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Generation of zonal flow by resonant Rossby-Haurwitz wave interactions

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Starting out with two interacting Rossby-Haurwitz waves, the generation of zonal flow is discussed. It is shown that zonal flow cannot be generated by first or second order interactions between two such waves, unless they are exchanging energy with a third wave within a resonant triad. The generation of zonal flow at second order through resonant triad interactions is subsequently established and studied.

Keywords: Zonal flow; Rossby-Haurwitz waves; Wave interactions

1. Introduction

Over the years there has been great interest in zonal flow and a number of mechanisms to generate zonal flow in different settings have been suggested. More recently it has been shown that Reynolds stresses between Rossby waves can excite zonal flow in a shallow rotating atmosphere (Shukla and Stenflo 2003, Onishchenko *et al.* 2004). This results in a transfer of energy from short scale Rossby waves to long scale zonal flows. Suggestions to generate zonal flow through the interaction of Rossby waves, however, go back much further. In 1969 Newell proposed a quartet mechanism to generate zonal flow through the interaction of Rossby waves at second order (Newell 1969). Newell's work in this area was later extended by Loesch (1977). Both these authors used the β-plane approximation instead of taking the spherical nature of the earth fully into account.

In this paper we re-examine Newell's mechanism for interacting Rossby waves by going beyond the β-plane approximation, working with spherical coordinates throughout. In this setting, the Rossby waves become the Rossby-Haurwitz waves. Our aim is to study how Rossby-Haurwitz waves can produce zonal flow through the resonant interaction energy transfer mechanism. It was previously shown that for meaningful interaction between Rossby-Haurwitz waves to occur a resonant triad should be considered (see, e.g., Reznik *et al.* 1993). Following on from these discoveries we found

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that if the second order of the equation under consideration is examined, zonal flow is produced. This method does not work in the case of a plane. To produce zonal flow in the β-plane approximation, i.e. for large wavenumbers, it is necessary to use a quartet mechanism (Newell 1969, Loesch 1977). In this paper we show that we require only a triad solution to produce zonal flow. Hence one less wave is required to be present, if the spherical form of the earth is important.

To indicate the range of applicability of our results, the non-divergent barotropic absolute vorticity equation will be discussed in the second section of the paper. In section 3, we examine Rossby-Haurwitz wave interactions. In this section we prove that zonal flow cannot be produced up to $O(\delta^2)$, if we start out with two Rossby-Haurwitz waves which do not interact resonantly. Section 4 shows that the generation of zonal flow occurs at $O(\delta^2)$ if we start out our calculations with two resonantly interacting Rossby-Haurwitz waves. Finally we illustrate this mechanism with a worked example in section 5.

Many global circulation models exhibit systematic deficiencies in representing the mean circulation. In particular, the distribution of the strength of the average westerly flow in the northern mid-latitudes appears to be difficult to simulate well. Van Ulden *et al.* (2006) analysed the characteristics of some nineteen global climate models. Most models had a significant westerly bias (too strong zonal flow) in winter, and had difficulties in getting the correct frequency for blocking situations. The onset and maintenance of blocks depends delicately on nonlinear energy exchanges from smaller scales (Buizza and Molteni 1996). Likewise, the accurate simulation of the mean zonal flow requires a correct representation of such exchanges. The mechanisms discussed in the present paper may be important for determining the strength of the mid-latitude westerly jet streams in the atmosphere. However, a more detailed examination of the mechanism in the context of a full global climate model, and comparison with observational data, which goes beyond the scope of the current study, would be required to confirm this.

2. Rossby-Haurwitz waves

In this section we will briefly discuss the derivation of the non-divergent barotropic absolute vorticity equation and its solutions, the Rossby-Haurwitz waves. We start with the two-dimensional mass conservation equation

$$\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}(\cos\phi v) = 0 \tag{1}$$

and momentum equations

$$\frac{\partial u}{\partial t} + \frac{1}{a\cos\phi} u\frac{\partial u}{\partial \lambda} - \frac{\tan\phi}{a} uv + \frac{1}{a} v\frac{\partial u}{\partial \phi} - 2\Omega v \sin\phi = -\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial \lambda}, \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{1}{a\cos\phi} u\frac{\partial v}{\partial \lambda} + \frac{\tan\phi}{a} u^2 + \frac{1}{a} v\frac{\partial v}{\partial \phi} + 2\Omega u \sin\phi = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi}. \tag{3}$$

Here λ represents the longitude ($-\pi \le \lambda < \pi$), ϕ represents the latitude ($-\pi/2 \le \phi \le \pi/2$), and t represents time. The constant ρ is the density, Ω is the angular frequency of the earth, a is its radius, and p is the pressure. The velocities in eastern and northern direction are given by u and v, respectively. Equations (1) to (3) are widely used in oceanography and meteorology.

