitude and phase changes induced by nonlinear interaction between the components. If the non-linear interactions are weak, the coefficients will vary slowly with time compared to the modal factors  $\exp(-i\sigma_{\gamma}t)$ .

Flows governed by the BVE conserve the total energy and total enstrophy, defined by

$$E = \frac{1}{4\pi a^2} \iint \frac{1}{2} \mathbf{V} \cdot \mathbf{V} d\lambda d\mu = -\frac{1}{4\pi a^2} \iint \frac{1}{2} \psi \zeta d\lambda d\mu,$$

$$S = \frac{1}{4\pi a^2} \iint \frac{1}{2} \zeta^2 d\lambda d\mu = -\frac{1}{4\pi a^2} \iint \frac{1}{2} \nabla \psi \cdot \nabla \zeta d\lambda d\mu.$$

In terms of the spectral coefficients, the constrained quantities may be written

$$E = \frac{1}{2} \sum_{\gamma} \kappa_{\gamma} |\zeta_{\gamma}|^2, \quad S = \frac{1}{2} \sum_{\gamma} |\zeta_{\gamma}|^2.$$

The constancy of energy and enstrophy profoundly influences the energetics of solutions of the BVE. As will be shown below (see eq. 17), analogues of these quantities are conserved for truncated solutions of the equation.

If the expansions (6) are substituted into the BVE (2), and the orthogonality condition (5) used, we obtain equations for the evolution of the spectral coefficients in time:

$$\frac{d\zeta_{\gamma}}{dt} = \frac{1}{2} i \sum_{\alpha, \beta} I_{\gamma\beta\alpha} \zeta_{\beta} \zeta_{\alpha} \exp(-i\sigma t), \quad (8)$$

where  $\sigma = \sigma_{\alpha} + \sigma_{\beta} - \sigma_{\gamma}$  and the ‘interaction coefficients’ are given by

$$I_{\gamma\beta\alpha} = (\kappa_{\beta} - \kappa_{\alpha}) K_{\gamma\beta\alpha}.$$

The ‘coupling integrals’  $K_{\gamma\beta\alpha}$  vanish unless  $m_{\alpha} + m_{\beta} = m_{\gamma}$ ; this follows from the separability of the spherical harmonics and the orthogonality of the exponential components for different  $m$ . In case  $m_{\alpha} + m_{\beta} = m_{\gamma}$ , they are given by

$$K_{\gamma\beta\alpha} = \frac{1}{2} \int_{-1}^{+1} P_{\gamma} \left( m_{\beta} P_{\beta} \frac{dP_{\alpha}}{d\mu} - m_{\alpha} P_{\alpha} \frac{dP_{\beta}}{d\mu} \right) d\mu.$$

Equation (8) is completely equivalent to (2.8) in Platzman (1962), although he writes the equation in a slightly different

(non-redundant) form. Silberman (1954), who first applied the spectral approach to solution of the vorticity equation, derived explicit formulae for the interaction coefficients. Using the properties of the spherical harmonics, it may be shown that the interaction coefficients vanish in most cases. For non-vanishing interaction, the following ‘selection rules’ must be satisfied:

$$\begin{aligned} m_{\alpha} + m_{\beta} &= m_{\gamma}, \\ m_{\alpha}^2 + m_{\beta}^2 &\neq 0, \\ n_{\gamma} n_{\beta} n_{\alpha} &\neq 0, \\ n_{\alpha} &\neq n_{\beta}, \\ n_{\alpha} + n_{\beta} + n_{\gamma} &\text{ is odd,} \\ (n_{\beta} - |m_{\beta}|)^2 + (n_{\alpha} - |m_{\alpha}|)^2 &\neq 0, \\ |n_{\alpha} - n_{\beta}| < n_{\gamma} < n_{\alpha} + n_{\beta}, \\ (m_{\beta}, n_{\beta}) \neq (-m_{\gamma}, n_{\gamma}) &\text{ and } (m_{\alpha}, n_{\alpha}) \neq (-m_{\gamma}, n_{\gamma}). \end{aligned} \quad (9)$$

