Margules' Tendency Equation and Richardson's Forecast

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Abstract

Max Margules contributed a short paper for the Festschrift published in 1904 to mark the sixtieth birthday of his former teacher, the renowned physicist Ludwig Boltzmann. Margules considered the possibility of predicting pressure changes by means of the continuity equation. He showed that, to obtain an accurate estimate of the pressure tendency, the winds would have to be known to a precision quite beyond the practical limit. He concluded that any attempt to forecast synoptic changes by this means was doomed to failure.

We re-examine the numerical weather forecast made by Lewis Fry Richardson in the light of Margules' findings. Richardson employed the method which Margules had shown to be problematical; as a result, his prediction was completely unrealistic. It appears that Richardson was unaware of Margules' paper, although a copy was received by the Met Office Library in 1905.

Figure 1: Max Margules (1856-1920). Photograph from the archives of Zentralanstalt für Meteorologie und Geodynamik, Wien.

1 Max Margules (1856–1920)

Many outstanding scientists were active in meteorological studies in Austria in the period 1890–1925, and great progress was made in dynamic and synoptic meteorology and in climatology during this time. Amongst the most important members of this 'Vienna School' were Julius Hann, Josef Pernter, Wilhelm Trabert, Felix Exner, Wilhelm Schmidt, Heinrich Ficker, Albert Defant and, of course, Max Margules. The Austrian Central Institute for Meteorology and Geodynamics (ZAMG) recently celebrated its 150th anniversary, in conjunction with which an attractive book has been produced (Hammerl, *et al.*, 2001) containing contributions on the work of the Vienna School and on the many scientists who worked there.

Margules, one of the founders of dynamical meteorology, was unquestionably a brilliant theoretician, the true value of whose work was adequately appreciated only after his death. The present biographical sketch is based on Khrgian (1959), Kutzbach (1979) and Gold (1920), and

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on several articles in Hammerl (2001) (see Davies, 2001; Fortak, 2001; Pichler, 2001). Margules was born in the town of Brody, in western Ukraine, in 1856. He studied mathematics and physics at Vienna University, and among his teachers was Ludwig Boltzmann. After a two-year spell as a Volunteer at the Meteorological Institute in Vienna, Margules went to Berlin University in 1879. He returned to Vienna University the following year as a lecturer in physics. In 1882 he rejoined the Meteorological Institute as an Assistant, and continued to work there for 24 years.

Margules studied the diurnal and semi-diurnal variations in atmospheric pressure due to solar radiative forcing, analyzing the Laplace tidal equations and deriving two species of solutions, which he called 'Wellen erster Art' and 'Wellen zweiter Art' (Margules, 1893). This was the first identification of the distinct types of waves now known as inertia-gravity waves and rotational waves. Margules turned next to the study of the source of energy of storms. He demonstrated that the available potential energy associated with horizontal temperature contrasts within a cyclone was, if converted to kinetic energy, sufficient to explain the observed winds. In the course of this work, he derived an expression for the slope of inclination of the boundary between two air masses, a formula which bears his name and is occasionally found in modern textbooks. This work overturned the convective theory of cyclones and adumbrated the frontal theory which emerged about a decade later.

Margules was an introverted and lonely man, who never married and worked in isolation, not collaborating with other scientists. He was disappointed and disillusioned at the lack of recognition of his work and retired from the Meteorological Institute in 1906, aged only fifty, on a modest pension. Its value was severely eroded during the First World War so that his 400 crowns per month was worth about one Euro, insufficient for more than the most meagre survival. His colleagues tried their best to help him, making repeated offers of help which Margules resolutely resisted. He died of starvation in 1920.

2 Hydrostatic Balance and Conservation of Mass

In his 1904 paper, Margules considered the possibility of predicting pressure changes by direct use of the mass conservation principle.[1] The pressure at a point is due to the weight of air in the column above it. Thus, it changes only if air flows into or out of the column. So, to calculate the pressure change, we just need to calculate this flow. However, the difficulty is that, normally, there is flow *through* the column, and the nett change depends on the balance between influx and outflow, a small difference between two large numbers. Margules showed that the calculation is very error-prone, and may give ridiculous results. He concluded that any attempt to forecast the weather was *immoral and damaging to the character of a meteorologist* (Fortak, 2001).

