The Sieve of Eratosthenes and a Partition of the Natural Numbers

Peter Lynch, UCD

Abstract. The sieve of Eratosthenes is a method for finding all the prime numbers less than some maximum value M by repeatedly removing multiples of the smallest remaining prime until no composite numbers less than or equal to M remain. The sieve provides a means of partitioning the natural numbers. We examine this partition and derive an expression for the densities of the constituent "Eratosthenes sets".

1 The Sieve of Eratosthenes

The *primorial*, P_K — often denoted $K\#$ — is defined to be the product of the first K primes:

$$P_K = \prod_{k=1}^{K} p_k \,.$$

The sequence of primorials is $\{2, 6, 30, 210, 2310, \dots\}$ and the terms of the sequence grow as K^K. It is convenient to set $M = P_K$. The algorithm of Eratosthenes goes as follows: starting from the set $I_M = \{1, 2, 3, \dots, M\}$,

- eliminate all multiples of 2 greater than 2;
- eliminate all remaining multiples of 3 greater than 3;
- eliminate all remaining multiples of 5 greater than 5;
- ... ...
- eliminate all remaining multiples of p_K greater than p_K.

All that remains is the set of the first m prime numbers, $\{2, 3, 5, \dots, p_m\}$, where p_m is the largest prime not exceeding $M = P_K$.

For all $k \in \mathbb{N}$, let D_k be the set of numbers in $\mathbb{N}$ that are divisible by p_k. For $k = 1, 2, \dots, K$, we define the set $D_{k,M} = D_k \cap I_M$ to be the set of numbers in I_M that are divisible by p_k. Thus, $D_{1,M}$ is the set of even numbers up to M, $D_{2,M}$ the multiples of 3 up to M, and so on.

The k-th "Eratosthenes set", E_k, is the set containing p_k together with all the numbers removed at stage k. Thus, E_1 is the set of all multiples of 2, that is, all the even numbers; E_2 is the set of odd multiples of 3; E_3 is the set of multiples of 5 not divisible by 2 or 3; E_4 is the set of multiples of 7 not divisible by 2, 3 or 5; and so on.

Table 1: Arrangement of the natural numbers as multiples of the prime numbers in sequence. Row 0, column 0 contains $E_0 = \{1\}$. The k-th row contains the "Eratosthenes set" E_k.

1																
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
	3	9	15	21	27	33	39	45	51	57	63	69	81	87	93	...
	5	25	35	55	65	85	95	115	125	145	155	175	185	205	215	...
	7	49	77	91	119	133	161	203	217	259	287	301	329	343	371	...
	11	121	143	187	209	253	319	341	407	451	473	517	583	649	671	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

The Eratosthenes set E_k may be defined symbolically:

$$E_k = \{n \in \mathbb{N} : (p_k \mid n) \wedge (p_\ell \nmid n \text{ for } \ell < k)\} \,.$$

Some initial values of E_k are shown in Table 1.

We denote by $E_{k,M}$ the set $E_k \cap I_M$. It is the set containing all multiples of p_k up to M that are *not multiples of any smaller prime*. We see immediately that $E_{1,M} = D_{1,M}$, that $E_{2,M} = D_{2,M} \setminus D_{1,M} = D_{2,M} \cap D^{\mathsf{c}}_{1,M}$ and, more generally, that

$$E_{k,M} = D_{k,M} \setminus (D_{1,M} \cup D_{2,M} \cup \cdots \cup D_{k-1,M}) = D_{k,M} \cap (D_{1,M} \cup D_{2,M} \cup \cdots \cup D_{k-1,M})^{\mathsf{c}} \,.$$

Using De Morgan's law, we may write

$$E_{k,M} = D_{k,M} \cap (D^{\mathsf{c}}_{1,M} \cap D^{\mathsf{c}}_{2,M} \cap \cdots \cap D^{\mathsf{c}}_{k-1,M}) \,. \tag{1}$$

Since all primes p_k for $k \le K$ divide M, the sizes of the D-sets are known: $|D_{k,M}| = M/p_k$ and so $|D^{\mathsf{c}}_{j,M}| = M - M/p_j = M(1 - 1/p_j)$.

