Generation of zonal flow by resonant Rossby-Haurwitz wave interactions

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Starting out with two interacting Rossby-Haurwitz waves, the generation of zonal flow is discussed. It is shown that zonal flow cannot be generated by first or second order interactions between two such waves, unless they are exchanging energy with a third wave within a resonant triad. The generation of zonal flow at second order through resonant triad interactions is subsequently established and studied.

Keywords: Zonal flow; Rossby-Haurwitz waves; Wave interactions

1. Introduction

Over the years there has been great interest in zonal flow and a number of mechanisms to generate zonal flow in different settings have been suggested. More recently it has been shown that Reynolds stresses between Rossby waves can excite zonal flow in a shallow rotating atmosphere (Shukla and Stenflo 2003, Onishchenko et al. 2004). This results in a transfer of energy from short scale Rossby waves to long scale zonal flows. Suggestions to generate zonal flow through the interaction of Rossby waves, however, go back much further. In 1969 Newell proposed a quartet mechanism to generate zonal flow through the interaction of Rossby waves at second order (Newell 1969). Newell’s work in this area was later extended by Loesch (1977). Both these authors used the $\beta$-plane approximation instead of taking the spherical nature of the earth fully into account.

In this paper we re-examine Newell’s mechanism for interacting Rossby waves by going beyond the $\beta$-plane approximation, working with spherical coordinates throughout. In this setting, the Rossby waves become the Rossby-Haurwitz waves. Our aim is to study how Rossby-Haurwitz waves can produce zonal flow through the resonant interaction energy transfer mechanism. It was previously shown that for meaningful interaction between Rossby-Haurwitz waves to occur a resonant triad should be considered (see, e.g., Reznik et al. 1993). Following on from these discoveries we found

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that if the second order of the equation under consideration is examined, zonal flow is produced. This method does not work in the case of a plane. To produce zonal flow in the \( \beta \)-plane approximation, i.e. for large wavenumbers, it is necessary to use a quartet mechanism (Newell 1969, Loesch 1977). In this paper we show that we require only a triad solution to produce zonal flow. Hence one less wave is required to be present, if the spherical form of the earth is important.

To indicate the range of applicability of our results, the non-divergent barotropic absolute vorticity equation will be discussed in the second section of the paper. In section 3, we examine Rossby-Haurwitz wave interactions. In this section we prove that zonal flow cannot be produced up to \( O(\delta^2) \), if we start out with two Rossby-Haurwitz waves which do not interact resonantly. Section 4 shows that the generation of zonal flow occurs at \( O(\delta^2) \) if we start out our calculations with two resonantly interacting Rossby-Haurwitz waves. Finally we illustrate this mechanism with a worked example in section 5.

Many global circulation models exhibit systematic deficiencies in representing the mean circulation. In particular, the distribution of the strength of the average westerly flow in the northern mid-latitudes appears to be difficult to simulate well. Van Ulden et al. (2006) analysed the characteristics of some nineteen global climate models. Most models had a significant westerly bias (too strong zonal flow) in winter, and had difficulties in getting the correct frequency for blocking situations. The onset and maintenance of blocks depends delicately on nonlinear energy exchanges from smaller scales (Buizza and Molteni 1996). Likewise, the accurate simulation of the mean zonal flow requires a correct representation of such exchanges. The mechanisms discussed in the present paper may be important for determining the strength of the mid-latitude westerly jet streams in the atmosphere. However, a more detailed examination of the mechanism in the context of a full global climate model, and comparison with observational data, which goes beyond the scope of the current study, would be required to confirm this.

2. Rossby-Haurwitz waves

In this section we will briefly discuss the derivation of the non-divergent barotropic absolute vorticity equation and its solutions, the Rossby-Haurwitz waves. We start with the two-dimensional mass conservation equation

\[
\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (\cos \phi v) = 0 \tag{1}
\]

and momentum equations

\[
\frac{\partial u}{\partial t} + \frac{1}{a \cos \phi} \frac{\partial u}{\partial \lambda} - \tan \phi \frac{u v}{a} + \frac{1}{a} \frac{\partial u}{\partial \phi} - 2\Omega \sin \phi = - \frac{1}{\rho a \cos \phi} \frac{\partial p}{\partial \lambda}. \tag{2}
\]

\[
\frac{\partial v}{\partial t} + \frac{1}{a \cos \phi} \frac{\partial v}{\partial \lambda} + \tan \phi \frac{u^2}{a} + \frac{1}{a} \frac{\partial v}{\partial \phi} + 2\Omega u \sin \phi = - \frac{1}{\rho a} \frac{\partial p}{\partial \phi}. \tag{3}
\]

Here \( \lambda \) represents the longitude \((-\pi \leq \lambda < \pi)\), \( \phi \) represents the latitude \((-\pi/2 \leq \phi \leq \pi/2)\), and \( t \) represents time. The constant \( \rho \) is the density, \( \Omega \) is the angular frequency of the
earth, $a$ is its radius, and $p$ is the pressure. The velocities in eastern and northern direction are given by $u$ and $v$, respectively. Equations (1) to (3) are widely used in oceanography and meteorology.