The solution of the mass conservation equation is given by

$$u = -\frac{1}{a}\frac{\partial \psi}{\partial \phi}, \quad v = \frac{1}{a\cos\phi}\frac{\partial \psi}{\partial \lambda}. \tag{4}$$

To reduce (2) and (3) down to a single equation, we apply the differential operators

$$D_\phi = \frac{\partial}{\partial \phi} - \tan\phi, \quad D_\lambda = \frac{1}{\cos\phi}\frac{\partial}{\partial \lambda} \tag{5}$$

to equation (2) and (3), respectively, and substract the resulting equations from each other. We also nondimensionalize our variables by setting $t' = t/T$, $\psi' = T\psi/(\alpha L^2)$, $\Omega' = T\Omega$, where T and L are typical time and length scales, respectively, and α is a small number reflecting our interest in waves of small amplitude. Dropping the primes, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\phi^2} - \tan\phi\frac{\partial\psi}{\partial\phi}\right) + 2\frac{\partial\psi}{\partial\lambda} &= \delta\left(\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\phi}\frac{\partial}{\partial\lambda} - \frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\phi}\right)\\ &\quad\times\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\phi^2} - \tan\phi\frac{\partial\psi}{\partial\phi}\right)\end{aligned} \tag{6}$$

with $\delta = \alpha L^2/a^2$. This is the non-divergent barotropic absolute vorticity equation whose solutions we will study.

Assuming $\delta \ll 1$ and expanding ψ about δ,

$$\psi(t,\phi,\lambda,\delta) = \psi_0(t,\phi,\lambda) + \delta\psi_1(t,\phi,\lambda) + \ldots, \tag{7}$$

the leading order equation of (6) is

$$\frac{\partial}{\partial t}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\psi_0}{\partial\lambda^2} + \frac{\partial^2\psi_0}{\partial\phi^2} - \tan\phi\frac{\partial\psi_0}{\partial\phi}\right) + 2\frac{\partial\psi_0}{\partial\lambda} = 0. \tag{8}$$

This equation is satisfied for the Rossby-Haurwitz waves (Haurwitz 1940)

$$\psi_0 = (A\mathrm{e}^{\mathrm{i}(m\lambda-\sigma t)} + \bar{A}\mathrm{e}^{-\mathrm{i}(m\lambda-\sigma t)})\mathrm{P}_n^m(\mu), \tag{9}$$

where

$$\mathrm{P}_n^m(\mu) = \sqrt{(2n+1)\frac{(n-m)!}{(n+m)!}}\mathrm{T}_n^m(\mu),$$

$$\mathrm{T}_n^m(\mu) = (-1)^m\frac{(1-\mu^2)^{m/2}}{2^n n!}\frac{\mathrm{d}^{n+m}}{\mathrm{d}\mu^{n+m}}(\mu^2-1)^n,$$

$$n = 1,2,3,\ldots, \quad m = -n,\ldots,n,$$

$$\mu = \sin\phi$$

and

$$\sigma = \frac{-2m}{n(n+1)}. \tag{10}$$

Note that $\psi = \psi_0$ is actually a solution of (6), since the quadratic term on the right-hand side of (6) vanishes for ψ_0 given in (9). Since the leading order equation (8) is linear, any linear superposition of Rossby-Haurwitz waves is also a solution of (8).

3. Wave interactions

To study the interaction of Rossby-Haurwitz waves we have to consider (6) for higher orders of δ. The O(δ) equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = J(\nabla^2\psi_0, \psi_0), \tag{11}$$

where

$$\nabla^2 = \left(\frac{1}{\cos 2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\phi^2} - \tan\phi\frac{\partial}{\partial\phi}\right)$$

and

$$J(a,b) = \frac{\partial a}{\partial\lambda}\frac{\partial b}{\partial\mu} - \frac{\partial a}{\partial\mu}\frac{\partial b}{\partial\lambda}.$$

In our calculations we will repeatedly use the identities

$$\mathrm{P}_n^{-m} = (-1)^m\mathrm{P}_n^m, \qquad \nabla^2\mathrm{Y}_n^m = -n(n+1)\mathrm{Y}_n^m,$$

where

$$\mathrm{Y}_n^m(\lambda,\mu) = \mathrm{e}^{\mathrm{i}m\lambda}\mathrm{P}_n^m(\mu).$$

We want to study the resonant generation of zonal flow, i.e., the generation of Rossby-Haurwitz waves of the form $\mathrm{e}^{\mathrm{i}(m\lambda-\sigma t)}\mathrm{P}_n^m$ where $m=0$. Firstly we examine a superposition of two solutions and start with

$$\psi_0 = (A_1\mathrm{e}^{\mathrm{i}(m_1\lambda-\sigma_1 t)} + \bar{A}_1\mathrm{e}^{-\mathrm{i}(m_1\lambda-\sigma_1 t)})\mathrm{P}_{n_1}^{m_1}(\mu) + (A_2\mathrm{e}^{\mathrm{i}(m_2\lambda-\sigma_2 t)} + \bar{A}_2\mathrm{e}^{-\mathrm{i}(m_2\lambda-\sigma_2 t)})\mathrm{P}_{n_2}^{m_2}(\mu), \tag{12}$$

where

$$\sigma_1 = \frac{-2m_1}{n_1(n_1+1)}, \quad \sigma_2 = \frac{-2m_2}{n_2(n_2+1)}.$$

