Boundary Filters using Half-sinc Functions

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1 Introduction

In this note we establish the mathematical framework for the filtering problem on a bounded interval, and provide a motivation for the use of the 'half-sinc' function as a boundary filter. A boundary filter is one which yields output valid at the start or end of the interval on which the input is available. Such filters have obvious relevance to the problem of initialization for NWP. We also discuss some fundamental difficulties encountered when filtering time-limited functions.

Let us assume that the function $f(t)$ is specified on an interval $t_1 \le t \le t_2$. We seek a function $\bar{f}(t)$ which is as close as possible to the given function, subject to its being of low frequency. We shall show that, with an obvious definition of 'low frequency' the method results in an approximation of the function by itself. This paradoxical result arises from the difficulty of defining 'low frequency' on a bounded domain (Slepian, 1976). We circumvent the difficulty by modifying the approximation method, and obtain applicable results.

The first idea that might spring to mind is straightforward Fourier filtering: the function $f(t)$ is analysed into its spectral components, the high frequency components are removed and the remainder re-synthesized to produce a low frequency approximation to the function. Thus, if we express $f(t)$ as a Fourier series

$$f(t) = \sum_{n=-\infty}^{+\infty} \varphi_n \exp\left[2\pi i n \left(\frac{t-t_1}{t_2-t_1}\right)\right] ,$$

the high frequencies are easily removed by restricting the series to values of $|n|$ less than some value n_0 determined by the desired cutoff frequency. There are two difficulties with this technique. First, if the interval (t_1, t_2) on which $f(t)$ is given is shorter than the cutoff period, the longest-period non-constant component appearing in the Fourier analysis is above the cutoff, so that all that remains is the constant component, a poor approximation to a slowly varying function. Second, expansion into a Fourier series involves an assumption that the function $f(t)$ is periodic, with a period $\tau = t_2 - t_1$; in general, this is not so, although we are told nothing about $f(t)$ outside the range (t_1, t_2). Moreover, if $f(t_1) \ne f(t_2)$, the assumption of periodicity implies a discontinuity, with the consequence that Gibbs oscillations are produced in the Fourier representation of $f(t)$. Convergence of the Fourier series is nonuniform near points of discontinuity, and the accuracy of approximation is generally poor there.

The filtering method derived below may be simply described: the function $\bar{f}(t)$ is expressed as an integral over all sinusoidal components with frequencies less than the cutoff. The minimization of the distance between $\bar{f}(t)$ and $f(t)$ (defined precisely below) leads to an equation for the amplitude of the components: the amplitude function must satisfy a Fredholm integral equation of the first kind. This equation may be solved by standard numerical techniques but, because of the nature of the kernel, the resulting algebraic system is highly ill-conditioned. The equation can also be solved by use of the eigenfunctions of the corresponding homogeneous Fredholm equation of the second kind. It transpires that these eigenfunctions are complete in $L_2(-\frac{1}{2}T, +\frac{1}{2}T)$ so that the filtering procedure makes no change to the input. Considering the discrete case, we formulate an explicit expression for the filter matrix and devise a means of modifying it, using the special characteristics of the eigenvalues, so that it has the desired filtering effect.

2 Mathematical Framework

In this section we examine the mathematical framework for the filtering problem on a bounded interval, and demonstrate why the 'half-sinc' function is a natural choice for use as a boundary filter. Readers content to accept the appropriateness of using the half-sinc function may prefer to skip immediately to §3 (but they will miss the austere beauty of the Fredholm equation, Mercer's theorem, the Dirichlet kernel and other mathematical delicacies).

2.1 Derivation of the Integral Equation

Let a function $f(t)$ be given on a finite interval $t_1 \le t \le t_2$. The derivation is simplified if we assume, without loss of generality, that the interval is symmetric about the origin: $t_1 = -T/2$ and $t_2 = +T/2$, where $T = t_2 - t_1$ is the duration or *span*. The problem is to find a low frequency approximation to $f(t)$. We assume that a cutoff frequency $\omega_c = 2\pi\nu_c$ is specified, and require the approximation to vary more slowly than this.

