On the Relationship between Barometric Variations and the Continuity Equation

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We denote by p_0 and p_h the pressure at two points in a vertical line, at the surface and at height h. We assume that the pressure difference, whether the air is in motion or stationary, is given by

$$\text{(A)} \qquad \mathfrak{p} = p_0 - p_h = \int_0^h g\mu \, dz \,,$$

where μ is density at height z. Because of the short distance, we assume that the acceleration due to gravity g is constant (Assumption B) and also that the earth's surface is an x–y plane (Assumption C).

The continuity equation is written

$$\frac{\partial \mu}{\partial t} + \frac{\partial(\mu u)}{\partial x} + \frac{\partial(\mu v)}{\partial y} + \frac{\partial(\mu w)}{\partial z} = 0$$

(where u, v are the horizontal velocity components, w the vertical velocity component and t the time). Multiplying by $g\,dz$ and vertical integration from the surface to height h, and introducing the quantities $\mathfrak{u}, \mathfrak{v}$ defined by

$$\mathfrak{p}\mathfrak{u} = \int_0^h g\mu u \, dz \,, \qquad \mathfrak{p}\mathfrak{v} = \int_0^h g\mu v \, dz \,,$$

we obtain the equation for the temporal variation of $\mathfrak{p}$:

$$\text{(1)} \qquad \frac{\partial \mathfrak{p}}{\partial t} + \frac{\partial(\mathfrak{p}\mathfrak{u})}{\partial x} + \frac{\partial(\mathfrak{p}\mathfrak{v})}{\partial y} + g\mu_h w_h = 0 \,.$$

This holds for all h. For very large height h we introduce the assumptions

$$\text{(D)}\quad p_h = 0\,, \qquad \text{(E)}\quad \mu_h w_h = 0\,.$$

Then $\mathfrak{p} = p_0$ is the pressure at the surface, and $\mathfrak{u}, \mathfrak{v}$ are the average horizontal wind components in a column of unit area at location (x, y) at time t. The average is formed by weighting (u, v) at every level in the column with the density at that level. It follows that

$$\frac{\partial p_0}{\partial t} + \frac{\partial(p_0\mathfrak{u})}{\partial x} + \frac{\partial(p_0\mathfrak{v})}{\partial y} = 0\,, \tag{2}$$

an equation of the same form as the continuity equation for horizontal motion. It can also be written as

$$\frac{\partial p_0}{\partial t} = -\frac{1}{\delta n}\frac{\partial(p_0 \cdot \mathfrak{c}\delta n)}{\partial s}\,. \tag{2*}$$

where

- $\mathfrak{c}$ is the resultant of $\mathfrak{u}$ and $\mathfrak{v}$ [actually, the magnitude of $(\mathfrak{u}, \mathfrak{v})$]
- s is the curve whose tangent is, at every point, in the direction of $\mathfrak{c}$ [actually, the direction of $(\mathfrak{u}, \mathfrak{v})$] at time t
- δn is the [infinitesimal] normal distance between s and a specific neighboring curve s' of the same type.

The tendency of p_0 depends on the spatial difference of $\mathfrak{c}$, δn and p_0 along the curve s. The influence of each individual factor is as follows:

p_0 and δn spatially constant on s : (2₁) $\quad \dfrac{\partial p_0}{\partial t} = -p_0\dfrac{\partial \mathfrak{c}}{\partial s}$

p_0 and $\mathfrak{c}$ spatially constant on s : (2_2) $\quad \dfrac{\partial p_0}{\partial t} = -\dfrac{p_0\mathfrak{c}}{\delta n}\cdot\dfrac{\partial \delta n}{\partial s}$

$\mathfrak{c}$ and δn spatially constant on s : (2_3) $\quad \dfrac{\partial p_0}{\partial t} = -\mathfrak{c}\dfrac{\partial p_0}{\partial s}\,.$

Following the assumption (2_1), the s-lines are parallel and coincide locally with the surface isobars, or they are in an area of uniform pressure.