2.1 Pressure of a Layer of Incompressible Fluid

The physical principle of mass conservation is expressed mathematically in terms of the continuity equation, a partial differential equation. However, it is possible to give a quantitative description of the essential physical processes which cause changes in surface pressure without recourse to complicated mathematics. All that is required is a knowledge of how powers of ten are manipulated. We first recall that the pressure at a point is determined by the mass of air above it. This is the hydrostatic approximation, and is found to hold to a high degree of accuracy for the atmosphere. The density of air decreases with height because of compressibility, and the atmosphere extends indefinitely in the vertical direction. However, in some ways its behaviour resembles that of an incompressible fluid layer of finite depth. The mean surface pressure is about 1000 mbar or 100,000 Pascals (10^5 Pa). The density at the surface is about one kilogram per cubic metre. The pressure of a layer of incompressible fluid is given by the

[1] A translation of Margules' paper, together with a short introduction, has been published as a Historical Note by Met Éireann, the Irish Meteorological Service (Lynch, 2001).

product of three quantities: the density (which is constant), the gravitational acceleration and the depth. We take the density to be numerically equal to one (comparable to air) and the gravitational acceleration as $10\,\mathrm{m\,s^{-2}}$. Then, if the depth is ten kilometres (or 10^4 m), the pressure is $1 \times 10 \times 10^4 = 100,000$ Pascals. In other words, this ten-kilometre layer of incompressible fluid of unit density gives rise to a surface pressure similar to that of the compressible atmosphere. So, for simplicity, we substitute for the compressible atmosphere a layer of incompressible fluid of mean depth ten kilometres.

The total mass of the atmosphere is constant. Let us consider a geographical region bound by the four sides of a square of side 10 kilometres. As an example, think of Central London from Notting Hill to Wapping and Holloway to Clapham (Fig. 2a). The area of the region is the square of the side, or 10^8 square metres. The column of fluid above this square forms a cube whose volume is the area multiplied by the depth, or $10^8 \times 10^4 = 10^{12}$ cubic metres. The total mass of fluid contained in the cube is easily calculated: since the density is unity, the mass in kilograms has the same numerical value as the volume. So, the mass is 10^{12} kilograms (or one thousand mega-tonnes).[2]

2.2 Convergence and Divergence

How does the pressure at a point change? Since pressure is due to the mass of fluid above the point in question, the only way it can change is through fluxes of air into or out of the column above the point. Nett inward and outward fluxes are respectively called convergence and divergence. Let us suppose that the movement of air is from west to east, so that no air flows through the north or south faces of our cubic column. Let us further assume for now that the wind speed has a uniform value of ten metres per second. Thus, in a single second, a slab of air of lateral extend 10 kilometres, of height ten kilo-

[2]The fluid exerts a force on the surface beneath it. The unit of force is the Newton. The force on the surface is simply pressure times area. Assuming the pressure to be 100,000 Pascals, this is $10^5 \times 10^8$, or 10^{13} Newtons. It can also be calculated, using Newton's second law of motion, as mass times (gravitational) acceleration, or 10^{12} by 10, giving the same result.