Let us consider a function $\bar{f}(t)$ which is a super-position of low frequency components:

$$\bar{f}(t) = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} \varphi(\omega')\exp(i\omega' t)\, d\omega' . \tag{1}$$

The L_2-norm of $f(t)$ on the given interval is defined by

$$\|f\|^2 = \int_{-T/2}^{+T/2} |f(t)|^2\, dt .$$

We may express the error E in the approximation of f by $\bar{f}$ as

$$E \equiv \|f - \bar{f}\|^2 = \int_{-T/2}^{+T/2} \left|f(t) - \bar{f}(t)\right|^2 dt .$$

The problem now is to choose $\varphi(\omega)$ in such a way as to minimize E. It is a problem in variational calculus. Let φ be replaced by $\varphi + \delta\varphi$; then $E \to E + \delta E$ where

$$\delta E = -2\int_{-T/2}^{+T/2} \left(f(t) - \bar{f}(t)\right)\left(\frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} \delta\varphi(\omega')\exp(i\omega' t)\, d\omega'\right) dt .$$

Let us suppose that $\delta\varphi(\omega') = \delta(\omega + \omega')$, a delta-function centered at a fixed point $-\omega$ such that $\omega \in (-\omega_c, +\omega_c)$. The variation of E then reduces to

$$\delta E = -\frac{1}{\pi}\int_{-T/2}^{+T/2} \left(f(t) - \bar{f}(t)\right)\exp(-i\omega t)\, dt .$$

For an extremum of E the variation vanishes, $\delta E = 0$ so that, for $|\omega| \le \omega_c$,

$$F(\omega) \equiv \int_{-T/2}^{+T/2} f(t)\exp(-i\omega t)\, dt = \int_{-T/2}^{+T/2} \bar{f}(t)\exp(-i\omega t)\, dt . \tag{2}$$

Using (1) to expand $\bar{f}$ yields

$$\begin{aligned} F(\omega) &= \int_{-T/2}^{+T/2} \left(\frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} \varphi(\omega')\exp(i\omega' t)\, d\omega'\right)\exp(-i\omega t)\, dt \\ &= \int_{-\omega_c}^{+\omega_c} \left(\frac{1}{2\pi}\int_{-T/2}^{+T/2} \exp\left[(i(\omega' - \omega)t\right] dt\right)\varphi(\omega')\, d\omega' . \end{aligned}$$

If we define the kernel $K(\omega, \omega')$ by

$$K(\omega, \omega') \equiv \frac{1}{2\pi}\int_{-T/2}^{+T/2} \exp\left[i(\omega' - \omega)t\right] dt = \frac{\sin(\omega' - \omega)T/2}{\pi(\omega' - \omega)} , \tag{3}$$

equation (3) may be written as

$$\int_{-\omega_c}^{+\omega_c} K(\omega, \omega')\varphi(\omega')\, d\omega' = F(\omega) . \tag{4}$$

This is a Fredholm integral equation of the first kind (Tricomi, 1985). Given $f(t)$, the forcing term $F(\omega)$ may be evaluated and the equation solved for $\varphi(\omega)$.

The kernel (3) is symmetric and of convolution type:

$$K(\omega, \omega') = K(\omega', \omega) , \qquad K(\omega, \omega') = k(\omega - \omega') . \tag{5}$$

The properties of the Fredholm equations having this kernel have been studied extensively; see, for example, Slepian and Pollak (1961). More general references on integral equations, in addition to Tricomi (1985), include Courant and Hilbert (1953) and Polyanin and Manzhirov (1998).