Further discussion of these rules may be found in Platzman (1962). Ellsaesser (1966) gives a detailed derivation of the selection rules and their implications. It is obvious that the following symmetries hold:

$$I_{\gamma\alpha\beta} = I_{\gamma\beta\alpha} \quad \text{and} \quad K_{\gamma\alpha\beta} = -K_{\gamma\beta\alpha}. \quad (10)$$

and the following ‘redundancy rules’ are easily proved by integration by parts:

$$K_{\alpha\bar{\beta}\gamma} = K_{\gamma\beta\alpha} \quad \text{and} \quad K_{\beta\gamma\bar{\alpha}} = K_{\gamma\beta\alpha}, \quad (11)$$

where  $\bar{\alpha} = (-m, n)$  when  $\alpha = (m, n)$ .

### 3. Resonant Rossby–Haurwitz triads

We now investigate solutions that are severely truncated, comprising only a small number of non-vanishing components. Of course, non-linear interactions between these may generate further components, so that the simple structure may not persist. However, under certain circumstances, the interactions are so weak that the simple low-order structure persists for a long time. We consider the case where there are just three non-vanishing spectral components. Actually, since the fields are assumed real, the complex conjugate components must also be present. Thus, we assume the streamfunction is of the form

$$\psi = \Re \left\{ \psi_{\alpha} Y_{\alpha} \exp(-i\sigma_{\alpha}t) + \psi_{\beta} Y_{\beta} \exp(-i\sigma_{\beta}t) + \psi_{\gamma} Y_{\gamma} \exp(-i\sigma_{\gamma}t) \right\}. \quad (12)$$

Without loss of generality, we assume that  $0 \leq m_{\alpha} \leq m_{\beta} \leq m_{\gamma}$ . For non-vanishing interactions, we then require  $m_{\alpha} + m_{\beta} = m_{\gamma}$ . The selection rules (9) then imply that the only non-vanishing interaction coefficients are as follows:

$$I_{\gamma\beta\alpha} = I_{\gamma\alpha\beta}, \quad I_{\beta\bar{\alpha}\gamma} = I_{\beta\gamma\bar{\alpha}}, \quad I_{\alpha\bar{\beta}\gamma} = I_{\alpha\gamma\bar{\beta}}.$$

Then, using the symmetries (10) and redundancy rules (11), we find that all the coefficients can be expressed in terms of a single

one, together with the quantities  $\kappa_\alpha$ ,  $\kappa_\beta$  and  $\kappa_\gamma$ . The spectral equations may then be written

$$\begin{aligned} i\dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma)K\zeta_\beta^*\zeta_\gamma \exp(+i\sigma t), \\ i\dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha)K\zeta_\gamma\zeta_\alpha^* \exp(+i\sigma t), \\ i\dot{\zeta}_\gamma &= +(\kappa_\alpha - \kappa_\beta)K\zeta_\alpha\zeta_\beta \exp(-i\sigma t), \end{aligned} \quad (13)$$

where  $K = K_{\gamma\beta\alpha}$  and  $\sigma = \sigma_\alpha + \sigma_\beta - \sigma_\gamma$ . These are equivalent to the system derived in section 8 of Platzman (1962) (see also Pedlosky, 1987, section 3.26). In general, the right-hand sides of these equations vary rapidly in time, due to the factors  $e^{i\sigma t}$ . Indeed, if the equations are averaged over a time  $\tau = 2\pi/\sigma$  and the variations of the spectral amplitudes during this time are negligible, the right-hand sides vanish, except in the special case where  $\sigma = 0$ . This is the case of ‘resonance’, and in this case, there is a possibility of strong interchange of energy between the modes.<sup>2</sup>