The solution of the mass conservation equation is given by

$$ u = -\frac{1}{a} \frac{\partial \psi}{\partial \phi}, \quad v = \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda}. \quad (4) $$

To reduce (2) and (3) down to a single equation, we apply the differential operators

$$ D_\phi = \frac{\partial}{\partial \phi} - \tan \phi, \quad D_\lambda = \frac{1}{\cos \phi} \frac{\partial}{\partial \lambda} \quad (5) $$

to equation (2) and (3), respectively, and subtract the resulting equations from each other. We also nondimensionalize our variables by setting $t' = t/T$, $\psi' = T \psi/\alpha L^2$, $\Omega' = T \Omega$, where $T$ and $L$ are typical time and length scales, respectively, and $\alpha$ is a small number reflecting our interest in waves of small amplitude. Dropping the primes, we obtain

$$ \frac{\partial}{\partial t} \left( \frac{1}{\cos^2 \phi} \frac{\partial^2 \psi}{\partial \lambda^2} + \frac{\partial^2 \psi}{\partial \phi^2} - \tan \phi \frac{\partial \psi}{\partial \phi} \right) + 2 \frac{\partial \psi}{\partial \lambda} = \delta \left( \frac{1}{\cos \phi} \frac{\partial \psi}{\partial \phi} \frac{\partial \psi}{\partial \phi} - \frac{1}{\cos \phi} \frac{\partial \psi}{\partial \phi} \frac{\partial \psi}{\partial \phi} \right) \times \left( \frac{1}{\cos^2 \phi} \frac{\partial^2 \psi}{\partial \lambda^2} + \frac{\partial^2 \psi}{\partial \phi^2} - \tan \phi \frac{\partial \psi}{\partial \phi} \right) \quad (6) $$

with $\delta = \alpha L^2/a^2$. This is the non-divergent barotropic absolute vorticity equation whose solutions we will study.

Assuming $\delta \ll 1$ and expanding $\psi$ about $\delta$,

$$ \psi(t, \phi, \lambda, \delta) = \psi_0(t, \phi, \lambda) + \delta \psi_1(t, \phi, \lambda) + \ldots, \quad (7) $$

the leading order equation of (6) is

$$ \frac{\partial}{\partial t} \left( \frac{1}{\cos^2 \phi} \frac{\partial^2 \psi_0}{\partial \lambda^2} + \frac{\partial^2 \psi_0}{\partial \phi^2} - \tan \phi \frac{\partial \psi_0}{\partial \phi} \right) + 2 \frac{\partial \psi_0}{\partial \lambda} = 0. \quad (8) $$

This equation is satisfied for the Rossby-Haurwitz waves (Haurwitz 1940)

$$ \psi_0 = (A e^{i(m\lambda - \sigma t)} + \bar{A} e^{-i(m\lambda - \sigma t)}) P_m^n(\mu), \quad (9) $$

where

$$ P_m^n(\mu) = \sqrt{(2n + 1) \frac{(n - m)!}{(n + m)!}} T_m^n(\mu), $$

$$ T_m^n(\mu) = (-1)^m \frac{(1 - \mu^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n, $$

$$ n = 1, 2, 3, \ldots, \quad m = -n, \ldots, n, \quad \mu = \sin \phi $$

and

$$ \sigma = \frac{-2m}{n(n + 1)}. \quad (10) $$
Note that \( \psi = \psi_0 \) is actually a solution of (6), since the quadratic term on the right-hand side of (6) vanishes for \( \psi_0 \) given in (9). Since the leading order equation (8) is linear, any linear superposition of Rossby-Haurwitz waves is also a solution of (8).

3. Wave interactions

To study the interaction of Rossby-Haurwitz waves we have to consider (6) for higher orders of \( \delta \). The \( O(\delta) \) equation is

\[
\frac{\partial}{\partial t} \nabla^2 \psi_1 + 2 \frac{\partial \psi_1}{\partial \lambda} = J(\nabla^2 \psi_0, \psi_0), \tag{11}
\]

where

\[
\nabla^2 = \left( \frac{1}{\cos 2\phi} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \phi^2} - \tan \phi \frac{\partial}{\partial \phi} \right)
\]

and

\[
J(a, b) = \frac{\partial a}{\partial \lambda} \frac{\partial b}{\partial \mu} - \frac{\partial a}{\partial \mu} \frac{\partial b}{\partial \lambda}.
\]

In our calculations we will repeatedly use the identities

\[
P_{n}^{-m} = (-1)^m P_n^m, \quad \nabla^2 Y_n^m = -n(n+1)Y_n^m,
\]

where

\[
Y_n^m(\lambda, \mu) = e^{im\lambda} P_n^m(\mu).
\]

We want to study the resonant generation of zonal flow, i.e., the generation of Rossby-Haurwitz waves of the form \( e^{i(m_\lambda - \sigma_\mu) t} P_n^m(\mu) \) where \( m = 0 \). Firstly we examine a superposition of two solutions and start with

\[
\psi_0 = (A_1 e^{i(m_\lambda - \sigma_\mu) t}) P_{n_1}^m(\mu) + (A_2 e^{-i(m_\lambda - \sigma_\mu) t}) P_{n_2}^m(\mu), \tag{12}
\]

where

\[
\sigma_1 = \frac{-2m_1}{n_1(n_1 + 1)}, \quad \sigma_2 = \frac{-2m_2}{n_2(n_2 + 1)}.
\]