2.2 Direct Numerical Solution

The integral in (4) may be approximated by a finite sum using one of many methods, such as Gaussian quadrature or the trapezoidal rule. We assume the interval $(-\omega_c, +\omega_c)$ is divided into N sub-intervals, the n-th containing the point ω_n. Then a finite approximation to (4) is given by

$$\sum_{n=1}^{N} K(\omega_m, \omega_n)\varphi(\omega_n)w_n = F(\omega_m)\,, \tag{6}$$

where w_n is the weight for the n-th interval. If we define a matrix and two vectors by

$$\mathbf{K}_{mn} = K(\omega_m, \omega_n)w_n \qquad \mathbf{\Phi}_n = \varphi(\omega_n) \qquad \mathbf{F}_n = F(\omega_n)\,,$$

equation (6) may be written as a standard system of algebraic equations

$$\mathbf{K\Phi} = \mathbf{F} \tag{7}$$

which can be solved for $\varphi(\omega_m)$ provided $\mathbf{K}$ is non-singular. We can use the chosen quadrature method for evaluating the forcing function $F(\omega)$ in (2), approximating the integral in (4) and evaluating the filtered result $\bar{f}(t)$ in (1).

The problem with this direct method is that the matrix $\mathbf{K}$ is ill-conditioned. We define the *Shannon Number*, Sh, as the duration-bandwidth product:

$$\mathrm{Sh} = 2\nu_c T = \frac{\omega_c T}{\pi} = \frac{2T}{\tau_c}$$

For $n \ll \mathrm{Sh}$ the eigenfunctions of $\mathbf{K}$ are close to unity; for $n \gg \mathrm{Sh}$, they decay rapidly with n (an example will be given below). Thus if, in the interests of accurate approximation, we choose n to be large, the condition number of $\mathbf{K}$ is large and the matrix is practically uninvertable.

2.3 Eigenfunction Analysis

Tricomi (*loc. cit.*, p143) writes: "Some mathematicians still have a kind of fear whenever they encounter a Fredholm integral equation of the *first* kind ... Today this fear is no longer justified, especially for the *symmetric case* $K(x, y) = K(y, x)$". So, we shall proceed fearlessly, but cautiously, to investigate (4). Caution is required because things are not quite as simple as they seem. For general $F(t)$, (4) may have no solution at all. If there is a solution, the fact that $F(t)$ can be expressed in the integral form of (4) guarantees, by the Hilbert-Schmidt theorem (Tricomi, p110), that it can also be expanded in the eigenfunctions of the kernel. These are the solutions of the homogeneous equation of the *second* kind:

$$\int_{-\omega_c}^{+\omega_c} K(\omega, \omega')\varphi(\omega')\,d\omega' = \lambda\varphi(\omega)\,. \tag{8}$$

Note that the historical convention, to write the equation $\lambda \int K(\omega, \omega')\varphi(\omega')\,d\omega' = \varphi(\omega)$, is used by many authors. But the convention here is more convenient as, in the discrete case λ corresponds to the matrix eigenvalues.

We shall give some properties of the solutions of (8). A more complete discussion can be found in Slepian and Pollak (1961). The eigenvalues λ_n form an infinite set

$$1 > \lambda_1 > \lambda_2 > \cdots > \lambda_k > \cdots > 0$$

whose only accumulation point is at zero. Because of the symmetry of the kernel, they are real. The eigenfunctions are doubly orthogonal. They may be assumed to be orthonormal on the interval $[-\omega_c, \omega_c]$:

$$\int_{-\omega_c}^{+\omega_c} \varphi_m(\omega)\varphi_n(\omega)\,d\omega = \delta_{mn}\,.$$

They are complete in the space of square-integrable functions on this interval. They are also orthogonal on the infinite interval:

$$\int_{-\infty}^{+\infty} \varphi_m(\omega)\varphi_n(\omega)\,d\omega = \frac{\delta_{mn}}{\lambda_n}\,.$$

The functions $\varphi_n(\omega)$ are bandlimited and span the space of bandlimited functions on the real line. The implication of these two orthogonality relationships is that, for small n ($n \ll \text{Sh}$), $\varphi_n(\omega)$ is large in $[-\omega_c, \omega_c]$ and small outside, whereas for large n ($n \gg \text{Sh}$) the opposite holds.

If the forcing function $F(\omega)$ in (4) is expanded in the eigenfunctions of (8) as

$$F(\omega) = \sum_{n=1}^{\infty} b_n \varphi_n(\omega)$$

and the solution is assumed to have a similar expansion with coefficients a_n it follows, *under certain convergence conditions*, that $a_n = b_n/\lambda_n$, so that

$$\varphi(\omega) = \sum_{n=1}^{\infty} (b_n/\lambda_n)\varphi_n(\omega)\,. \tag{9}$$

Thus, if we can solve (8) for the eigenvalues and eigenfunctions, (9) gives a formal solution of our equation of the *first* kind, (4).