In the sequel, we consider exclusively the resonant case. The condition for resonance,  $\sigma = 0$ , may be written

$$m_\alpha\kappa_\alpha + m_\beta\kappa_\beta = m_\gamma\kappa_\gamma.$$

This, together with the selection condition  $m_\alpha + m_\beta = m_\gamma$ , implies that either  $\kappa_\alpha \geq \kappa_\gamma \geq \kappa_\beta$  or  $\kappa_\alpha \leq \kappa_\gamma \leq \kappa_\beta$ . For definiteness, let us assume  $\kappa_\alpha \geq \kappa_\gamma \geq \kappa_\beta$ . The cases where equality holds are easily disposed of: when all  $\kappa$ -factors are equal, all  $\zeta$ -coefficients are constants. When two are equal, say  $\kappa_\gamma = \kappa_\beta$ , then the third coefficient,  $\zeta_\alpha$  is constant, and  $\zeta_\beta$  and  $\zeta_\gamma$  vary sinusoidally. Let us therefore consider the generic case,

$$\kappa_\alpha > \kappa_\gamma > \kappa_\beta. \quad (14)$$

Thus,  $n_\alpha < n_\gamma < n_\beta$ , so that the component  $\zeta_\gamma$  is of a scale intermediate between the two components that are interacting to modify it (Fjørtoft, 1953). The eqs. (13) may then be written

$$\begin{aligned} i\dot{\zeta}_\alpha &= k_\alpha\zeta_\beta^*\zeta_\gamma, \\ i\dot{\zeta}_\beta &= k_\beta\zeta_\gamma\zeta_\alpha^*, \\ i\dot{\zeta}_\gamma &= k_\gamma\zeta_\alpha\zeta_\beta, \end{aligned} \quad (15)$$

where, assuming  $K > 0$ , the coefficients

$$k_\alpha = (\kappa_\gamma - \kappa_\beta)K, \quad k_\beta = (\kappa_\alpha - \kappa_\gamma)K, \quad k_\gamma = (\kappa_\alpha - \kappa_\beta)K$$

are all positive and  $k_\alpha + k_\beta = k_\gamma$ . If  $K < 0$ , we reverse the signs of the definitions to make the  $k$ s positive. The energy and enstrophy of the triad may be written

$$\begin{aligned} E &= \frac{1}{2}(\kappa_\alpha|\zeta_\alpha|^2 + \kappa_\beta|\zeta_\beta|^2 + \kappa_\gamma|\zeta_\gamma|^2), \\ S &= \frac{1}{2}(|\zeta_\alpha|^2 + |\zeta_\beta|^2 + |\zeta_\gamma|^2). \end{aligned}$$

We now introduce the transformation

$$\eta_\alpha = \sqrt{k_\beta k_\gamma} \zeta_\alpha, \quad \eta_\beta = \sqrt{k_\gamma k_\alpha} \zeta_\beta, \quad \eta_\gamma = \sqrt{k_\alpha k_\beta} \zeta_\gamma,$$

to recast eqs. (15) in the standard form of the ‘three-wave equations’:

$$\begin{aligned} i\dot{\eta}_\alpha &= \eta_\beta^*\eta_\gamma, \\ i\dot{\eta}_\beta &= \eta_\gamma\eta_\alpha^*, \\ i\dot{\eta}_\gamma &= \eta_\alpha\eta_\beta. \end{aligned} \quad (16)$$

There are two conservation laws for (16), corresponding to the energy and enstrophy of the full system. The ‘Manley–Rowe’ quantities are defined as

$$\begin{aligned} N_1 &= |\eta_\alpha|^2 + |\eta_\gamma|^2, \\ N_2 &= |\eta_\beta|^2 + |\eta_\gamma|^2, \\ J &= |\eta_\alpha|^2 - |\eta_\beta|^2. \end{aligned} \quad (17)$$

It is immediate from (16) that these are all constants of the motion, any two of them being independent. The system (16) may be shown to be the canonical equations arising from the Hamiltonian  $H = \Re\{\eta_\alpha\eta_\beta\eta_\gamma^*\}$  (Holm and Lynch, 2002; Lynch and Houghton, 2004).