Calculating the right-hand side of (11) we obtain

$$\begin{aligned} J(\nabla^2\psi_0,\psi_0) = \mathrm{i}l_{n_1n_2}\Bigg\{&A_1A_2\mathrm{e}^{\mathrm{i}[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}\left[m_2\mathrm{P}_{n_2}^{m_2}\frac{\mathrm{d}\mathrm{P}_{n_1}^{m_1}}{\mathrm{d}\mu} - m_1\mathrm{P}_{n_1}^{m_1}\frac{\mathrm{d}\mathrm{P}_{n_2}^{m_2}}{\mathrm{d}\mu}\right] \\ &- A_1\bar{A}_2\mathrm{e}^{\mathrm{i}[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}\left[m_2\mathrm{P}_{n_2}^{m_2}\frac{\mathrm{d}\mathrm{P}_{n_1}^{m_1}}{\mathrm{d}\mu} + m_1\mathrm{P}_{n_1}^{m_1}\frac{\mathrm{d}\mathrm{P}_{n_2}^{m_2}}{\mathrm{d}\mu}\right] + \text{c.c.}\Bigg\}, \end{aligned} \tag{13}$$

where

$$l_{n_in_j} = n_i(n_i+1) - n_j(n_j+1).$$

By examining (13) it is clear that none of these terms can be a Rossby-Haurwitz wave and at the same time correspond to zonal flow. Zonal flow on the sphere corresponds to Rossby-Haurwitz waves whose λ derivative terms are zero. For (13) this means that $m_1 = \pm m_2$. However, for a Rossby-Haurwitz wave we must also have the condition that $\sigma_1 = \pm\sigma_2$. This condition will immediately force $n_1 = n_2$. If this were true $l_{n_1 n_2}$ would vanish. Therefore it can be concluded that it is not possible to produce resonant zonal flow at $O(\delta)$ when starting out with two Rossby-Haurwitz waves.

The next possibility to consider is whether zonal flow can be generated at $O(\delta^2)$. Before examining this equation, ψ_1 needs to be determined. To solve (11) and find ψ_1, we expand (13) in terms of Legendre functions. To do this we shall use the following expansion:

$$\psi(\lambda, \mu, t) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \psi_n^m(t) \mathrm{Y}_n^m(\lambda, \mu), \tag{14}$$

where

$$\psi_n^m(t) = \frac{1}{4\pi} \int_{-1}^{1} \int_{-\pi}^{\pi} \bar{\mathrm{Y}}_n^m(\lambda, \mu) \psi(\lambda, \mu, t)\, \mathrm{d}\lambda\, \mathrm{d}\mu. \tag{15}$$

From this expansion we seet that each term on the right-hand side of (11) can be written as a sum of Legendre functions. In fact it can be shown (Infeld and Hull 1951) that only a finite number of terms in this sum are nonzero, since

$$\mathrm{i}m_k \mathrm{P}_{n_k}^{m_k} \frac{\mathrm{dP}_{n_j}^{m_j}}{\mathrm{d}\mu} - \mathrm{i}m_j \mathrm{P}_{n_j}^{m_j} \frac{\mathrm{dP}_{n_k}^{m_k}}{\mathrm{d}\mu} = \sum_{n=|n_k-n_j|}^{n_j+n_k} B_{n_j n_k n}^{m_j m_k} \mathrm{P}_n^{m_j+m_k},$$

where

$$B_{n_j n_k n}^{m_j m_k} = \frac{1}{2} \int_{-1}^{1} \mathrm{P}_n^{m_j+m_k} \left(\mathrm{i}m_k \mathrm{P}_{n_k}^{m_k} \frac{\mathrm{dP}_{n_j}^{m_j}}{\mathrm{d}\mu} - \mathrm{i}m_j \mathrm{P}_{n_j}^{m_j} \frac{\mathrm{dP}_{n_k}^{m_k}}{\mathrm{d}\mu} \right) \mathrm{d}\mu. \tag{16}$$

There are further conditions on n_{i} and m_{i} which, unless satisfied, will result in $B_{n_j n_k n}^{m_j m_k}$ being zero. These conditions are discussed by Silberman (1954).