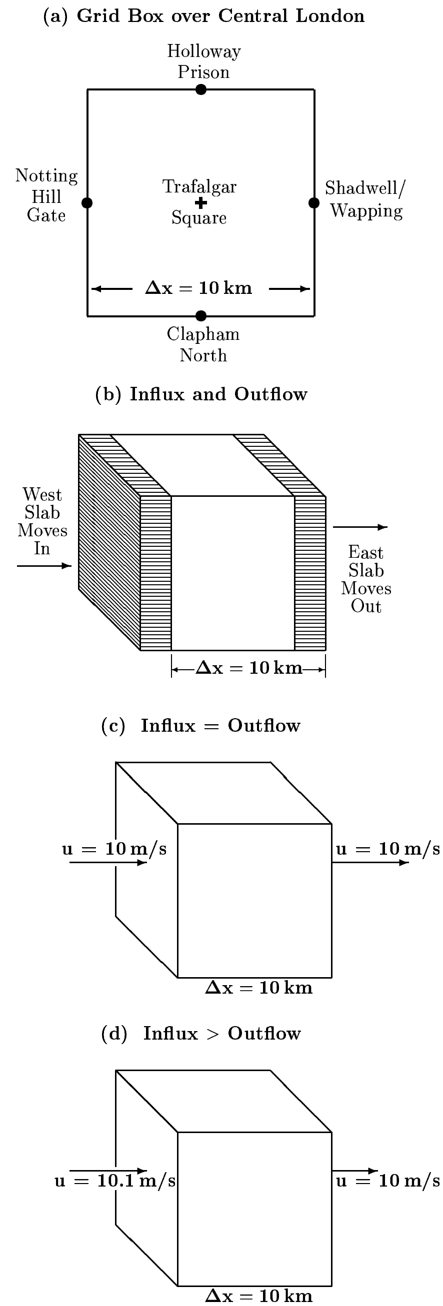

Figure 2: (a) Sample grid box over London. (b) Illustration of influx on western face and outflow on eastern face. (c) No convergence: influx equal to outflow. (d) Convergence: Influx exceeds outflow.

metres and of thickness ten metres, moves into the cube through its western face (Fig. 2b). This slab has total volume of 10^9 cubic metres. Its mass in kilograms has the same numerical value (the density is unity). So the mass of the cube would increase by 10^9 kilograms or one million tonnes in a single second if this were the only flux. However, there is a corresponding flux of air outward through the eastern face of the cube, with precisely the same value (Fig. 2c). So, the *nett* flux is zero, the total mass of air in the cube remains unchanged, and the pressure at the surface remains constant.

Now suppose that the flow speed inward through the western face of the cube is slightly greater, let us say 1% greater at 10.1 metres per second, while the speed outward through the eastern face remains at 10 metres per second (Fig. 2d). There is thus more fluid flowing into the cube than out of it: there is a nett convergence of mass into the cube. We may expect a pressure rise; let us now calculate it. The additional inward flux of mass is 1% of the total inward flux, or 10^7 kilograms per second. We saw that, initially, the total mass of the cube was 10^{12} kilograms, so the *fractional* increase in mass is just the ratio of these two numbers, or $10^7/10^{12} = 10^{-5}$.

To get the rate of increase in mass, we multiply the total mass by 10^{-5}. Since pressure is proportional to mass, its fractional increase is precisely the same: to get the rate of pressure increase, we multiply the total pressure (10^5 Pa) by 10^{-5}, yielding one Pascal per second. A pressure tendency of $1\ \mathrm{Pa\,s^{-1}}$ may not sound impressive, but we shall soon see that, if the relatively small difference between inflow and outflow is sustained over a long period of time, the resulting pressure rise is dramatic. The situation is reminiscent of the song from the musical *The Pajama Game*:

> Seven and a half cents doesn't buy a heck-of-a-lot,
> Seven and a half cents doesn't mean a thing.
> But give it to me every hour, forty hours of every week,
> That's enough for me to be livin' like a king.

We recall that there are 86,400 seconds in a day. To confine the arithmetic to powers of ten, let us redefine a *day* (in italics) to be a period of 100,000 seconds (about 27 hours 46 minutes). So, our *day* is 10^5 s. The pressure change in a *day* will then be the tendency ($1\ \mathrm{Pa\,s^{-1}}$) multiplied by the number of seconds in a *day*. Thus, the pressure will increase by 10^5 Pascals. But this is the same as its initial value, so the pressure will increase by 100% in a *day*!

An even more paradoxical conclusion is reached if we consider the speed at the western face to be 1% *less than*, rather than greater than, the outflow speed at the eastern face. The above reasoning would suggest a decrease of pressure by 100% in a *day*, resulting in a total vacuum and leaving the citizens of London quite breathless. Of course, there is a blunder in the reasoning: as the mass in the 'cube' decreases, its volume must decrease in proportion, since the density is constant. In effect, the fluid depth, which we have taken to be constant, must decrease.