Now for the paradox: the eigenfunctions are complete in $L_2(-\frac{1}{2}T, +\frac{1}{2}T)$. Thus *any* L_2-function can be expanded in these eigenfunctions. But the eigenfunctions themselves are band-limited (Slepian and Pollak, 1961). Consequently, any L_2-function on a bounded interval can be approximated arbitrarily closely by a band-limited function. This is the surprise: our intuitve notion of a band-limited function on a time-limited or bounded domain is defective.

We are led to a strange conclusion: we have tried to filter $f(t)$, but it remains resolutely unchanged! How is this so? The problem is that the notion of bandlimitedness is unequivocally defined only for functions on an infinite domain. Our intuitive notion of bandlimited functions on a bounded interval is misleading: any L_2-function given on a finite interval can be approximated arbitrarily closely on that interval by a function which has a natural extension to the entire real line as a bandlimited function (Slepian and Pollak, 1961). For a general discussion of the paradox, see Slepian's (1976) paper *On Bandwidth.* It is possible to show explicitly that $\bar{f}(t) = f(t)$, but this is best done for the discrete case, which we shall examine after a brief digression.

2.4 The Bilinear Series

Before proceeding to the discrete case, we derive an expansion of the kernel in terms of its eigenfunctions, the so-called bilinear formula. Considering the kernel $K(\omega, \omega')$ for fixed ω as a function of ω', let us assume it can be expanded in the eigenfunctions:

$$K(\omega, \omega') = \sum_{m=1}^{\infty} c_m(\omega)\varphi_m(\omega')\,,$$

where $c_m(\omega)$ are to be found. Multiply by $\varphi_n(\omega')$ and integrate:

$$\int_{-\omega_c}^{+\omega_c} K(\omega, \omega')\varphi_n(\omega')\,d\omega' = \sum_{m=1}^{\infty} c_m(\omega) \int_{-\omega_c}^{+\omega_c} \varphi_m(\omega')\varphi_n(\omega')\,d\omega'\,.$$

But then, from the definition and orthonormality of the eigenfunctions, it follows that

$$c_n(\omega) = \lambda_n \varphi_n(\omega)$$

and this leads immediately to the bilinear series expansion:

$$K(\omega, \omega') = \sum_{n=1}^{\infty} \lambda_n \varphi_n(\omega)\varphi_n(\omega')\,. \tag{10}$$

This is Mercer's Theorem. For our kernel it can be shown that the series converges absolutely and uniformly. We shall see the convenience of abandoning the historical convention, and writing the eigenvalue on the right side of (8), when the discrete analogue of (10) appears below.

2.5 The Discrete Case

We now replace $f(t)$ by a function of a discrete variable, and assume that we have knowledge of it only on an index-limited set, $\{f_0, f_1, \ldots, f_{N-1}\}$. It will prove convenient to assume that N is an odd number and to re-index:

$$\{f_{-M}, f_{-M+1}, \ldots, f_{+M}\}$$

where $2M+1 = N$. We shall minimise the difference $E = \|f - \bar{f}\|^2 = \sum_{m=-M}^{M} |f_m - \bar{f}_m|^2$ where

$$\bar{f}_m = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \varphi(\omega) e^{im\omega} \, d\omega \,. \tag{11}$$

By the usual variational procedure, we find that E is a minimum if

$$\sum_{m=-M}^{M} \bar{f}_m e^{-im\omega} = \sum_{m=-M}^{M} f_m e^{-im\omega} \equiv F(\omega) \,. \tag{12}$$