Assuming that the Rossby-Haurwitz waves are not resonantly interacting, we can determine ψ_1. Using (11) and (12) we obtain

$$\begin{aligned}\psi_1 = &\sum_{n=|n_2-n_1|}^{n_1+n_2} A_1 A_2 B_{n_1 n_2 n}^{m_1 m_2} b_{n_1 n_2 n}^{m_1 m_2} \mathrm{e}^{\mathrm{i}[(m_1+m_2)\lambda - (\sigma_1+\sigma_2)t]} \mathrm{P}_n^{m_1+m_2} \\ &+ (-1)^{m_2} \sum_{n=|n_2-n_1|}^{n_1+n_2} A_1 \bar{A}_2 B_{n_1 n_2 n}^{m_1 -m_2} b_{n_1 n_2 n}^{m_1 -m_2} \mathrm{e}^{\mathrm{i}[(m_1-m_2)\lambda - (\sigma_1-\sigma_2)t]} \mathrm{P}_n^{m_1-m_2} + \text{c.c.},\end{aligned} \tag{17}$$

where

$$b_{n_j n_k n}^{m_j m_k} = \frac{-\mathrm{i} l_{n_j n_k}}{n(n+1)(\sigma_j+\sigma_k) + 2(m_j+m_k)}.$$

With ψ_1 now determined, the $O(\delta^2)$ equation

$$\frac{\partial}{\partial t} \nabla^2 \psi_2 + 2\frac{\partial \psi_2}{\partial \lambda} = J(\nabla^2 \psi_1, \psi_0) + J(\nabla^2 \psi_0, \psi_1) \tag{18}$$

is studied. Examining the exponential terms obtained when the right-hand side of (18) is calculated we see that we get zonal flow if $m_1 \pm m_j \pm m_k = 0$ and if $\sigma_l \pm \sigma_j \pm \sigma_k = 0$ in the case of a resonant interaction. This is true if and only if $m_1 = \pm 2m_2$ and $\sigma_1 = \pm 2\sigma_2$ or $2m_1 = \pm m_2$ and $2\sigma_1 = \pm\sigma_2$. In either case this implies $n_1 = n_2$, which leads to $l_{n_1n_2} = 0$ and ψ_1 to be zero. Therefore we can see that zonal flow cannot be produced up to $O(\delta^2)$ by starting out with two waves that do not interact resonantly.

4. Resonant zonal flow

The next option to be considered is to start out with two resonantly interacting waves. Starting out with the superposition (12), which this time consists of two resonantly interacting waves, we examine equation (11), i.e.,

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial\psi_1}{\partial\lambda} = l_{n_1n_2}\Bigg(\sum_{n=|n_2-n_1|}^{n_1+n_2} A_1A_2B_{n_1n_2n}^{m_1m_2}\mathrm{e}^{\mathrm{i}[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}\mathrm{P}_n^{m_1+m_2}$$
$$+(-1)^{m_2}\sum_{n=|n_2-n_1|}^{n_1+n_2} A_1\bar{A}_2B_{n_1n_2n}^{m_1-m_2}\mathrm{e}^{\mathrm{i}[(m_1-m_2)\lambda-(\sigma_1-\sigma_2)t]}\mathrm{P}_n^{m_1-m_2} + \text{c.c.}\Bigg). \tag{19}$$

The right-hand side of (19) will now contain a Rossby-Haurwitz wave because of the resonant interactions. If the corresponding n is called n_3 and if we set $m_1 + m_2 = m_3$ and $\sigma_1 + \sigma_2 = \sigma_3$, we have

$$\frac{m_1}{n_1(n_1+1)} + \frac{m_2}{n_2(n_2+1)} = \frac{m_3}{n_3(n_3+1)}. \tag{20}$$

Without loss of generality we have assumed that the Rossby-Haurwitz wave is one of the terms in the first sum; if it is not, we simply replace m_2 by $-m_2$ in (12).

Since a Rossby-Haurwitz wave is present on the right-hand side of the equation, ψ_1 contains a term of the form

$$tB_{n_1n_2n}^{m_1m_2}\mathrm{e}^{\mathrm{i}(m_3\lambda-\sigma_3t)}\mathrm{P}_{n_3}^{m_3}. \tag{21}$$

Such a term grows linearly in time, indicating resonance. We now insist that the expansion (7) is valid for times of $O(1/\delta)$, in the sense that the $O(\delta)$ term in (7) is much smaller than the leading term, for time t of order $1/\delta$. This means that we cannot tolerate terms of the form (21) in ψ_1. This third wave must be included in our original leading order solution to ensure that the asymptotic approximation remains valid. The amplitudes must also be made time-dependent to ensure the validity of the approximation. Therefore the calculations now start with the triad