Calculating the right-hand side of (11) we obtain

\[
J(\nabla^2 \psi_0, \psi_0) = i l_{n_1 n_2} \left\{ A_1 A_2 e^{i(m_1 \lambda - \sigma_1) t} \left[ m_2 P_{n_2}^m \frac{dP_{n_1}^m}{d\mu} - m_1 P_{n_1}^m \frac{dP_{n_2}^m}{d\mu} \right] + c.c. \right\}, \tag{13}
\]

where

\[
l_{n_1 n_2} = n_1(n_1 + 1) - n_2(n_2 + 1).
\]
By examining (13) it is clear that none of these terms can be a Rossby-Haurwitz wave and at the same time correspond to zonal flow. Zonal flow on the sphere corresponds to Rossby-Haurwitz waves whose \( \lambda \) derivative terms are zero. For (13) this means that \( m_1 = \pm m_2 \). However, for a Rossby-Haurwitz wave we must also have the condition that \( \sigma_1 = \pm \sigma_2 \). This condition will immediately force \( n_1 = n_2 \). If this were true \( l_{n_1 n_2} \) would vanish. Therefore it can be concluded that it is not possible to produce resonant zonal flow at \( \mathcal{O}(\delta) \) when starting out with two Rossby-Haurwitz waves.

The next possibility to consider is whether zonal flow can be generated at \( \mathcal{O}(\delta^2) \). Before examining this equation, \( \psi_1 \) needs to be determined. To solve (11) and find \( \psi_1 \), we expand (13) in terms of Legendre functions. To do this we shall use the following expansion:

\[
\psi(\lambda, \mu, t) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \psi_n^m(t) Y_n^m(\lambda, \mu),
\]

where

\[
\psi_n^m(t) = \frac{1}{4\pi} \int_{-1}^{1} \int_{-1}^{1} Y_n^m(\lambda, \mu) \psi(\lambda, \mu, t) \, d\lambda \, d\mu.
\]

From this expansion we see that each term on the right-hand side of (11) can be written as a sum of Legendre functions. In fact it can be shown (Infeld and Hull 1951) that only a finite number of terms in this sum are nonzero, since

\[
im_k P_{n_1}^{m_1} \frac{dP_{n_1}^{m_1}}{d\mu} - \im j P_{n_2}^{m_2} \frac{dP_{n_2}^{m_2}}{d\mu} = \sum_{n_1, n_2} l_{n_1 n_2} B_{n_1 n_2}^{m_1 m_2} P_{n_1 + n_2}^{m_1 + m_2},
\]

where

\[
B_{n_1 n_2}^{m_1 m_2} = \frac{1}{2} \int_{-1}^{1} P_{n_1 + n_2}^{m_1 + m_2} \left( \im k P_{n_1}^{m_1} \frac{dP_{n_1}^{m_1}}{d\mu} - \im j P_{n_2}^{m_2} \frac{dP_{n_2}^{m_2}}{d\mu} \right) \, d\mu.
\]

There are further conditions on \( n_i \) and \( m_i \) which, unless satisfied, will result in \( B_{n_1 n_2}^{m_1 m_2} \) being zero. These conditions are discussed by Silberman (1954).

Assuming that the Rossby-Haurwitz waves are not resonantly interacting, we can determine \( \psi_1 \). Using (11) and (12) we obtain

\[
\psi_1 = \sum_{n=|n_2-n_1|}^{n_1+n_2} A_1 A_2 B_{n_1 n_2}^{m_1 m_2} b_{n_1 n_2}^{m_1 m_2} e^{i(n_1+m_1)\lambda-(\sigma_1+\sigma_2)\theta} P_{n_1 + n_2}^{m_1 + m_2}
\]

\[
+ (-1)^{m_2} \sum_{n=|n_2-n_1|}^{n_1+n_2} A_1 A_2 B_{n_1 n_2}^{m_1 m_2} b_{n_1 n_2}^{m_1 m_2} e^{i(n_1-m_1)\lambda-(\sigma_1-\sigma_2)\theta} P_{n_1 + n_2}^{m_1 - m_2} + \text{c.c.,}
\]

where

\[
b_{n_1 n_2}^{m_1 m_2} = \frac{-i l_{n_1 n_2}}{n(n+1)(\sigma_j + \sigma_k) + 2(m_j + m_k)}.
\]

With \( \psi_1 \) now determined, the \( \mathcal{O}(\delta^2) \) equation

\[
\frac{\partial}{\partial t} \nabla^2 \psi_2 + 2 \frac{\partial \psi_2}{\partial \lambda} = J(\nabla^2 \psi_1, \psi_0) + J(\nabla^2 \psi_0, \psi_1)
\]
is studied. Examining the exponential terms obtained when the right-hand side of (18) is calculated we see that we get zonal flow if \( m_1 \pm m_2 \pm m_3 = 0 \) and if \( \sigma_1 \pm \sigma_2 \pm \sigma_3 = 0 \) in the case of a resonant interaction. This is true if and only if \( m_1 = \pm 2 m_2 \) and \( \sigma_1 = \pm 2 \sigma_2 \) or \( 2 m_1 = \pm m_2 \) and \( 2 \sigma_1 = \pm \sigma_2 \). In either case this implies \( n_1 = n_2 \), which leads to \( l_{n_1 n_2} = 0 \) and \( \psi_1 \) to be zero. Therefore we can see that zonal flow cannot be produced up to \( O(\delta^2) \) by starting out with two waves that do not interact resonantly.