The Inclusion-Exclusion Principle

The inclusion-exclusion principle provides a valuable means of calculating the sizes of unions of sets (Bajnok, 2013). We denote the cardinality of a finite set A by $|A|$. The size of the union of two finite sets is

$$|A \cup B| = |A| + |B| - |A \cap B| \,, \tag{2}$$

where the intersection term prevents double counting. For the union of three sets,

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C| \,. \tag{3}$$

This idea can be generalised using the inclusion-exclusion principle to give the magnitude of the union of n finite sets:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\leqslant i<j\leqslant n} |A_i \cap A_j| + \sum_{1\leqslant i<j<k\leqslant n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \,. \quad (4)$$

Thus, the size of the union of sets is expressed as a combination of sizes of intersections.

Density

We define the density of a set $A \subseteq I_M$ (relative to M) to be $\rho(A) = |A|/M$. Then $\rho(D_{k,M}) = 1/p_k$ and $\rho(D^{\complement}_{j,M}) = (1 - 1/p_j)$. Clearly, density is additive for disjoint sets. Thus,

$$\rho(D_{k,M}) = \rho(D_{k,M} \cap (D_{\ell,M} \uplus D^{\complement}_{\ell,M})) = \rho(D_{k,M} \cap D_{\ell,M}) + \rho(D_{k,M} \cap D^{\complement}_{\ell,M})$$

and, if p_k and p_ℓ are coprime, $\rho(D_{k,M} \cap D_{\ell,M}) = 1/p_k p_\ell$ and $\rho(D_{k,M} \cap D^{\complement}_{\ell,M}) = (p_\ell - 1)/p_k p_\ell$, so that

$$\rho(D_{k,M} \cap D_{\ell,M}) = \rho(D_{k,M})\rho(D_{\ell,M}) \qquad \text{and} \qquad \rho(D_{k,M} \cap D^{\complement}_{\ell,M}) = \rho(D_{k,M})\rho(D^{\complement}_{\ell,M}) \quad (5)$$

Moreover,

$$\begin{aligned}
\rho(D^{\complement}_{k,M} \cap D^{\complement}_{\ell,M}) &= \rho((D_{k,M} \cup D_{\ell,M})^{\complement}) = 1 - \rho(D_{k,M} \cup D_{\ell,M}) \\
&= 1 - [\rho(D_{k,M}) + \rho(D_{\ell,M}) - \rho(D_{k,M} \cap D_{\ell,M})] \\
&= 1 - \left(\frac{1}{p_k} + \frac{1}{p_\ell}\right) + \frac{1}{p_k p_\ell} \;=\; \frac{p_k - 1}{p_k}\frac{p_\ell - 1}{p_\ell} \\
&= \rho(D^{\complement}_{k,M})\rho(D^{\complement}_{\ell,M}) \,. \qquad (6)
\end{aligned}$$

We note that division of equations (2)–(4) by M converts the cardinalities to densities. Thus, for example, (2) becomes

$$\rho(A \cup B) = \rho(A) + \rho(B) - \rho(A \cap B) \,.$$

By means of the inclusion-exclusion principle, we easily extend the product relationships (5) and (6) to show that the density of the set $E_{k,M}$ in (1) is the product of the densities of the component sets on the right side:

$$\rho(E_{k,M}) = \rho(D_{k,M})\rho(D^{\complement}_{1,M})\rho(D^{\complement}_{2,M}) \ldots \rho(D^{\complement}_{k-1,M}) \,. \quad (7)$$

Using explicit expressions for the terms on the right, the density of the set $E_{k,N}$ is

$$\rho(E_{k,M}) = \frac{1}{p_k}\frac{(p_1 - 1)}{p_1}\frac{(p_2 - 1)}{p_2} \;\ldots\; \frac{(p_{k-1} - 1)}{p_{k-1}} = \frac{1}{P_k}\prod_{j=1}^{k-1}(p_j - 1) \,, \quad (8)$$

where $P_k = p_1 p_2 \ldots p_k$. We observe that the numbers $\rho_{k,M} := \rho(E_{k,M})$ are generated by a recurrence relation

$$\rho_{k+1,M} = \left(\frac{p_k - 1}{p_{k+1}}\right)\rho_{k,M} \,, \quad (9)$$

with initial value $\rho_{1,M} = \frac{1}{2}$. This enables us to compute the sequence $\{\rho_{k,M}\}$. The first eight density values are given in Table 2.