$$\begin{aligned}\psi_0 = {} & A_1(\tau_1)\mathrm{e}^{\mathrm{i}(m_1\lambda-\sigma_1t)}\mathrm{P}_{n_1}^{m_1}(\mu) + \bar{A}_1(\tau_1)\mathrm{e}^{-\mathrm{i}(m_1\lambda-\sigma_1t)}\mathrm{P}_{n_1}^{m_1}(\mu)\\ & + A_2(\tau_1)\mathrm{e}^{\mathrm{i}(m_2\lambda-\sigma_2t)}\mathrm{P}_{n_2}^{m_2}(\mu) + \bar{A}_2(\tau_1)\mathrm{e}^{-\mathrm{i}(m_2\lambda-\sigma_2t)}\mathrm{P}_{n_2}^{m_2}(\mu)\\ & + A_3(\tau_1)\mathrm{e}^{\mathrm{i}(m_3\lambda-\sigma_3t)}\mathrm{P}_{n_3}^{m_3}(\mu) + \bar{A}_3(\tau_1)\mathrm{e}^{-\mathrm{i}(m_3\lambda-\sigma_3t)}\mathrm{P}_{n_3}^{m_3}(\mu).\end{aligned} \tag{22}$$

The triad will, of course, interact to produce once again the unwanted terms similar to (21) in ψ_1. To avoid this problem it is assumed that the amplitudes A_1, A_2 and A_3 are slowly varying in time, such that their time derivatives are of $O(\delta)$. We implement this idea by introducing a slow time $\tau_1 = \delta t$. Note that this asymptotic scheme generalizes the asymptotic expansion (7) since $\psi_0,\ \psi_1, \ldots$ themselves are now functions of δ. As a bookkeeping device the time derivative shall be rewritten as

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \tau_1}.$$

Using these new slowly time dependent amplitudes, (11) is replaced by

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + 2\frac{\partial \psi_1}{\partial \lambda} = J(\nabla^2\psi_0, \psi_0) - \frac{\partial}{\partial \tau_1}\nabla^2\psi_0. \tag{23}$$

The term $J(\nabla^2\psi_0,\ \psi_0)$ will now generate three Rossby-Haurwitz waves. However, with the assumption that the amplitudes are slowly time dependent, some extra terms are also produced. These terms can cancel the unwanted terms on the right-hand side of (23). To ensure they do, the amplitudes must satisfy the following set of ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}A_1}{\mathrm{d}\tau_1} &= \frac{l_{n_2 n_3}}{n_1(n_1+1)} B^{m_1 m_2}_{n_1 n_2 n_3} \bar{A}_2 A_3, \\
\frac{\mathrm{d}A_2}{\mathrm{d}\tau_1} &= \frac{l_{n_3 n_1}}{n_2(n_2+1)} B^{m_1 m_2}_{n_1 n_2 n_3} \bar{A}_1 A_3, \\
\frac{\mathrm{d}A_3}{\mathrm{d}\tau_1} &= \frac{l_{n_2 n_1}}{n_3(n_3+1)} B^{m_1 m_2}_{n_1 n_2 n_3} A_1 A_2.
\end{aligned} \tag{24}$$

These equations have been studied as far back as 1969 (Newell 1969). Taking these amplitude conditions into account, the O(δ) equation is reexamined and it is found that

$$\begin{aligned}
\psi_1 = \sum_{j=1}^{3}\sum_{k=j+1}^{3}\Bigg(&\sum_{\substack{n=|n_j-n_k| \\ n\neq n_{jk}}}^{n_j+n_k} A_j A_k B^{m_j m_k}_{n_j n_k n} b^{m_j m_k}_{n_j n_k n} \mathrm{e}^{\mathrm{i}[(m_j+m_k)\lambda-(\sigma_j+\sigma_k)t]} \mathrm{P}_n^{m_j+m_k} \\
&+ (-1)^{m_k} \sum_{\substack{n=|n_j-n_k| \\ n\neq \tilde{n}_{jk}}}^{n_j+n_k} A_j \bar{A}_k B^{m_j-m_k}_{n_j n_k n} b^{m_j-m_k}_{n_j n_k n} \mathrm{e}^{\mathrm{i}[(m_j-m_k)\lambda-(\sigma_j-\sigma_k)t]} \mathrm{P}_n^{m_j-m_k}\Bigg) + \text{c.c.},
\end{aligned} \tag{25}$$

where

$$n_{jk} = \begin{cases} n_3, & j=1, k=2, \\ n_j+n_k+1, & \text{otherwise}, \end{cases} \qquad \tilde{n}_{jk} = \begin{cases} n_1, & j=2, k=3, \\ n_2, & j=1, k=3, \\ n_j+n_k+1, & \text{otherwise}. \end{cases}$$

All the conditions $n \neq n_{jk}$ and $n \neq \tilde{n}_{jk}$ do is take out the resonant term. Once again none of these terms correspond to zonal flow so we examine the O(δ^2) equation.