4. Resonant zonal flow

The next option to be considered is to start out with two resonantly interacting waves. Starting out with the superposition (12), which this time consists of two resonantly interacting waves, we examine equation (11), i.e.,

\[
\frac{\partial}{\partial t} \nabla^2 \psi_1 + 2 \frac{\partial \psi_1}{\partial \lambda} = l_{n_1 n_2} \left( \sum_{n=[n_2-n_1]}^{n_1+n_2} A_1 A_2 B_{n_1 n_2 n}^{m_1 m_2} e^{i[(m_1+m_2)\lambda -(\sigma_1+\sigma_2)\mu]} p_{n_3}^{m_1+m_2} 
+ (-1)^{m_2} \sum_{n=[n_2-n_1]}^{n_1+n_2} A_1 A_2 B_{n_1 n_2 n}^{m_1-m_2} e^{i[(m_1-m_2)\lambda -(\sigma_1-\sigma_2)\mu]} p_{n_3}^{m_1-m_2} + \text{c.c.} \right). \tag{19}
\]

The right-hand side of (19) will now contain a Rossby-Haurwitz wave because of the resonant interactions. If the corresponding \( n \) is called \( n_3 \) and if we set \( m_1 + m_2 = m_3 \) and \( \sigma_1 + \sigma_2 = \sigma_3 \), we have

\[
\frac{m_1}{n_1(n_1+1)} + \frac{m_2}{n_2(n_2+1)} = \frac{m_3}{n_3(n_3+1)}. \tag{20}
\]

Without loss of generality we have assumed that the Rossby-Haurwitz wave is one of the terms in the first sum; if it is not, we simply replace \( m_3 \) by \(-m_3\) in (12).

Since a Rossby-Haurwitz wave is present on the right-hand side of the equation, \( \psi_1 \) contains a term of the form

\[
t B_{n_1 n_2 n}^{m_1 m_2} e^{i(m_1 \lambda - \sigma_3 \mu)} p_{n_3}^{m_3}. \tag{21}
\]

Such a term grows linearly in time, indicating resonance. We now insist that the expansion (7) is valid for times of \( O(1/\delta) \), in the sense that the \( O(\delta) \) term in (7) is much smaller than the leading term, for time \( t \) of order \( 1/\delta \). This means that we cannot tolerate terms of the form (21) in \( \psi_1 \). This third wave must be included in our original leading order solution to ensure that the asymptotic approximation remains valid. The amplitudes must also be made time-dependent to ensure the validity of the approximation. Therefore the calculations now start with the triad

\[
\psi_0 = A_1(\tau_1) e^{i(m_1 \lambda - \sigma_1 \mu)} p_{n_1}^{m_1}(\mu) + A_2(\tau_1) e^{-i(m_1 \lambda - \sigma_1 \mu)} p_{n_1}^{m_1}(\mu) 
+ A_3(\tau_1) e^{i(m_2 \lambda - \sigma_2 \mu)} p_{n_2}^{m_2}(\mu) + A_4(\tau_1) e^{-i(m_2 \lambda - \sigma_2 \mu)} p_{n_2}^{m_2}(\mu) 
+ A_5(\tau_1) e^{i(m_3 \lambda - \sigma_3 \mu)} p_{n_3}^{m_3}(\mu) + A_6(\tau_1) e^{-i(m_3 \lambda - \sigma_3 \mu)} p_{n_3}^{m_3}(\mu). \tag{22}
\]

The triad will, of course, interact to produce once again the unwanted terms similar to (21) in \( \psi_1 \). To avoid this problem it is assumed that the amplitudes \( A_1, A_2 \) and \( A_3 \) are slowly varying in time, such that their time derivatives are of \( O(\delta) \). We implement this...
idea by introducing a slow time $\tau_1 = \delta t$. Note that this asymptotic scheme generalizes the asymptotic expansion (7) since $\psi_0$, $\psi_1$, ... themselves are now functions of $\delta$. As a bookkeeping device the time derivative shall be rewritten as

$$ \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \tau_1}. $$

Using these new slowly time dependent amplitudes, (11) is replaced by

$$ \frac{\partial}{\partial t} \nabla^2 \psi_1 + 2 \frac{\partial \psi_1}{\partial \lambda} = J(\nabla^2 \psi_0, \psi_0) - \frac{\partial}{\partial \tau_1} \nabla^2 \psi_0. $$

(23)

The term $J(\nabla^2 \psi_0, \psi_0)$ will now generate three Rossby-Haurwitz waves. However, with the assumption that the amplitudes are slowly time dependent, some extra terms are also produced. These terms can cancel the unwanted terms on the right-hand side of (23). To ensure they do, the amplitudes must satisfy the following set of ordinary differential equations

$$ \frac{dA_1}{d\tau_1} = \frac{l_{n_1n_2}}{n_1(n_1+1)} B^{n_1n_2}_{n_1n_2n_3} A_2 A_3, $$

$$ \frac{dA_2}{d\tau_1} = \frac{l_{n_2n_3}}{n_2(n_2+1)} B^{n_2n_3}_{n_1n_2n_3} A_1 A_3, $$

$$ \frac{dA_3}{d\tau_1} = \frac{l_{n_3n_1}}{n_3(n_3+1)} B^{n_3n_1}_{n_1n_2n_3} A_1 A_2. $$

(24)