### 3.1. Numerical example

To illustrate the role of resonant triads in instability of RH waves, we integrate the BVE on the sphere using a grid-point model with  $61 \times 31$  points ( $6^\circ$  resolution) and a 3-hour time step. The initial conditions are dominated by mode RH(4,5), with the remaining modes having amplitudes up to 5% of its amplitude and random phases. This is the mode that Hoskins (1973) suggested was stable, but that Thuburn and Li (2000) found to be unstable. The triad (4, 5), (1, 3), (3, 7) comes close to satisfying the frequency criterion for resonance. The respective frequencies (normalized by  $2\Omega$ ) are

$$\sigma_5^4 = -0.13333, \quad \sigma_3^1 = -0.08333, \quad \sigma_7^3 = -0.05357,$$

so that  $\sigma_5^4 \approx \sigma_3^1 + \sigma_7^3$ . The initial amplitude of the streamfunction component  $\psi_5^4$  corresponded to a height amplitude of 100 m. In Fig. 1, we show the evolution of the component amplitudes over a period of 80 d. The triad members are indicated in the figure. For the first half of the integration, the predominating energy exchange is from RH(4,5) to mode RH(1,3), with less energy flowing to RH(3,7) and to other modes. During the second half of the integration, the bulk of the energy returns to RH(4,5). When the integration is continued, the distribution of energy becomes more chaotic (not shown) as many other modes enter the picture. It is clear from these results that triad resonance is crucially involved in the breakdown of the primary RH mode.

## 4. Forcing and damping

We consider the consequences of including forcing by orography and damping toward a reference state with potential vorticity

<sup>2</sup> A table of low-order resonant triads (for  $n \leq 20$ ) is given in Burzlaff et al. (2008).

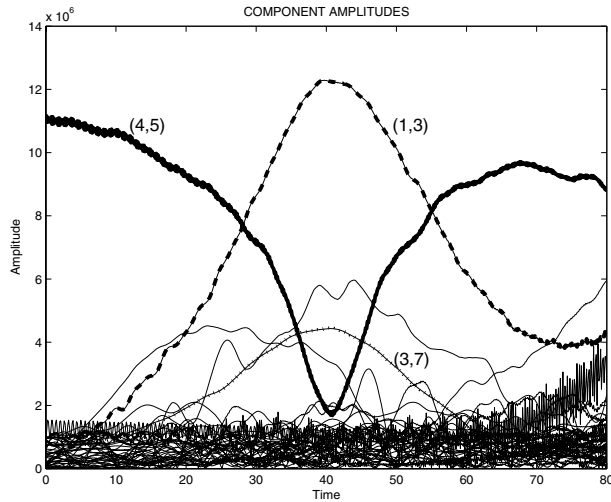


Fig. 1. Evolution of component amplitudes over an 80-day period. Mode RH(4,5) is initially dominant. Modes RH(1,3) and RH(3,7), which form a near-resonant triad with RH(4,5), are also indicated.

$f/H$ . The barotropic potential vorticity equation may be written

$$\frac{d}{dt} \left( \frac{\zeta + f}{H - h_0} \right) = -\nu \left( \frac{\zeta + f}{H - h_0} - \frac{f}{H} \right), \quad (18)$$

where  $H$  is the (constant) height of the upper surface,  $h_0$  is the elevation of the orography and  $\nu$  is the damping coefficient. We remark that variation of the upper surface may be accommodated, but at the expense of complicating the mathematical analysis: the eigensolutions become spheroidal wave functions rather than the simpler spherical harmonics (Longuet-Higgins, 1968).