We now use the replacement

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \delta\frac{\partial}{\partial \tau_1} + \delta^2\frac{\partial}{\partial \tau_2} + \cdots,$$

again purely as a bookkeeping device. Including the new slowly time dependent amplitudes into the O(δ^2) equation it reads

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + 2\frac{\partial \psi_2}{\partial \lambda} = J(\nabla^2\psi_1, \psi_0) + J(\nabla^2\psi_0, \psi_1) - \frac{\partial}{\partial \tau_1}\nabla^2\psi_1 - \frac{\partial}{\partial \tau_2}\nabla^2\psi_0. \tag{26}$$

Examining the first two terms on the right-hand side of (26) we see that zonal flow is indeed produced at this order. Combining, for example, $\mathrm{e}^{\mathrm{i}(m_3\lambda-\sigma_3 t)}$ from ψ_0 with $\mathrm{e}^{-\mathrm{i}[(m_1+m_2)\lambda-(\sigma_1+\sigma_2)t]}$ from ψ_1 will produce zonal flow. These Rossby-Haurwitz waves generated at $O(\delta^2)$ are now included in our leading order solution from the outset, i.e.,

$$\psi_0 = \sum_{j=1}^{N} \psi_0^j, \tag{27}$$

where

$$\begin{aligned}
\psi_0^j &= (A_j(\tau_1,\tau_2)\mathrm{e}^{\mathrm{i}(m_j\lambda-\sigma_j t)} + \bar{A}_j(\tau_1,\tau_2)\mathrm{e}^{-\mathrm{i}(m_j\lambda-\sigma_j t)})\mathrm{P}_{n_j}^{m_j}(\mu),\\
\sigma_j &= \frac{-2m_j}{n_j(n_j+1)},\\
m_j &= 0, \quad A_j = \bar{A}_j, \quad j = 4,5,\ldots,N.
\end{aligned}$$

Again we have to make sure that Rossby-Haurwitz waves do not occur on either the right-hand side of (23) or on the right-hand side of (26). The conditions on the amplitudes necessary to ensure this are

$$\begin{aligned}
\frac{\mathrm{d}A_1}{\mathrm{d}\tau_1} &= \frac{l_{n_2n_3}B^{m_1m_2}_{n_1n_2n_3}}{n_1(n_1+1)}\bar{A}_2A_3 + 2\sum_{k=4}^{N}\frac{l_{n_kn_1}B^{m_10}_{n_1n_kn_1}}{n_1(n_1+1)}A_1A_k,\\
\frac{\mathrm{d}A_2}{\mathrm{d}\tau_1} &= \frac{l_{n_3n_1}B^{m_1m_2}_{n_1n_2n_3}}{n_2(n_2+1)}\bar{A}_1A_3 + 2\sum_{k=4}^{N}\frac{l_{n_kn_2}B^{m_20}_{n_2n_kn_2}}{n_2(n_2+1)}A_2A_k,\\
\frac{\mathrm{d}A_3}{\mathrm{d}\tau_1} &= \frac{l_{n_2n_1}B^{m_1m_2}_{n_1n_2n_3}}{n_3(n_3+1)}A_1A_2 + 2\sum_{k=4}^{N}\frac{l_{n_kn_3}B^{m_30}_{n_3n_kn_3}}{n_3(n_3+1)}A_3A_k,\\
\frac{\mathrm{d}A_k}{\mathrm{d}\tau_1} &= 0 \quad \text{for } k=4,5,\ldots,N.
\end{aligned} \tag{28}$$

We see that the amplitudes of the zonal flow do not gain or lose energy at this time scale. At this order the zonal flow actually acts as a catalyst, helping the other waves to exchange energy between themselves. The next order of the equation must be studied to establish conditions for the zonal flow amplitudes. Note that we do not treat t, τ_1, τ_2 as independent variables, and therefore use ordinary derivatives in (28). The ordinary differential equations (28) have to be solved to find the amplitudes as functions of $\tau_1 = \delta t$ and N integration constants $c_1, c_2, \ldots, c_N$. Obviously, $A_k = c_k$ for $k = 4,5,\ldots,N$.

To study the next order of the equation, an expression is required for ψ_1. Taking into account the above conditions, it can be deduced that

$$\begin{aligned}
\psi_1 = \sum_{j=1}^{3}\sum_{k=j+1}^{N}\Bigg(&\sum_{\substack{n=|n_j-n_k|\\ n\neq n_{jk}}}^{n_j+n_k} A_jA_kB^{m_jm_k}_{n_jn_kn}b^{m_jm_k}_{n_jn_kn}\mathrm{e}^{\mathrm{i}[(m_j+m_k)\lambda-(\sigma_j+\sigma_k)t]}\mathrm{P}_n^{m_j+m_k}\\
&+(-1)^{m_k}\sum_{\substack{n=|n_j-n_k|\\ n\neq \tilde{n}_{jk}}}^{n_j+n_k} A_j\bar{A}_kB^{m_j-m_k}_{n_jn_kn}b^{m_j-m_k}_{n_jn_kn}\mathrm{e}^{\mathrm{i}[(m_j-m_k)\lambda-(\sigma_j-\sigma_k)t]}\mathrm{P}_n^{m_j-m_k}\Bigg) + \text{c.c.},
\end{aligned} \tag{29}$$