These equations have been studied as far back as 1969 (Newell 1969). Taking these amplitude conditions into account, the $O(\delta)$ equation is reexamined and it is found that

$$ \psi_1 = \sum_{j=1}^{3} \sum_{k=j+1}^{3} \left( \sum_{n=n_jn_k} \left( A_j A_k B_{n_jn_kn_3}^{n_jn_k} p^{n_jn_k} e^{i(m_j-m_k)\lambda - (\sigma_j+\sigma_k)\beta} \right) A_j A_k B_{n_jn_kn_3}^{n_jn_k} p^{n_jn_k} \right) + c.c., $$

(25)

$$ + (-1)^{n_j} \sum_{n=n_jn_k} \left( A_j A_k B_{n_jn_kn_3}^{n_jn_k} p^{n_jn_k} e^{i(m_j-m_k)\lambda - (\sigma_j+\sigma_k)\beta} \right) + c.c., $$

where

$$ n_{jk} = \begin{cases} n_3, & j = 1, k = 2, \\ n_j + n_k + 1, & \text{otherwise,} \\ \tilde{n}_{jk} = \begin{cases} n_1, & j = 2, k = 3, \\ n_2, & j = 1, k = 3, \\ n_j + n_k + 1, & \text{otherwise.} \end{cases} \end{cases} $$

All the conditions $n \neq n_{jk}$ and $n \neq \tilde{n}_{jk}$ do is take out the resonant term. Once again none of these terms correspond to zonal flow so we examine the $O(\delta^2)$ equation.

We now use the replacement

$$ \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \tau_1} + \delta^2 \frac{\partial}{\partial \tau_2} + \cdots, $$

again purely as a bookkeeping device. Including the new slowly time dependent amplitudes into the $O(\delta^2)$ equation it reads

$$ \frac{\partial}{\partial t} \nabla^2 \psi_2 + 2 \frac{\partial \psi_2}{\partial \lambda} = J(\nabla^2 \psi_1, \psi_0) + J(\nabla^2 \psi_0, \psi_1) - \frac{\partial}{\partial \tau_1} \nabla^2 \psi_1 - \frac{\partial}{\partial \tau_2} \nabla^2 \psi_0. $$

(26)
Examining the first two terms on the right-hand side of (26) we see that zonal flow is indeed produced at this order. Combining, for example, $e^{i(m_j \lambda - \sigma_j \ell)}$ from $\psi_0$ with $e^{-i(m_j + m_k) \lambda - (\sigma_j + \sigma_k) \ell}$ from $\psi_1$ will produce zonal flow. These Rossby-Haurwitz waves generated at $O(\delta^2)$ are now included in our leading order solution from the outset, i.e.,

$$\psi_0 = \sum_{j=1}^{N} \psi_0^j,$$  \hspace{1cm} (27)

where

$$\psi_0^j = (A_j(\tau_1, \tau_2)e^{i(m_j \lambda - \sigma_j \ell)} + \tilde{A}_j(\tau_1, \tau_2)e^{-i(m_j \lambda - \sigma_j \ell)}P_{n_j}^m(\mu),$$

$$\sigma_j = \frac{-2m_j}{n_j(n_j + 1)},$$

$$m_j = 0, \quad A_j = \tilde{A}_j, \quad j = 4, 5, \ldots, N.$$

Again we have to make sure that Rossby-Haurwitz waves do not occur on either the right-hand side of (23) or on the right-hand side of (26). The conditions on the amplitudes necessary to ensure this are

$$\frac{dA_1}{d\tau_1} = \sum_{n_2 = 0}^{n_2(n_2 + 1)} l_{n_2 n_1} B_{n_2 n_1}^{m_1 m_2} A_2 A_3 + 2 \sum_{k=4}^{N} l_{n_k n_1} B_{n_k n_1 n_1}^{m_1 n_1} A_1 A_k,$$

$$\frac{dA_2}{d\tau_1} = \sum_{n_3 = 0}^{n_3(n_3 + 1)} l_{n_3 n_2} B_{n_3 n_2}^{m_1 m_2} \tilde{A}_1 A_3 + 2 \sum_{k=4}^{N} l_{n_k n_2} B_{n_k n_2 n_2}^{m_1 n_2} A_2 A_k,$$

$$\frac{dA_3}{d\tau_1} = \sum_{n_4 = 0}^{n_4(n_4 + 1)} l_{n_4 n_3} B_{n_4 n_3}^{m_1 m_2} A_1 A_2 + 2 \sum_{k=4}^{N} l_{n_k n_3} B_{n_k n_3 n_3}^{m_1 n_3} A_3 A_k,$$

$$\frac{dA_k}{d\tau_1} = 0 \quad \text{for} \quad k = 4, 5, \ldots, N. \hspace{1cm} (28)$$

We see that the amplitudes of the zonal flow do not gain or lose energy at this time scale. At this order the zonal flow actually acts as a catalyst, helping the other waves to exchange energy between themselves. The next order of the equation must be studied to establish conditions for the zonal flow amplitudes. Note that we do not treat $t, \tau_1, \tau_2$ as independent variables, and therefore use ordinary derivatives in (28). The ordinary differential equations (28) have to be solved to find the amplitudes as functions of $\tau_1 = \delta t$ and $N$ integration constants $c_1, c_2, \ldots, c_N$. Obviously, $A_k = c_k$ for $k = 4, 5, \ldots, N$.