The astute reader will realize that $\bar{f}_m = f_m$ satisfies (12), but the question is: can $\bar{f}_m = f_m$ be expressed in the form (11)? This is not obvious. Substituting (11) in (12) and manipulating, we find that

$$\int_{-\omega_c}^{+\omega_c} K(\omega, \omega') \varphi(\omega') \, d\omega' = F(\omega) \,, \tag{13}$$

where the kernel is now of the form

$$K(\omega, \omega') \equiv \frac{1}{2\pi} \sum_{m=-M}^{M} e^{-im\omega} e^{im\omega'} = \frac{N}{2\pi} \mathcal{D}_N(\omega - \omega') \tag{14}$$

where $\mathcal{D}_N$ is the Dirichlet kernel

$$\mathcal{D}_N(\omega) \equiv \frac{\sin N\omega/2}{N \sin \omega/2} \,.$$

Although (13) is formally similar to (4), there is a crucial difference: the kernel is now degenerate: it is a finite sum of products of the form $g(\omega)h(\omega')$. As a result, the integral equation of the second kind

$$\int_{-\omega_c}^{+\omega_c} K(\omega, \omega') \varphi(\omega') \, d\omega' = \lambda \varphi(\omega) \ . \tag{15}$$

can be reduced to a classical algebraic eigenvalue problem. We define

$$\xi_m = \int_{-\omega_c}^{+\omega_c} e^{im\omega'} \varphi(\omega') \, d\omega' \,, \quad m = -M, \ldots, +M$$

and substitute in (15), using (14) to obtain

$$\lambda \varphi(\omega) = \frac{1}{2\pi} \sum_{m=-M}^{M} e^{-im\omega} \xi_m \,. \tag{16}$$

Now multiplying by $e^{in\omega}$ and integrating over ω, we arrive at

$$\lambda \xi_n = \sum_{m=-M}^{M} K_{nm} \xi_m \tag{17}$$

where K_{nm} is an element of the symmetric matrix $\mathbf{K}$ defined by

$$[\mathbf{K}]_{mn} = K_{mn} = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{-im\omega} e^{+in\omega} \, d\omega = \frac{\sin(m-n)\omega_c}{\pi(m-n)} \,. \tag{18}$$

Thus, equation (15) is completely equivalent to the standard eigenvalue problem (17).

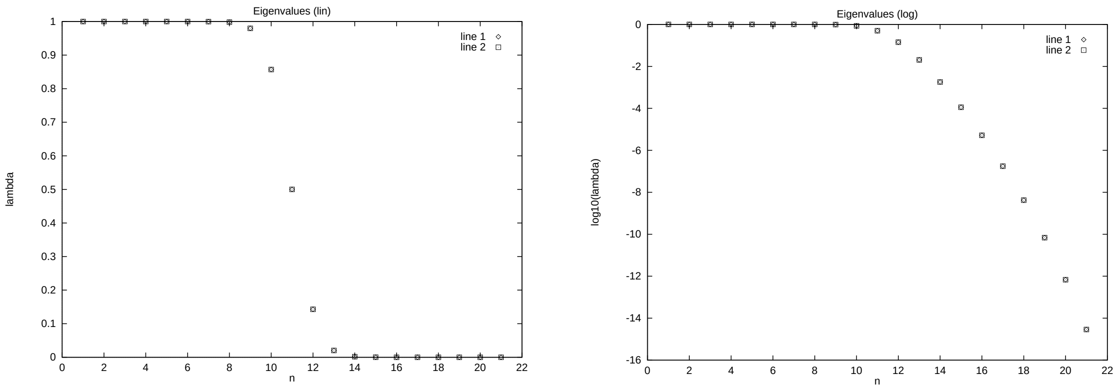

Fig. 1. Eigenvalues λ_k of the Matrix $\mathbf{K}$. The parameter values are $N = 21$, $T = 3$ hours and $\tau_c = 0.6$ hours, so that the Shannon number Sh=10. Left: linear vertical scale. Right: logarithmic vertical scale.