where

$$n_{jk} = \begin{cases} n_3, & j = 1, k = 2, \\ n_j, & j = 1, 2, 3 \text{ and } k = 4, 5, \ldots, N, \\ n_j + n_k + 1, & \text{otherwise,} \end{cases}$$

and

$$\tilde{n}_{jk} = \begin{cases} n_1, & j = 2, k = 3, \\ n_2, & j = 1, k = 3, \\ n_j, & j = 1, 2, 3 \text{ and } k = 4, 5, \ldots, N, \\ n_j + n_k + 1, & \text{otherwise.} \end{cases}$$

When the O(δ^2) equation is now examined for these values of ψ_0 and ψ_1, it will once again be found that resonance terms will be produced which invalidate our generalized expansion, that is the expansion (7) where now $\psi_0, \psi_1, \ldots$ have become δ dependent. To overcome the same problem at order δ, we imposed the conditions (28) on the amplitudes. Solving these equations will make the amplitudes functions of the form $A(\delta, t, c_1, c_2, \ldots, c_N)$, where $c_1, c_2, \ldots, c_N$ are the constants of integration. To overcome the problem with resonances at O(δ^2), we now make the integration constants t dependent, such that their t derivatives will first appear in the O(δ^2) equation. The derivatives of the integration constants will always occur in the form

$$\frac{\partial A_j}{\partial c_1}\frac{\mathrm{d}c_1}{\mathrm{d}t} + \cdots + \frac{\partial A_j}{\partial c_N}\frac{\mathrm{d}c_N}{\mathrm{d}t} =: \delta^2 \frac{\partial A_j}{\partial \tau_2}. \tag{30}$$

It is important to note that in our asymptotic scheme τ_1 and τ_2 are not independent variables, and therefore the right-hand side of (30) is just a shorthand notation for the left-hand side. We will explain later that treating τ_1 and τ_2 as independent variables leads to an inconsistent asymptotic scheme. Using the notation defined in (30), we obtain the next set of amplitude equations,

$$\begin{aligned} \frac{\partial A_1}{\partial \tau_2} = \frac{1}{n_1(n_1+1)}\Bigg[&- A_1 A_3 \bar{A}_3 \Bigg(\sum_{\substack{n=|n_3-n_1| \\ n\neq n_2}}^{n_1+n_3} l_{nn_3} (B^{m_1-m_3}_{n_1 n_3 n})^2 b^{m_1-m_3}_{n_1 n_3 n} + \sum_{n=|n_3-n_1|}^{n_1+n_3} l_{nn_3} (B^{m_1 m_3}_{n_1 n_3 n})^2 b^{m_1 m_3}_{n_1 n_3 n} \Bigg) \\ &- A_1 A_2 \bar{A}_2 \Bigg(\sum_{n=|n_2-n_1|}^{n_1+n_2} l_{nn_2} (B^{m_1-m_2}_{n_1 n_2 n})^2 b^{m_1-m_2}_{n_1 n_2 n} + \sum_{\substack{n=|n_2-n_1| \\ n\neq n_3}}^{n_1+n_2} l_{nn_2} (B^{m_1 m_2}_{n_1 n_2 n})^2 b^{m_1 m_2}_{n_1 n_2 n} \Bigg) \\ &- \sum_{k=4}^{N} 2\bar{A}_2 A_3 A_k \Bigg(\sum_{\substack{n=|n_1-n_3| \\ n\neq n_2}}^{n_1+n_3} l_{nn_3} (-1)^{m_3} (-1)^{m_2} B^{m_1-m_3}_{n_1 n_3 n} B^{m_2 0}_{n_2 n_k n} b^{m_2 0}_{n_2 n_k n} \\ &+ \sum_{\substack{n=|n_2-n_1| \\ n\neq n_3}}^{n_1+n_2} l_{nn_2} B^{m_1 m_2}_{n_1 n_2 n} B^{m_3 0}_{n_3 n_k n} b^{m_3 0}_{n_3 n_k n} + \sum_{\substack{n=|n_2-n_3| \\ n\neq n_1}}^{n_2+n_3} l_{nn_k} (-1)^{m_3} (-1)^{m_1} B^{m_1 0}_{n_1 n_k n} B^{m_2-m_3}_{n_2 n_3 n} b^{m_2-m_3}_{n_2 n_3 n} \Bigg) \\ &- \sum_{k=4}^{N} \sum_{l=4}^{N} 4 A_1 A_k A_l \sum_{\substack{n=|n_2-n_3| \\ n\neq n_1}}^{n_2+n_3} l_{nn_k} B^{m_1 0}_{n_1 n_l n} B^{m_1 0}_{n_1 n_k n} b^{m_1 0}_{n_1 n_k n} \Bigg]. \end{aligned} \tag{31}$$