2.3 Gravity Waves and Numerical Prediction

While the calculated pressure tendency is arithmetically correct, the resulting pressure change over a *day* is meteorologically preposterous. Why? We have extrapolated the *instantaneous* pressure change, assuming it to remain constant over a long time period. The assumption that the convergence of fluid into the cubic column remains unchanged for a *day* is quite unrealistic. It ignores the propensity of the atmosphere to respond rapidly to changes. An increase of pressure within the cube causes an immediate outward pressure gradient which acts to resist further change. Indeed, the result of this negative feedback is for over-compensation, so that the pressure falls below its initial value, and a cycle of pressure oscillations ensues. These oscillations are known as gravity waves, and they radiate outwards with high speed from a localised disturbance, dispersing it over a wider area. The detailed discussion of gravity waves requires a level of mathematics beyond what is appropriate to this article (Margules, 1893, was the first comprehensive study of their dynamics). We note only that, as soon as an imbalance arises in the atmosphere, these waves act in such a way as to restore balance. Since they are of high frequency, they result in pressure changes which may be large but which oscillate rapidly in time: *the instantaneous rate of change is not a reliable* *indicator of the long-term change in pressure.*

To obtain an accurate prediction, it is necessary to proceed in stages. From the initial tendency, we calculate the pressure a short time later. The motion fields must also be advanced in this way, using the momentum equations to compute the accelerations. Then, using the updated fields, new values of the tendencies and accelerations are calculated and another short step forward is made. By this means, the mutual adjustments between the fields of mass and velocity can be accommodated. The time step has to be short enough to allow this adjustment to take place. Gravity-wave oscillations may be present, but they need not spoil the forecast: they may be regarded as noise super-imposed on the long-term synoptic evolution. They may also be effectively removed by a minor adjustment of the initial data; this process is called initialization (Lynch, 1987). Modern numerical forecasts are made using the continuity equation in the manner that Margules regarded as impossible, but initialization controls excessive gravity wave noise and a small time step ensures that the calculations remain stable.

3 Richardson's Forecast

During the First World War, Lewis Fry Richardson carried out a manual calculation of the change in pressure over Central Europe (Richardson, 1922). His initial data were based on a series of synoptic charts published in Leipzig by Vilhelm Bjerknes. Richardson chose the date 20 May, 1910, which had the best observational coverage available at that time. Using Bjerknes' charts, he extracted the relevant values on a discrete grid. From these data, he was able to compute the rate of change of pressure for a region in Southern Germany. To do this, he used the continuity equation, employing precisely the method which Margules had shown more than ten years earlier to be seriously problematical. As is well known, the resulting prediction of pressure change was completely unrealistic.

Richardson divided the compressible atmosphere into five discrete layers in the vertical. From the initial data given on page 185 of *Weather Prediction by Numerical Process*, we can easily extract the values of momentum at the four points adjacent to the central point where pressure is given. The values are presented in Table 1. Note that Richardson's values have been converted to SI units, and some signs changed so that positive values represent influx. For simplicity, we shall consider only the vertically summed values. These are given on the last line of Table 1.

Table 1: Richardson's Initial Values of Mass Flux ($\mathrm{kg\,m^{-1}\,s^{-1}}$).

Level	North	South	East	West
I	$+1800$	$+8000$	0	$+2700$
II	$+6200$	$+4100$	$+16600$	-32800
III	-2900	$+15000$	$+9500$	-13600
IV	-5800	$+8000$	$+1900$	-3300
V	-5500	$+4000$	$+6500$	$+4800$
Total	-6200	$+39100$	$+34500$	-42200

Table 2: Calculation of mass influx for central grid box.

	Mass Flux per Unit Length ($\mathrm{kg\,m^{-1}\,s^{-1}}$)	Width of Grid Face (metres)	Mass Flux Rate (kg/s) ($10^9\,\mathrm{kg\,s^{-1}}$)
North	-6200	424,950	-2.635
South	$+39100$	456,360	$+17.844$
East	$+34500$	400,000	$+13.800$
West	-42200	400,000	-16.880
Total			$+12.129$

To compute pressure tendency, it is necessary to calculate the rate at which air is flowing into or out of the grid box. The totals from Table 1 are repeated in the first numerical column of Table 2. The width of each face (in metres) is given in the following column. The mass influx through the face is the product of these, given in the last column. Finally, the fluxes through the four faces are summed to give the total rate of mass influx.