The flow is separated into a super-rotation  $\bar{u} = a \cos \phi \bar{\omega}$ , with constant  $\bar{\omega}$ , and a perturbation  $(u, v)$ . Assuming that the orography is small,  $h_0 \ll H$ , we can write (18) in the form

$$\left( \frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda} \right) \zeta + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} + \frac{1}{a^2} \frac{\partial(\psi, \zeta)}{\partial(\lambda, \mu)} - \frac{\bar{\omega} f}{H} \frac{\partial h_0}{\partial \lambda} = -\nu \left( \zeta - \frac{f h_0}{H} \right). \quad (19)$$

This is the generalization of (2) for the forced-damped case. We have omitted non-linear terms involving  $h_0$ , assuming them to be small.

The linear normal modes have streamfunctions with spherical harmonic structure and eigenfrequencies

$$\sigma_n^m = \bar{\omega} - \frac{(2\Omega + \bar{\omega})m}{n(n+1)}. \quad (20)$$

If  $\bar{\omega}$  is such that  $\sigma_n^m$  vanishes for some  $(m, n)$ , the orographic forcing leads to a solution that initially grows linearly with time, until equilibrated by the damping. In the absence of damping, this mode grows without limit. However, as the amplitude increases, non-linear interactions transfer energy to other modes, and it

is possible to have a bounded response to constant orographic forcing. This is the case we study below.

We now seek a solution of (19) in the form of a resonant triad (12), with  $\sigma_\alpha + \sigma_\beta = \sigma_\gamma$ . Assuming that the solution is of small amplitude  $\epsilon$ , we expand the streamfunction as

$$\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + \dots$$

With the assumption that the linear forcing term does not enter at  $O(\epsilon)$ , we find that the non-linear term involving  $J(\psi, fh_0/H)$  does not enter at  $O(\epsilon^2)$ , justifying its omission from (19). We also assume that the damping coefficient  $\nu$  is  $O(\epsilon)$ . We perform a multiple-timescale perturbation analysis, similar to that in (section 3.26 Pedlosky, 1987) but including the forcing and damping. We assume that the orography (actually,  $fh_0$ ) has the same spatial structure  $Y_\gamma(\lambda, \phi)$  as the  $\gamma$ -term in (12), and that the mean flow is such as to render this component stationary:

$$\bar{\omega} = \frac{(2\Omega + \bar{\omega})m_\gamma}{n_\gamma(n_\gamma + 1)} \quad \text{or} \quad \bar{\omega} = \frac{2\Omega m_\gamma \kappa_\gamma}{1 - m_\gamma \kappa_\gamma}.$$

Thus, the  $\gamma$ -term resonates with the orography. At order  $\epsilon$ , the equations are linear and unforced; so, the three components evolve independently. At order  $\epsilon^2$ , the forcing, damping and non-linearity enter, and the equations at this level of approximation are

$$\begin{aligned} \dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma) K \zeta_\beta^* \zeta_\gamma^* - \nu \zeta_\alpha, \\ \dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha) K \zeta_\gamma^* \zeta_\alpha^* - \nu \zeta_\beta, \\ \dot{\zeta}_\gamma &= +(\kappa_\alpha - \kappa_\beta) K \zeta_\alpha^* \zeta_\beta^* - \nu \zeta_\gamma + F/\sqrt{k_\alpha k_\beta}, \end{aligned}$$

where the coefficient  $F$  is a constant proportional to the magnitude of the orographic forcing. Introducing a transformation as in Section 3, we arrive at the ‘forced-damped three-wave equations’:

$$\begin{aligned} i\dot{\eta}_\alpha &= \eta_\beta^* \eta_\gamma - i\nu \eta_\alpha, \\ i\dot{\eta}_\beta &= \eta_\gamma \eta_\alpha^* - i\nu \eta_\beta, \\ i\dot{\eta}_\gamma &= \eta_\alpha \eta_\beta - i\nu \eta_\gamma + iF. \end{aligned} \quad (21)$$

Defining the quantities  $N = N_1 + N_2$  and  $J$  as in (17) and  $H = \Re\{\eta_\alpha \eta_\beta \eta_\gamma^*\}$  as before, we find that they are no longer conserved quantities but obey the evolution equations

$$\begin{aligned} \dot{J} &= -2\nu J, \\ \dot{N} &= -2\nu N + 2\Re\{F^* \eta_\gamma\}, \\ \dot{H} &= -3\nu H + 2\Re\{F^* \eta_\alpha \eta_\beta\}. \end{aligned} \quad (22)$$

Note that the energy quantity  $N$  may increase or decrease in response to the forcing  $F$ , depending on the phase relationship between  $F$  and  $\eta_\gamma$ .