We denote the eigenvalue/eigenvector pairs of (17) by (λ_k, ξ^k). The eigenvector matrix Ξ is defined by $\Xi_{nk} = \xi_n^k$ and the eigenvalue matrix by $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Then the eigenvector equation (17) may be written

$$\mathbf{K}\Xi = \Xi\mathbf{\Lambda}\,. \tag{19}$$

The eigenvalues and eigenvectors corresponding to the Dirichlet kernel have been discussed in detail by Slepian (1978) and by Percival and Walden (1993). The eigenfunctions of (15) are called discrete prolate spheroidal wave functions, and the eigenvectors of (17) or (19) are known as discrete prolate spheroidal sequences (dpss). They have many interesting properties and a wide range of applications in digital signal processing. We shall assume a normalization different to that in the continuous case. For each pair (λ_k, ξ^k), there is a corresponding eigenfunction $\varphi^k(\omega)$ of (15) given by (16). We shall assume that the eigenvectors ξ^k are orthonormal; this means $\Xi^{\mathrm{T}} = \Xi^{-1}$. It also constrains the norms of the eigenfunction $\varphi^k(\omega)$, as the relationship $\lambda_k\|\varphi^k\|^2 = \|\xi^k\|^2$ is easily shown to hold.

To illustrate the interesting properties of the eigenvalues, we set $N = 21$ and choose the parameters $T = 3$ hours and $\tau_c = 0.6$ hours, so that the Shannon number Sh=10. We plot the eigenvalues in Fig. 1. For $k \ll$ Sh, the eigenvalues are close to unity, and for $k \gg$ Sh they are very small. We may note that $\lambda_1 = 0.999999999999997$ and $\lambda_{21} = 2.88657986402541 \times 10^{-15}$, so the condition number of the matrix is 3.5×10^{14}, a warning signal that caution is required in its numerical analysis.

We now use the eigenvectors to solve (13) and then use (11) to find $\bar{f}_m$. Let us expand $\varphi(\omega)$ and $F(\omega)$ as

$$\varphi(\omega) = \sum_{k=1}^{N} a_k\varphi^k(\omega)\,, \qquad F(\omega) = \sum_{k=1}^{N} b_k\varphi^k(\omega) \tag{20}$$

It follows from (13) that $a_k = b_k/\lambda_k$. But now using (12) to express b_k in terms of f_n, and (16) to substitute ξ^k for $\varphi(\omega)$ we arrive, after some algebra, at the result

$$\bar{f}_m = \sum_{n=-M}^{M} \left\{ \sum_{k=1}^{N} \xi_m^k \xi_n^k \right\} f_n\,. \tag{21}$$

This is a matrix product and the matrix in braces is just $\mathbf{M} = \Xi\Xi^{\mathrm{T}}$. However, since the eigenvectors ξ^k are orthonormal we know that $\Xi^{\mathrm{T}}\Xi = \mathbf{I}$. Since Ξ^{T} is a left inverse, it is also a right inverse of Ξ, and $\mathbf{M} = \mathbf{I}$. Thus, $\bar{f}_n = f_n$ and no filtering is achieved.

2.6 The Flight of the Phœnix

Our filtering strategy using (11) is unsuccessful. But all is not lost and we can still salvage something of value from the ashes. We know that the eigenfunctions $\varphi^k(\omega)$ oscillate more rapidly with increasing index: $\varphi_k(\omega)$ has k zeros in $(-\omega_c, \omega_c)$ and, similarly, the eigenvector ξ^k has k zeros. Since rapid oscillations run counter to our intuitive notion of low frequency, it makes sense to eliminate them. We can do this by truncating the expansion (21) in ξ^k. A convenient way to do this is to exploit the properties of the eigenvalues by using them as weighting coefficients. The natural truncation point is given by the Shannon number. We know that λ_k is near unity for $k < \mathrm{Sh}$ and small for $k > \mathrm{Sh}$. So the sum in (21) is replaced by

$$\sum_{k=1}^{[\mathrm{Sh}]} \xi_m^k \xi_n^k \approx \sum_{k=1}^{N} \lambda_k \xi_m^k \xi_n^k \,.$$

But this is just the (m, n)-th element of the matrix $\mathbf{\Xi\Lambda\Xi}^{\mathrm{T}}$ which, by (19), is nothing other than the original matrix:

$$\mathbf{K} = \mathbf{\Xi\Lambda\Xi}^{\mathrm{T}} \tag{22}$$

This expansion is just a discrete analogue of the bilinear series (10). We reach the delightful conclusion that $\mathbf{K}$ can be used directly to act on the input signal f_n to produce a filtered output $\bar{f}_n$. Despite our long mathematical peregrinations in the frequency domain and our sojourn amongst the integral equations, we arrive back in the time domain and find that all that is needed to filter the signal is the matrix $\mathbf{K}$ defined by (18). We do not need to solve any integral equations or calculate any eigenvalues or eigenfunctions explicitly.