To determine the value of $\partial \mathfrak{c}/\partial s$ which occurs under typical changes of pressure, we postulate that the barometer rises by 1 mm in one hour and fix $p_0 = 760\,\mathrm{mm\ Hg}$; then

$$\frac{\partial \mathfrak{c}}{\partial s}\left[= -\frac{1}{p_0}\frac{\partial p_0}{\partial t}\right] = -\frac{1}{760}\cdot\frac{1}{3600\,\mathrm{s}} = -3.65\times 10^{-7}\,\mathrm{s}^{-1}\,,$$

and with $ds = 10^5\,\mathrm{m}$ we get $d\mathfrak{c} \approx -0.04\,\mathrm{m\,s^{-1}}$. If the resulting velocity at a point on the s-line is greater by 0.04 m/s than at a point 100 km downstream on the same s-line and the gradient is constant, then the barometric pressure increases along the whole distance by 1 mm per hour.

(At the same time, the velocity of the wind can vary from 0 to $40\,\mathrm{m\,s^{-1}}$, from different directions, at different heights. How accurate must the wind conditions be known if the continuity equation is used to determine if the barometer is going to rise or fall.) [Margules poses a question here—without a question-mark—but provides no answer.]

Eqn. (2_2) determines the pressure tendency which is caused by the divergence of s-lines at constant $\mathfrak{c}$ in an area of uniform pressure. It can be written as

$$\frac{1}{p_0}\frac{\partial p_0}{\partial t}\left[= -\mathfrak{c}\left(\frac{1}{\delta n}\frac{\partial \delta n}{\partial s}\right)\right] = -\mathfrak{c}\frac{\partial \alpha}{\partial n}\,,$$

were α denotes the angle between a given direction on the plane and the tangent on s. Using the above equation and setting $\mathfrak{c} = 1\,\mathrm{m\,s^{-1}}$ and $dn = 1\,\mathrm{km}$, it follows that $d\alpha = -1.26'$ [minutes of arc corresponding to $d\alpha = -3.65\times 10^{-4}$ radians]

Eqn. (2_3) holds for parallel s-lines and with $\mathfrak{c}$ constant along every s-line. The resulting velocity can be a function of the parameters of the family of s-lines [*i.e.*, can vary from one s-line to another]. A pressure change occurs where the direction of $\mathfrak{c}$ is different from that of the isobar at the surface; because of the difference in p_0, the incoming and outgoing air masses in δn, $\delta n'$ [*i.e.*, normals between s and s' at two different points] are different. If $\mathfrak{c}$ is constant in time, then (2_3) has the general integral

$$p_0 = f(s - \mathfrak{c}t)$$

and if, in addition, $\mathfrak{c}$ assumes the same value in the entire region, there is a parallel displacement of the isobaric system in the s-direction with velocity $\mathfrak{c}$. Similar displacements occur which do not necessarily require that $\mathfrak{c}$ be constant

The pressure tendency at the surface is completely determined if p_0, $\mathfrak{u}$ and $\mathfrak{v}$ are known as functions of position; however, $\mathfrak{u}$ and $\mathfrak{v}$ are not uniquely determined by p_0 and $\partial p_0/\partial t$. For every family of [curves] s_j, one can select a $\mathfrak{c}_j$ so that along every curve [the product] $p_0 \cdot \mathfrak{c}_j \cdot \delta n_j$ is contant; this component gives no contribution to $\partial p_0/\partial t$ [see (2*)]. If the continuity equation alone is used, then all non-contributing components of $\mathfrak{c}$ to the pressure tendency can be neglected. [For example,] the expression $(\bar{p}/p_0)\mathfrak{c}$ is such a component, where $\bar{p}$ denotes constant pressure (say, standard pressure), corresponding to parallel displacement with constant $\mathfrak{c}$. The same displacement of the isobars as obtained above with (2_3) is also obtained from (2^*) with parallel s [curves], with the resulting velocity

$$\mathfrak{c}' = \mathfrak{c}\left(1 - \frac{\bar{p}}{p_0}\right) .$$

The velocity distribution is now similar to that of a travelling wave: motion opposite to the propagation direction at locations of low pressure, and motion in the same direction as propagation at locations of high pressure; and $\mathfrak{c}'$ is small compared to the propagation velocity $\mathfrak{c}$. The same displacement of the isobar system can also be produced in infinitely many different ways.