#### 4.1. Numerical example

We integrated the BVE (2) with orographic forcing of a single spectral component, RH(3,9). The mean flow  $\bar{\omega}$  is set so that this mode is stationary. Mode RH(3,9) forms a resonant triad with

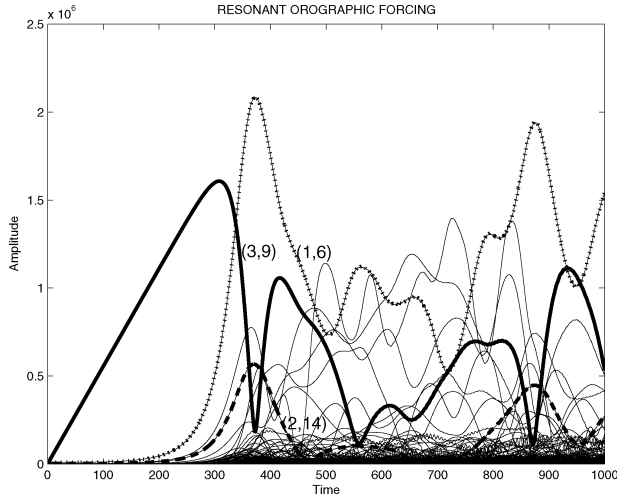


Fig. 2. Component response to orographic forcing of mode RH(3,9). Modes RH(1,6) and RH(2,14), which form a resonant triad with RH(3,9) are also indicated (heavy lines).

RH(1,6) and RH(2,14). Initially, all modes have very small amplitudes, representing background noise. In Fig. 2, we show the component amplitudes for weak orographic forcing (mountain height just 1 m). Initially, the directly forced mode, RH(3,9), grows linearly with time. When it reaches a substantial amplitude, non-linear interactions lead to growth of other modes. At 380 d, mode RH(1,6) dominates, with substantial energy also in mode RH(2,14), and the primary mode has collapsed. However, it grows rapidly again and oscillates in amplitude thereafter. Despite the absence of damping, the response to a constant forcing is bounded; extended integrations confirm this. We will see that this behaviour can be understood in the context of a simple mechanical model, the dynamics of which are described by (21).

### 5. Forced-damped swinging spring

Lynch (2003) showed that free Rossby wave triads in the atmosphere could be modelled by an elastic pendulum or swinging spring. At a certain level of approximation, the equations of the two systems are mathematically isomorphic. Thus, behaviour such as the precession of successive horizontal excursions of the spring indicated similar behaviour in the atmosphere. We extend this correspondence here to include forcing and damping.

We consider a swinging spring whose point of suspension oscillates vertically with the period of the elastic oscillations (Fig. 3). We assume an unstretched spring length  $\ell_0$ , length  $\ell$  at equilibrium, spring constant  $k$  and unit mass  $m = 1$ . The Lagrangian, approximated to cubic order in the amplitudes, is

$$\mathcal{L} = \frac{1}{2}[x^2 + y^2 + (\dot{z}^2 + 2z\dot{\zeta} + \dot{\zeta}^2)] - \frac{1}{2}[\omega_R^2(x^2 + y^2) + \omega_Z^2 z^2] - \frac{1}{2}\lambda(x^2 + y^2)z. \quad (23)$$