To study the next order of the equation, an expression is required for $\psi_1$. Taking into account the above conditions, it can be deduced that

$$\psi_1 = \sum_{j=1}^{3} \sum_{k=j+1}^{N} \left( \sum_{n = |n_j - n_k|}^{n_j + n_k} A_j A_k B_{n_j n_k}^{m_j m_k} \bar{P}_{n_j n_k}^{m_j m_k} e^{i[(m_j + m_k) \lambda - (\sigma_j + \sigma_k) \ell]} + (-1)^{m_j} \sum_{n = |n_j - n_k|}^{n_j + n_k} A_j \tilde{A}_k B_{n_j n_k}^{m_j m_k} \bar{P}_{n_j n_k}^{m_j m_k} e^{i[(m_j - m_k) \lambda - (\sigma_j - \sigma_k) \ell]} + \text{c.c.} \right), \hspace{1cm} (29)$$
where

\[
    n_{jk} = \begin{cases} 
        n_3, & j = 1, k = 2, \\
        n_j, & j = 1, 2, 3 \text{ and } k = 4, 5, \ldots, N, \\
        n_j + n_k + 1, & \text{otherwise,}
    \end{cases}
\]

and

\[
    \tilde{n}_{jk} = \begin{cases} 
        n_1, & j = 2, k = 3, \\
        n_2, & j = 1, k = 3, \\
        n_j, & j = 1, 2, 3 \text{ and } k = 4, 5, \ldots, N, \\
        n_j + n_k + 1, & \text{otherwise.}
    \end{cases}
\]

When the \(O(\delta^2)\) equation is now examined for these values of \(\psi_0\) and \(\psi_1\), it will once again be found that resonance terms will be produced which invalidate our generalized expansion, that is the expansion (7) where now \(\psi_0, \psi_1, \ldots\) have become \(\delta\) dependent. To overcome the same problem at order \(\delta\), we imposed the conditions (28) on the amplitudes. Solving these equations will make the amplitudes functions of the form \(A(\delta, t, c_1, c_2, \ldots, c_N)\), where \(c_1, c_2, \ldots, c_N\) are the constants of integration. To overcome the problem with resonances at \(O(\delta^2)\), we now make the integration constants \(t\) dependent, such that their \(t\) derivatives will first appear in the \(O(\delta^2)\) equation. The derivatives of the integration constants will always occur in the form

\[
    \frac{\partial A_j}{\partial t} + \cdots + \frac{\partial A_j}{\partial c_N} \frac{dc_N}{dt} = \delta^2 \frac{\partial A_j}{\partial \tau_2}. \tag{30}
\]

It is important to note that in our asymptotic scheme \(\tau_1\) and \(\tau_2\) are not independent variables, and therefore the right-hand side of (30) is just a shorthand notation for the left-hand side. We will explain later that treating \(\tau_1\) and \(\tau_2\) as independent variables leads to an inconsistent asymptotic scheme. Using the notation defined in (30), we obtain the next set of amplitude equations,

\[
    \frac{\partial A_j}{\partial \tau_2} = \frac{1}{n_1(n_1 + 1)} \left[ -A_1 A_2 A_3 \left( \sum_{\substack{n_1+n_3 \in \mathbb{N} \setminus \{n_2-n_1\} \atop n_2 \neq n_1}} l_{n_1}(B_{n_1,n_3}^{m_1,m_3})^2 h_{n_1,n_3}^{m_1,m_3} + \sum_{\substack{n_1+n_3 \in \mathbb{N} \setminus \{n_2-n_1 \atop n_2 \neq n_1}} l_{n_1}(B_{n_1,n_3}^{m_1,m_3})^2 h_{n_1,n_3}^{m_1,m_3} \right) \right. \\
    - A_1 A_2 A_3 \left( \sum_{\substack{n_2 \in \mathbb{N} \setminus \{n_1-n_2\} \atop n_2 \neq n_1}} l_{n_2}(B_{n_1,n_2}^{m_1,m_2})^2 h_{n_1,n_2}^{m_1,m_2} + \sum_{\substack{n_2 \in \mathbb{N} \setminus \{n_1-n_2\} \atop n_2 \neq n_1}} l_{n_2}(B_{n_1,n_2}^{m_1,m_2})^2 h_{n_1,n_2}^{m_1,m_2} \right) \\
    - \sum_{k=4}^{N} 2A_1 A_2 A_3 A_k \left( \sum_{\substack{n_2 \in \mathbb{N} \setminus \{n_1-n_2\} \atop n_2 \neq n_1}} l_{n_2}(-1)^{m_2}(-1)^{m_3} B_{n_1,n_2,n_3}^{m_1-m_3} B_{n_2,n_3,n_3}^{m_1-m_3} h_{n_2,n_3}^{m_1-m_3} \right. \\
    + \sum_{\substack{n_2 \in \mathbb{N} \setminus \{n_1-n_2\} \atop n_2 \neq n_1}} l_{n_2}(-1)^{m_2}(-1)^{m_3} B_{n_1,n_2}^{m_1-m_3} B_{n_2,n_3}^{m_1-m_3} h_{n_2,n_3}^{m_1-m_3} \right) \\
    - \sum_{k=4}^{N} \sum_{l=4}^{N} 4A_1 A_2 A_3 A_l \left. \sum_{\substack{n_2 \in \mathbb{N} \setminus \{n_1-n_2\} \atop n_2 \neq n_1}} l_{n_2} B_{n_1,n_2,n_3}^{m_1,m_2} B_{n_2,n_3,n_3}^{m_1,m_2} h_{n_2,n_3}^{m_1-m_2} h_{n_2,n_3}^{m_1-m_2} \right]. \tag{31}
\]
To determine the analogous equation for $A_2$ apply the substitutions

$$m_1 \leftrightarrow m_2, \quad n_1 \leftrightarrow n_2, \quad A_1 \leftrightarrow A_2$$

to equation (31), and to produce the equation for $A_3$ use

$$m_1 \leftrightarrow -m_3, \quad n_1 \leftrightarrow n_3, \quad A_1 \rightarrow (-1)^{m_1} \tilde{A}_3, \quad A_3 \rightarrow (-1)^{m_1} \tilde{A}_1$$