Table 2: Density of the Eratosthenes sets E_k for $k \leq 8$.

k	1	2	3	4	5	6	7	8	...
p_k	2	3	5	7	11	13	17	19	...
P_k	2	6	30	210	2310	30,030	510,510	9,699,690	...
ρ_k	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{15}$	$\frac{4}{105}$	$\frac{8}{385}$	$\frac{16}{1001}$	$\frac{192}{17{,}017}$	$\frac{3072}{323{,}323}$	...

2 Passage from I_N to $\mathbb{N}$

For arbitrary $N \in \mathbb{N}$, let $I_N = \{1, 2, \ldots, N\}$ and let $D_{k,N}$ denote $D_k \cap \{1, 2, ..., N\}$, the set of all multiples of p_k not exceeding N. Then $|D_{k,N}| = [\![N/p_k]\!]$ and $\rho(D_{k,N}) = [\![N/p_k]\!]/N$. Since, for any real x, we have $x - 1 < [\![x]\!] \leq x$, it follows that $(N/p_k) - 1 < [\![N/p_k]\!] \leq N/p_k$, and thus $(1/p_k) - (1/N) < \rho(D_{k,N}) \leq 1/p_k$. Therefore, the limit of $\rho(D_{k,N})$ exists, so that

$$\rho(D_k) := \lim_{N\to\infty} \rho(D_{k,N}) = \frac{1}{p_k} \qquad \text{and also} \qquad \rho(D_k^c) = 1 - \rho(D_k) = 1 - \frac{1}{p_k}\,.$$

In this way, we can pass from I_N to $\mathbb{N}$, obtaining the densities of all the Eratosthenes sets in $\mathbb{N}$. In particular, the values of ρ_k in Table 2 are also the densities of the first eight (infinite) Eratosthenes sets relative to the natural numbers. Equations (7) and (8) remain valid in the limit $M \to \infty$, as does the recurrence relation for $\rho_k := \lim_{M\to\infty} \rho_{k,M}$. Thus,

$$\rho_{k+1} = \left(\frac{p_k - 1}{p_{k+1}}\right) \rho_k\,. \tag{10}$$

Convergence

We now show that the series $\sum \rho_n$ converges. The simple ratio test is inadequate, as $\lim \rho_{n+1}/\rho_n = 1$, telling us nothing. A more subtle and discriminating test is required.

In his classical text, *Introduction to the Theory of Infinite Series*, Bromwich (1926, §12.1) describes an extension of the ratio test, originating with Ernst Kummer and refined by Ulisse Dini. To test a series $\sum a_n$ for convergence, we select a sequence $\{d_n\}$ such that the series $\sum d_n^{-1}$ is divergent. The criterion is as follows.

$$\text{Let}\quad t_n = d_n \left[\frac{a_n}{a_{n+1}}\right] - d_{n+1}\,. \quad \text{Then} \begin{cases} \text{if } \lim t_n > 0, & \sum a_n \text{ converges;} \\ \text{if } \lim t_n < 0, & \sum a_n \text{ diverges.} \end{cases} \tag{11}$$

If $\lim t_n = 0$, there is no conclusion and another choice of $\{d_n\}$ is required. The selection of the sequence $\{d_n\}$ depends on the series being tested.