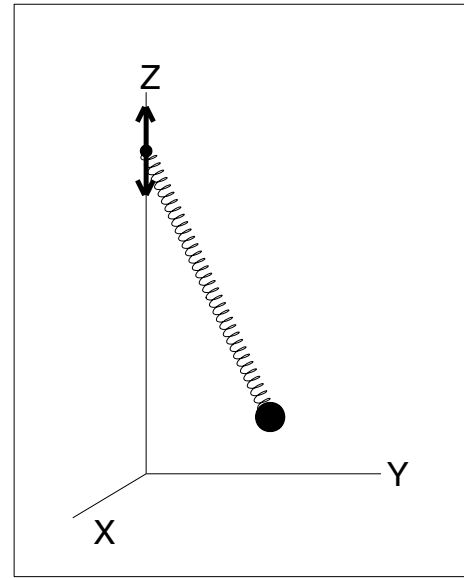


Fig. 3. The swinging spring: the point of suspension is forced periodically in a vertical direction.

where  $x, y$  and  $z$  are Cartesian coordinates centred at the point of equilibrium;  $\zeta(t) = \Re\{\zeta_0 \exp(i\omega_Z t)\}$  is the displacement of the point of suspension;  $\omega_R = (g/\ell)^{1/2}$  is the frequency of linear pendular motion;  $\omega_Z = (k/m)^{1/2}$  is the frequency of its elastic oscillations and  $\lambda = \ell_0 \omega_Z^2 / \ell^2$ . If damping is introduced through a Rayleigh dissipation function

$$\mathcal{F} = \frac{1}{2}v(x^2 + y^2 + \dot{z}^2),$$

(e.g. José and Saletan, 1998), Lagrange's equations are

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} + \frac{\partial \mathcal{F}}{\partial \dot{\mathbf{q}}} = 0, \quad (24)$$

where  $\mathbf{q} = (x, y, z)$ . The motion of the suspension point introduces an inhomogeneous term  $-\ddot{\zeta}$  into the  $z$ -equation. We employ the average Lagrangian technique to obtain an approximate solution. Details may be found in Holm and Lynch (2002) and Lynch and Houghton (2004). We confine attention to the resonant case  $\omega_Z = 2\omega_R$ . The solution is assumed to be of the form

$$\begin{aligned} x &= \Re\{a(t) \exp(i\omega_R t)\}, \\ y &= \Re\{b(t) \exp(i\omega_R t)\}, \\ z &= \Re\{c(t) \exp(i\omega_Z t)\}. \end{aligned}$$

The coefficients  $a, b$  and  $c$  are assumed to vary on a timescale much longer than the timescale of the oscillations  $\tau = 2\pi/\omega_R$ . If the Lagrangian and the dissipation function are averaged over time  $\tau$ , the Lagrange equations for the modulation amplitudes are

$$i\dot{a} = -\mu a^* c - i\nu a, \quad (25)$$

$$i\dot{b} = -\mu b^* c - i\nu b, \quad (26)$$

$$i\dot{c} = -\frac{1}{4}\mu(a^2 + b^2) - i\nu c + \frac{1}{2}\omega_Z \zeta_0. \quad (27)$$

where  $\mu = \lambda/4\omega_R$ . Defining new variables by

$$\alpha = \frac{1}{2}\mu(a + ib), \quad \beta = \frac{1}{2}\mu(a - ib), \quad \gamma = \mu c,$$

the equations for the envelope dynamics become

$$\begin{aligned} i\dot{\alpha} &= \beta^* \gamma - i\nu \alpha, \\ i\dot{\beta} &= \gamma \alpha^* - i\nu \beta, \\ i\dot{\gamma} &= \alpha \beta - i\nu \gamma + iF, \end{aligned} \quad (28)$$

where  $F = -\frac{1}{2}i\mu\omega_Z \zeta_0$  represents the external forcing. This system is mathematically isomorphic to the system (21) for a forced-damped resonant Rossby triad.