It is not expected that the continuity equation by itself can lead far. The motivation for collecting these considerations was given by two publications[1] which attempted to find a connection between this equation and certain hypotheses so that one could predict the pressure fluctuations or the weather during a day. This depends very much on the hypotheses, which are not discussed here.

Of the assumptions introduced above, (B) and (C) are for convenience only, and they could be disregarded. Assumption (A) has the following implication: It is very probable[2] that the static pressure difference does not deviate more than 1 mm Hg from the true value $p_0 - p_h$, even at the largest height difference; at least, not persisting for a [full] day. The changes of p_0 or $\mathfrak{p}$ often reach 10 to 20 mm Hg at the same time. Considering large [scale] changes, (A) can be assumed to be reasonably accurate. Assumptions (D) and (E) can be omitted if one confines attention to Equation (1). In this case, however, a large portion of the changes in $\mathfrak{p}$ are caused by inflow and outflow of air at the upper surface.

[1]Felix M. Exner, Sitzungsber. d. k. Akad. d. Wissensch. zu Wien. **111**. p. 707. 1902; W. Trabert, Meteorolog. Zeitschr. **38**, p. 231. 1903.

[2]A. Sprung, Lehrb. d. Meteorologie, p. 160. Hamburg, 1885.

Assuming that the change is caused by vertical motion only, then

$$\mu_h w_h = -\frac{1}{g}\frac{\partial \mathfrak{p}}{\partial t}\,,$$

and a rise of $\mathfrak{p}$ by 1 mm Hg/hour corresponds to[3]

$$\left[-\frac{1}{g}\frac{\partial \mathfrak{p}}{\partial t} =\right] \qquad -\frac{10\,333}{760}\cdot\frac{1}{3600} = -0.00378\ \mathrm{kg\,m^{-2}\,s^{-1}}\,.$$

At the heights of 10, 20 [and] 30 km, the density is

$$0.42\,, \quad 0.089\,, \quad 0.0067\ \mathrm{kg\,m^{-3}}\,;$$

then w_h has the values

$$-0.009\,, \quad -0.042\,, \quad -0.56\ \mathrm{m\,s^{-1}}\,,$$

which are small downward velocities.

Large changes in $\mathfrak{p}$ are caused by persistent small differences in horizontal air inflow or outflow, and also by small values of the vertical component of velocity. The effect of these two [factors] can be cancelled in different ways to produce an unchanged $\mathfrak{p}$; from the continuity equation alone, it is not possible to determine if rising or sinking air motion will occur.

In equation (1) it is assumed that steady distributed sources and sinks occur only at the upper boundary surface. If air (water vapour) leaves the lower surface or is absorbed, then the appropriate term has to be added. If condensation of vapour takes place, then there are sinks within the air mass too, and a term must be added to the left hand side of equation (1) to account for the weight of condensed mass in the unit column between 0 and h per unit time.

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[3][Margules does not explain the origin of the factor 10 333 here. We recall that $\frac{4}{3}\times 760 = 1013.33$ is an excellent approximation to the pressure in hectoPascals corresponding to 760 mm Hg. Taking the acceleration of gravity to be $g = 9.8066\,\mathrm{m\,s^{-2}}$, we find that $100 \times 1013.33/g = 10\,333\,\mathrm{kg\,m^{-2}}$.]