3 Properties of the Half-sinc Function

An ideal low-pass filter with cut-off θ_c has frequency response

$$H(\theta) = \begin{cases} 1 & \text{for } |\theta| < \theta_c \\ 0 & \text{for } |\theta| > \theta_c \end{cases}$$

The corresponding impulse response is that of a non-causal FIR filter

$$h_n = \frac{\sin n\theta_c}{n\pi} = \left(\frac{\theta_c}{\pi}\right) \mathrm{sinc}\left(\frac{n\theta_c}{\pi}\right)$$

where $\mathrm{sinc}\,\alpha = \sin(\pi\alpha)/\pi\alpha$ and $n \in Z$. In practice, we must restrict n to some finite range. Moreover, for a causal filter we require $n \geq 0$. Then the coefficients are

$$h_n = \frac{\sin n\theta_c}{n\pi}\,, \qquad n = 0, 1, \ldots, N-1\,. \tag{23}$$

We refer to this sequence as a *half-sinc* sequence. We note that the elements of the matrix $\mathbf{K}$ studied in §2 are $\mathbf{K}_{mn} = h(m-n)$. Boundary filters, that is, those which yield output valid at the extremity of the time-range, must be either causal or anti-causal. They are represented by the first and last rows of $\mathbf{K}$. We examine these cases below.

The frequency response may be written

$$\sum_{n=0}^{N-1} h_n e^{in\theta} = H(\theta) = M(\theta) e^{i\varphi(\theta)}\,. \tag{24}$$

The response to a signal of vanishingly small frequency (the DC component) is $H(0)$. We always normalize so that $H(0) = 1$. Thus, the sum of the filter weights h_n is unity. The *group delay* is defined as $\delta = -d\varphi/d\theta$. Taking the derivative of (24) and recalling that h_n are real, we easily see that $\delta_0 = \delta(0) = \sum n h_n$. For the output of a boundary filter to apply at the start of the time interval it must be 'zero-delay'. That is, we require $\delta_0 = 0$. For the half-sinc sequence, this can be satisfied if we truncate after an exact number of wavelengths:

$$\sum_{n=0}^{N-1} n h_n = \frac{1}{\pi} \sum_{n=0}^{N-1} \sin n\theta_c = 0 \tag{25}$$

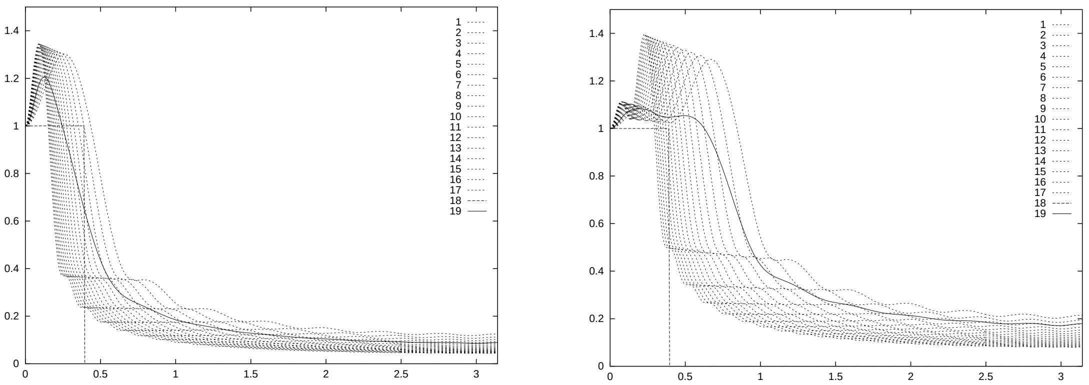

Fig. 2. Dashed curves: Frequency responses $H(\theta)$ for seventeen half-sincs with varying spans. Parameters are $\Delta t = 225\,\text{s}$, $N \in \{17, 19, \ldots, 49\}$, $T_S = (N-1)\Delta t \in \{1\text{h}, 1\frac{1}{8}\text{h}, \ldots, 3\text{h}\}$ and $\tau_c = T_S$. Solid curve: weighted sum of seventeen half-sincs, to reduce intermediate frequency boost. Left panel: $K = 1$. Right panel: $K = 2$.