The extra force due to the influx is the product of the additional mass and the acceleration of gravity. Since pressure is force per unit area, the pressure tendency over the grid box can be calculated as the product of the mass influx rate and gravitational acceleration divided by the area of the grid box. The rate at which mass is flowing into the grid-box is 12.129×10^9 kg per second. We multiply this by the acceleration of gravity (9.81 m s^{-2}) and divide by the area of the grid-box (1.763×10^{11} m^2). The result is 0.675 Pa s^{-1} or 145.8 mbar in six hours. This is almost exactly equal to the value (145.1 mbar/6h) given by Richardson on page 211 of his book.

4 Margules and Richardson

One especially interesting question about Margules' results is what influence, if any, they had on Richardson's approach to forecasting. Margules' sent a copy of his 1904 paper to the Met Office, where it was lodged in the library.[3] Thus, it was available for consultation by scientists such as Sir Napier Shaw and William H. Dines. If Napier Shaw, who was fully aware of Richardson's weather prediction project and indeed supported it strongly, knew of Margules' work, he would surely have alerted Richardson to its existence. There was ample opportunity for this between 1913, when Richardson was appointed Superintendent of Eskdalemuir Observatory, and 1916, when he resigned in order to work with the Friends Ambulence Unit in France.

Richardson ascribed the difficulties with his predicted tendency to spurious values of divergence arising from errors in the wind observations. This explanation, while incomplete, is consistent with the analysis of Margules. Had Richardson been aware, at an earlier stage, of Margules' results, he might well have decided not to proceed with his trial forecast, or sought a radically different approach (Platzman, 1967).

There is no reason to believe that Richardson was aware of Margules' paper; certainly, he makes no reference to it in his book. As pointed out by Platzman (1967), Margules' results were summarised by Exner in his textbook *Dynamische Meteorologie*, which Richardson does cite, but without explicit reference to the relevant section.[4] Since this book was published in Leipzig in 1917, Richardson could not have seen it until his return to Britain in 1919, after the First World War and after his trial forecast had been completed (this was done while he was still in France).

It is possible that Richardson may have realized the significance of Margules' results when he read Exner's book, in 1919 or 1920. But in this case, it is inexplicable that he did not refer to Margules, or to the relevant section of Exner, explicitly. He had finished a Homeric numerical forecast and included it in his book, and Margules' results showed that his approach was, from the outset, doomed to failure. Although such a realization would have been devastating, one cannot doubt that Richardson would have faced it with honesty. It appears more likely that Richardson overlooked the relevant section of Exner. He was busy at this time with his strange and wonderful instruments for upper-air measurements at Benson, in collaboration with Dines (Ashford, 1985). His explanation of the failure of his forecast, while compatible with Margules' results, was incomplete. In his book, he writes that smoothing of the data would yield a realistic forecast. This is, of course, questionable.

Shortly after Margules published his paper, his colleague Felix Exner presented a method of forecasting the weather using dynamical methods (Exner, 1908). Exner assumed geostrophically balanced flow and thermal forcing constant in time. He derived a prediction equation for advection of the pressure pattern with constant westerly speed, modified by the effects of diabatic heating. Exner did *not* make use of the continuity equation, so his forecasts were not affected by the difficulties associated with it. His method, while of limited applicability, yielded a realistic forecast in the case illustrated in the paper.

[3]Graham Bartlett of the National Meteorological Library and Archive has kindly checked the records and has informed me that Margules' article was received and catalogued by the Met Office Library in March, 1905.

[4]Oliver Ashford, who holds Richardson's copy of Exner's book, has kindly checked the relevant section (§31) and has confirmed that there are no annotations, whereas Richardson liberally annotated several other sections of the book.

Acknowledgement

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