The equations for $A_k$ are given by

$$\frac{\partial A_k}{\partial \tau_2} = (A_1 A_2 \bar{A}_3 - \bar{A}_1 A_2 A_3) \frac{1}{n_k(n_k + 1)} \left( - \sum_{n=n_1-n_3, n \neq n_3} l_{n_1} B_{n_1 n_2 n_3}^{m_1 0} B_{n_2 n_3 n_1}^{m_2-m_3} l_{n_1 n_3 n_2}^{m_2-m_3} (-1)^{m_3} (-1)^{m_1} + \sum_{n=n_1-n_3, n \neq n_3} l_{n_1} B_{n_2 n_1 n_3}^{m_1 0} B_{n_1 n_3 n_2}^{m_2-m_3} l_{n_1 n_3 n_2}^{m_2-m_3} (-1)^{m_3} (-1)^{m_2} \right)$$

$$k = 4, \ldots, N.$$

These are the ordinary differential equations the functions $c_1(t), \ldots, c_N(t)$ have to satisfy. If we had treated $\tau_1$ and $\tau_2$ as independent variables, the two sets of partial differential equations would be inconsistent. This is easy to see for the zonal flow amplitudes, but it is also true for the amplitudes of the triad if the zonal flow amplitudes are assumed to be of order $\mathcal{O}(\delta)$. In our asymptotic scheme, which is a variation of parameter method, the integration constants at a certain order are made time dependent to remove the secular terms at the next order. The questions of consistency of the usual multi-time-scale method do not arise. Our scheme leads to the result that starting out with two resonantly interacting waves, zonal flow is generated at $\mathcal{O}(\delta^2)$ and the change in the zonal flow amplitudes is given by the system (32) of ordinary differential equations for $A_1(t) = c_1(t), \ldots, A_N(t) = c_N(t)$.

5. Worked example

Applying all the required conditions necessary for the occurrence of a Rossby-Haurwitz wave triad, the first thirteen triads for the lowest wavenumbers are given in table 1.

<table>
<thead>
<tr>
<th>$(m_1, n_1)$</th>
<th>$(m_2, n_2)$</th>
<th>$(m_3, n_3)$</th>
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<tr>
<td>(1, 6)</td>
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<td>(3, 9)</td>
</tr>
<tr>
<td>(1, 6)</td>
<td>(11, 20)</td>
<td>(12, 15)</td>
</tr>
<tr>
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<td>(3, 8)</td>
<td>(5, 7)</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>(4, 14)</td>
<td>(6, 9)</td>
</tr>
<tr>
<td>(2, 7)</td>
<td>(11, 20)</td>
<td>(13, 14)</td>
</tr>
<tr>
<td>(2, 14)</td>
<td>(17, 20)</td>
<td>(19, 19)</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>(6, 14)</td>
<td>(9, 9)</td>
</tr>
<tr>
<td>(3, 9)</td>
<td>(8, 20)</td>
<td>(11, 14)</td>
</tr>
<tr>
<td>(3, 14)</td>
<td>(1, 20)</td>
<td>(4, 15)</td>
</tr>
<tr>
<td>(4, 12)</td>
<td>(5, 14)</td>
<td>(9, 13)</td>
</tr>
<tr>
<td>(6, 14)</td>
<td>(2, 20)</td>
<td>(8, 15)</td>
</tr>
<tr>
<td>(6, 18)</td>
<td>(7, 20)</td>
<td>(13, 19)</td>
</tr>
<tr>
<td>(9, 14)</td>
<td>(3, 20)</td>
<td>(12, 15)</td>
</tr>
</tbody>
</table>
In order to show the workings of this theory an example is explicitly studied to generate numerical results. We take the first triad, namely, \((1,6), (2,14)\) and \((3,9)\). Therefore for the rest of this section we have

\[(m_1, n_1) = (1,6), \ (m_2, n_2) = (2,14), \ (m_3, n_3) = (3,9).\]

Firstly we examine the numbers involved for our \(\psi_1\) solution. From this we can examine the \(O(\delta^2)\) equation and determine how many zonal flow terms (if any) are created. Due to the aforementioned problem with resonance all these terms will form part of the leading order solution. Using this leading order solution the required amplitude conditions are determined for these numbers.

To determine \(\psi_1\) it should be noted that \(B_{n_1 n_2 n_3}\) is only not zero for \(n\) between 9 and 19, with \(n\) odd. \(B_{n_1 n_2 n_3}\) is only not zero for \(n\) between 4 and 14, where \(n\) is even, and \(B_{n_1 n_2 n_3}\) is only not zero for \(n\) between 6 and 22, with \(n\) also even. The numbers obtained for these calculations are given in the following tables, table 2 for \(B_{n_1 n_2 n_3}\), table 3 for \(B_{n_1 n_2 n_3}\) and finally table 4 for \(B_{n_1 n_2 n_3}\).