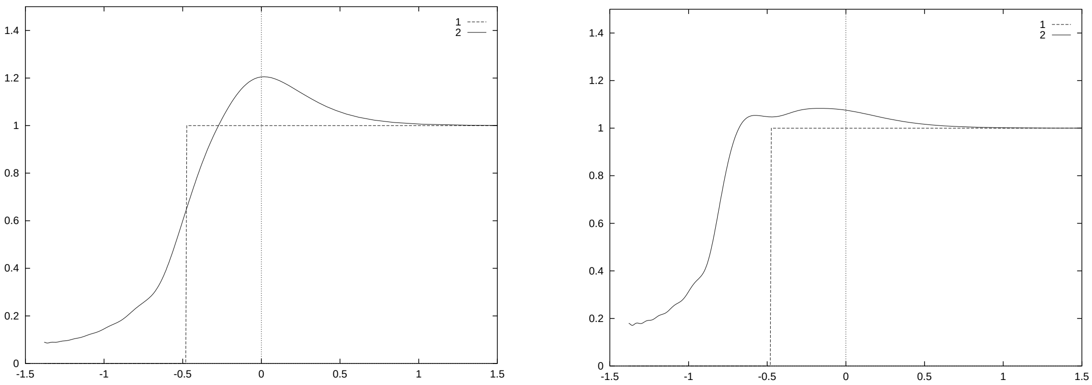

Fig. 3. Frequency response of weighted sums, as in Fig. 2, plotted against a horizontal axis $\mu = \log_{10}(\tau/\tau_c)$ where $\tau_c = 3\,\text{h}$ and $\tau = 2\pi\Delta t/\theta$ is the period. The square response is for an ideal one-hour cut-off filter. Left panel: $K = 1$. Right panel: $K = 2$.

provided $(N-1)\theta_c = 2\pi K$ for some integer K.

We now examine the frequency response of some boundary filters based on zero-delay half-sinc sequences. In Fig. 2, the responses $H(\theta)$ are shown for a selection of (seventeen) half-sincs with $K = 1$ and varying spans. The parameters are $\Delta t = 225\,\text{s} = \frac{1}{16}$ hours, filter order $N \in \{17, 19, \ldots, 49\}$, span $T_S = (N-1)\Delta t \in \{1\text{h}, 1\frac{1}{8}\text{h}, \ldots, 3\text{h}\}$ and cut-off $\tau_c = T_S$. All are normalized so that $H(0) = 1$. The square wave shows an ideal response with cut-off period of one hour. All responses are low-pass, as high frequencies are strongly attenuated. However, all half-sincs have an over-shoot or *boost* for some intermediate frequencies. To reduce this boost, we construct a weighted combination, the response of which is shown by the solid curve. The corresponding response curves for $K = 2$ are shown in the right-hand panel of Fig. 2. The boost is further reduced in this case, but at the expense of widening the pass-band.

In Fig. 3, the responses are plotted with horizontal axis $\mu = \log_{10}(\tau/\tau_c)$ where $\tau_c = 3\,\text{h}$ and $\tau = 2\pi\Delta t/\theta$ is the period. Again, the square response is for an ideal one-hour cut-off filter. The left panel is for $K = 1$ and the right one for $K = 2$; the choice of K represents a compromise between reducing boost and optimising high frequency cut-off. So far, we have not succeeded in obtaining a boundary filter which is free from some boost at intermediate frequencies. It is planned to test the half-sinc filter in an incremental initialization scheme. Meanwhile, the search for a better filter goes on!

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