One consequence of (22) is that, with damping but no forcing, the quantity

$$\Delta\varphi = \tan^{-1} \left[ \sqrt{\frac{N}{8} \frac{J}{H}} \right],$$

which represents the ‘precession angle’, is constant (Lynch and Houghton, 2004). This is surprising, considering that the angular momentum  $J$  decays exponentially to zero.

### 5.1. Numerical example

We integrated the system (28) over 30 time units, with unit forcing  $F = 1$  and no damping, from the following initial conditions:

$$\alpha_0 = (+0.0005, 0.0000),$$

$$\beta_0 = (-0.0005, 0.0005),$$

$$\gamma_0 = (+0.0000, 0.0000).$$

The amplitudes of the components (real and imaginary parts) are shown in Fig. 4. Only the components  $\Im\{\alpha\}$ ,  $\Re\{\beta\}$  and  $\Re\{\gamma\}$  have substantial amplitudes. These are shown in the figure. We see that, initially, the forced component,  $\gamma$ , grows linearly. As it gains energy, there is a sudden surge of energy into the other two components,  $\alpha$  and  $\beta$ . This is the ‘pulsation’ phenomenon (Lynch and Houghton, 2004). However, these components quickly reach their peak, and the energy flows back to the primary mode, ‘but now its phase is reversed’. As a result, the effect of the forcing is to remove energy from the system, so the forced mode decreases linearly back to its initial zero value. This cycle of alternate forcing and pulsing is then repeated indefinitely. We see that the qualitative features of the response to orographic forcing found in Section 4 (Fig. 2), at least for the initial growth and pulsation, can be explained by the interplay between the forced response and non-linear triad interaction. In particular, we see how a constant resonant forcing can result in a bounded response, even in the absence of damping.

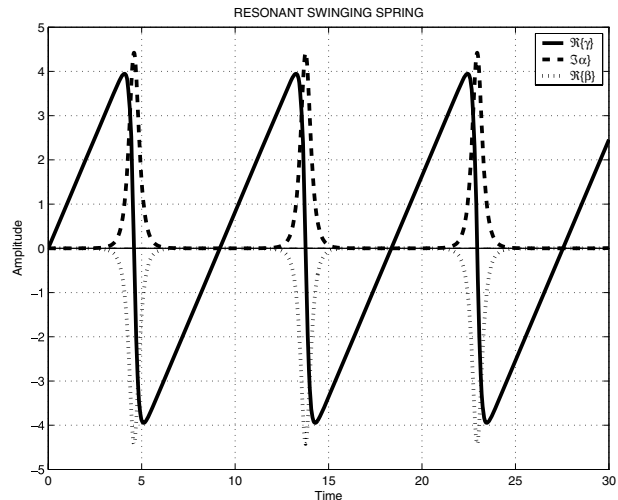


Fig. 4. Amplitudes of  $\alpha$ ,  $\beta$  and  $\gamma$ . The components  $\Im\{\alpha\}$ ,  $\Re\{\beta\}$  and  $\Re\{\gamma\}$  are shown bold. The remaining amplitudes remain small. The resonant forcing is constant, and there is no damping.

## 6. Concluding remarks

The dynamics of non-linear interactions between wave components of a barotropic fluid on the sphere have been reviewed. We have presented new evidence that the resonant triad mechanism can explain the instability of large-scale RH waves. We have shown that the equations governing triad interactions are mathematically equivalent to the envelop equations for a simple mechanical system, the forced-damped swinging spring. This equivalence, not previously discussed in the literature, enables us to analyse triad interactions in the simplest possible context. In particular, the surprising behaviour of a forced triad, in which a constant forcing can lead to a periodic response even in the absence of damping, can be understood in terms of the behaviour of the mechanical system.

Triad interactions are also important in establishing and maintaining the atmospheric energy spectrum. Such interactions can account for quasi-periodic variations of long timescale (Kartashova and L’vov, 2007). An examination of the spectral characteristics of reanalysis data such as ERA40, with a focus on the variation of triad amplitudes, would be of great interest in clarifying the role of this mechanism in atmospheric dynamics.

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