### Table 2. Coupling constants \(B_{n_1 n_2 n_3}\) and \(b_{n_1 n_2 n_3}\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(B_{n_1 n_2 n_3})</th>
<th>(b_{n_1 n_2 n_3})</th>
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<tbody>
<tr>
<td>11</td>
<td>17.2301i</td>
<td>-60i</td>
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<tr>
<td>13</td>
<td>12.4548i</td>
<td>-27.3913i</td>
</tr>
<tr>
<td>15</td>
<td>8.29596i</td>
<td>-16.8i</td>
</tr>
<tr>
<td>17</td>
<td>4.96375i</td>
<td>-11.6667i</td>
</tr>
<tr>
<td>19</td>
<td>2.34338i</td>
<td>-8.68966i</td>
</tr>
</tbody>
</table>

### Table 3. Coupling constants \(B_{n_1 n_2 n_3}\) and \(b_{n_1 n_2 n_3}\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(B_{n_1 n_2 n_3})</th>
<th>(b_{n_1 n_2 n_3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-8.84799i</td>
<td>-13.2632i</td>
</tr>
<tr>
<td>6</td>
<td>-9.22981i</td>
<td>-15i</td>
</tr>
<tr>
<td>8</td>
<td>-6.00544i</td>
<td>-18.2609i</td>
</tr>
<tr>
<td>10</td>
<td>1.35831i</td>
<td>-25.2i</td>
</tr>
<tr>
<td>12</td>
<td>11.9378i</td>
<td>-46.6667i</td>
</tr>
</tbody>
</table>

### Table 4. Coupling constants \(B_{n_1 n_2 n_3}\) and \(b_{n_1 n_2 n_3}\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(B_{n_1 n_2 n_3})</th>
<th>(b_{n_1 n_2 n_3})</th>
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<tr>
<td>8</td>
<td>14.5228i</td>
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</tr>
<tr>
<td>10</td>
<td>0.162338i</td>
<td>-37.0588i</td>
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<tr>
<td>12</td>
<td>-15.9295i</td>
<td>-22.1053i</td>
</tr>
<tr>
<td>14</td>
<td>-27.7451i</td>
<td>-15i</td>
</tr>
<tr>
<td>16</td>
<td>-29.5888i</td>
<td>-10.9565i</td>
</tr>
<tr>
<td>18</td>
<td>-17.2662i</td>
<td>-8.4i</td>
</tr>
<tr>
<td>20</td>
<td>9.12981i</td>
<td>-6.66667i</td>
</tr>
<tr>
<td>22</td>
<td>37.8582i</td>
<td>-5.43103i</td>
</tr>
</tbody>
</table>
With the coupling constants in $\psi_1$ for the triad determined we must examine the $O(\delta^2)$ equation to see what zonal flow terms are created. It is found that this example produces zonal flow at $O(\delta^2)$ for $n$ ranging between 3 and 27, $n$ odd. Therefore the solution which we must consider to examine the equation up to $O(\delta^2)$ is

$$\psi_0 = \sum_{j=1}^{16} \psi_j^i,$$

where

$$\psi_j^i = (A_j(\tau_1, \tau_2)e^{i(m_j\lambda - \sigma_j\lambda)} + \tilde{A}_j(\tau_1, \tau_2)e^{-i(m_j\lambda - \sigma_j\lambda)}P_{nj}(\mu),$$

$$\sigma_j = \frac{-2m_j}{nj(n_j + 1)},$$

$$m_j = 0, \quad A_j = \tilde{A}_j, \quad j = 4, 5, \ldots, 16,$$

and

$$n_4 = 3, \quad n_5 = 5, \quad n_6 = 7, \ldots n_{16} = 27.$$

Using this new leading order solution we calculate the additional terms in $\psi_1$. The numbers produced for the $j = 3$ case are given in table 5. Band structures are similarly produced for both $B_{n_1n_2n}^{m_10}$ and $B_{n_2n_3n}^{m_20}$.

Using all these numbers we can calculate $\partial A_k/\partial \tau_2$ from the equation

$$\frac{\partial A_k}{\partial \tau_2} = \Lambda_k(A_1A_2\bar{A}_3 - \bar{A}_1A_2A_3), \quad k = 4, 5, \ldots, 16,$$

where $\Lambda_k, k = 4, 5, \ldots, 16$, are given in table 6.

<table>
<thead>
<tr>
<th>$k$</th>
<th>11</th>
<th>13</th>
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<th>17</th>
<th>19</th>
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<td>5</td>
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</table>
6. Conclusions

This paper describes the generation of zonal flow through the resonant interactions of Rossby-Haurwitz waves. We have proved that this generation cannot take place at $O(\delta)$. Furthermore, if we start out with two nonresonantly interacting Rossby-Haurwitz waves then zonal flow will not be generated up to $O(\delta^2)$. However, two resonantly interacting Rossby-Haurwitz waves will generate zonal flow at order $O(\delta^2)$. This effect disappears in the $\beta$-plane approximation, i.e. in the limit of large wavenumbers for the triad. In this limit, a quartet mechanism is required.

Acknowledgement

The authors wish to thank E. Benilov for his helpful advice.

References


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