This test can be used to show that the series $\sum \rho_n$ converges. From (9), the ratio of successive terms is $\rho_n/\rho_{n+1} = p_{n+1}/(p_n - 1)$. In his paper on infinite series, Euler (1737) showed that the series $\sum 1/p_n$ diverges. Choosing $d_n = p_n$, we have

$$t_n = p_n \left[\frac{p_{n+1}}{p_n - 1}\right] - p_{n+1} = \left[\frac{p_n p_{n+1} - (p_n - 1)p_{n+1}}{p_n - 1}\right] = \left[\frac{p_{n+1}}{p_n - 1}\right] > 1\,,$$

which fulfils the convergence criterion $\lim t_n > 0$, so the series converges. We will show below that the sum to infinity is 1, but the convergence rate is quite slow. Writing $\sigma_N = \sum_{k=1}^{N} \rho_k$ we have $\sigma_{10} = 0.842$, $\sigma_{1,000} = 0.938$, and $\sigma_{100,000} = 0.960$.

Partitioning the Natural Numbers

Defining $E_0 = \{1\}$, we obtain a partition of the natural numbers $\mathbb{N}$:

$$\mathbb{N} = \biguplus_{n=0}^{\infty} E_n\,, \tag{12}$$

where the sets E_n may be listed explicitly:

$$\begin{array}{rcl}
E_0 &=& \langle 1 \rangle \\
E_1 &=& \langle 2, 4, 6, 8, 10, 12, \dots \rangle \\
E_2 &=& \langle 3, 9, 15, 21, 27, \dots \rangle \\
E_3 &=& \langle 5, 25, 35, 55, 65, 85, \dots \rangle \\
 & & \dots\dots\dots\dots\dots\dots\dots\dots \\
E_K &=& \langle p_K, p_K^2, p_K p_{K+1}, \dots \rangle \\
 & & \dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

The disjoint union in (12) contains all the positive integers, each occurring just once, providing a partition of $\mathbb{N}$.

Totient Function Expression for ρ_k

Euler's totient function $\varphi(n)$ counts the natural numbers up to n that are coprime to n. In other words, $\varphi(n)$ is the number of integers k in the range $1 \le k \le n$ for which the greatest common divisor $\gcd(k, n)$ is equal to 1. Clearly, for prime numbers, $\varphi(p) = p - 1$. Gauss first proved a result presented as Theorem 63 in Hardy and Wright (1960):

$$\sum_{d|N} \varphi(d) = N\,.$$

This states that the sum of the numbers $\varphi(d)$, extended over all the divisors d of any number N, is equal to N itself.

The number of values x coprime to $\prod_{j=1}^{k} m_j$ is, by definition, given by $\varphi(m_1 m_2 \dots m_k)$. But Euler's function is multiplicative for products of mutually coprime numbers $\{m_1, m_2, \dots, m_k\}$:

$$\varphi(m_1 m_2 \dots m_k) = \prod_{i=1}^{k} \varphi(m_j) \,.$$

Thus, for $M = P_K$, we have

$$\varphi(P_K) = \varphi\left(\prod_{j=1}^{K} p_j\right) = \prod_{j=1}^{K} \varphi(p_j) = \prod_{j=1}^{K} (p_j - 1) = P_K \prod_{j=1}^{K} \left(1 - \frac{1}{p_j}\right) \,.$$

Now, using (8), we can write the density in terms of the totient function:

$$\rho_k = \frac{1}{P_k} \prod_{j=1}^{k-1} (p_j - 1) = \frac{1}{P_k} \prod_{j=1}^{k-1} \varphi(p_j) = \frac{\varphi(P_{k-1})}{P_k} \,. \qquad (13)$$

For example, for $K = 4$ we have $p_K = 7$, $P_K = 210$ and $P_{K-1} = 30$ and, counting explicitly, $\varphi(30) = |\{1, 7, 11, 13, 17, 19, 23, 29\}| = 8$. Thus,

$$\rho_4 = \frac{\varphi(P_3)}{P_4} = \frac{8}{210} = \frac{4}{105} \,,$$

as already shown in Table 2.

An Interesting Result

Defining the cumulative density $\sigma_k = \sum_{j=1}^{k} \rho_k$ and noting that, as the sets E_k are mutually disjoint, σ_k must approach 1, we obtain the relationship

$$\sum_{k=1}^{\infty} \left[\frac{\varphi(P_{k-1})}{P_k}\right] = 1 \,.$$

This result must be well known, although it has not been found in a cursory search of the literature.

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Sources

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