To determine the analogous equation for A_2 apply the substitutions

$$m_1 \leftrightarrow m_2, \quad n_1 \leftrightarrow n_2, \quad A_1 \leftrightarrow A_2$$

to equation (31), and to produce the equation for A_3 use

$$m_1 \leftrightarrow -m_3, \quad n_1 \leftrightarrow n_3, \quad A_1 \to (-1)^{m_3}\bar{A}_3, \quad A_3 \to (-1)^{m_1}\bar{A}_1$$

The equations for A_k are given by

$$\frac{\partial A_k}{\partial \tau_2} = (A_1 A_2 \bar{A}_3 - \bar{A}_1 \bar{A}_2 A_3)\frac{1}{n_k(n_k+1)}\Bigg(- \sum_{\substack{n=|n_2-n_3| \\ n\neq n_1}}^{n_2+n_3} l_{nn_1} B^{m_1 0}_{n_1 n_k n} B^{m_2-m_3}_{n_2 n_3 n} b^{m_2-m_3}_{n_2 n_3 n}(-1)^{m_3}(-1)^{m_1}$$
$$- \sum_{\substack{n=|n_1-n_3| \\ n\neq n_2}}^{n_1+n_3} l_{nn_2} B^{m_2 0}_{n_2 n_k n} B^{m_1-m_3}_{n_1 n_3 n} b^{m_1-m_3}_{n_1 n_3 n}(-1)^{m_3}(-1)^{m_2} + \sum_{\substack{n=|n_1-n_2| \\ n\neq n_3}}^{n_1+n_2} l_{nn_3} B^{m_3 0}_{n_3 n_k n} B^{m_1 m_2}_{n_1 n_2 n} b^{m_1 m_2}_{n_1 n_2 n}\Bigg)$$
$$k = 4, \ldots, N. \tag{32}$$

These are the ordinary differential equations the functions $c_1(t), \ldots, c_N\,(t)$ have to satisfy. If we had treated τ_1 and τ_2 as independent variables, the two sets of partial differential equations would be inconsistent. This is easy to see for the zonal flow amplitudes, but it is also true for the amplitudes of the triad if the zonal flow amplitudes are assumed to be of order δ. In our asymptotic scheme, which is a variation of parameter method, the integration constants at a certain order are made time dependent to remove the secular terms at the next order. The questions of consistency of the usual multi-time-scale method do not arise. Our scheme leads to the result that starting out with two resonantly interacting waves, zonal flow is generated at $O(\delta^2)$ and the change in the zonal flow amplitudes is given by the system (32) of ordinary differential equations for $A_1(t) = c_1(t), \ldots, A_N(t) = c_N(t)$.

5. Worked example

Applying all the required conditions necessary for the occurrence of a Rossby-Haurwitz wave triad, the first thirteen triads for the lowest wavenumbers are given in table 1.

Table 1. The first 13 triads.

(m_1, n_1)	(m_2, n_2)	(m_3, n_3)
(1, 6)	(2,14)	(3, 9)
(1, 6)	(11, 20)	(12, 15)
(2, 6)	(3, 8)	(5, 7)
(2, 6)	(4, 14)	(6, 9)
(2, 7)	(11, 20)	(13, 14)
(2, 14)	(17, 20)	(19, 19)
(3, 6)	(6, 14)	(9, 9)
(3, 9)	(8, 20)	(11, 14)
(3, 14)	(1, 20)	(4, 15)
(4, 12)	(5, 14)	(9, 13)
(6, 14)	(2, 20)	(8, 15)
(6, 18)	(7, 20)	(13, 19)
(9, 14)	(3, 20)	(12, 15)

In order to show the workings of this theory an example is explicitly studied to generate numerical results. We take the first triad, namely, (1,6), (2,14) and (3,9). Therefore for the rest of this section we have

$$(m_1, n_1) = (1, 6), \quad (m_2, n_2) = (2, 14), \quad (m_3, n_3) = (3, 9).$$

Firstly we examine the numbers involved for our ψ_1 solution. From this we can examine the O(δ^2) equation and determine how many zonal flow terms (if any) are created. Due to the aforementioned problem with resonance all these terms will form part of the leading order solution. Using this leading order solution the required amplitude conditions are determined for these numbers.

To determine ψ_1 it should be noted that $B^{m_1 m_2}_{n_1 n_2 n}$ is only not zero for n between 9 and 19, with n odd. $B^{m_1 -m_3}_{n_1 n_3 n}$ is only not zero for n between 4 and 14, where n is even, and $B^{m_2 -m_3}_{n_2 n_3 n}$ is only not zero for n between 6 and 22, with n also even. The numbers obtained for these calculations are given in the following tables, table 2 for $B^{m_1 m_2}_{n_1 n_2 n}$, table 3 for $B^{m_1 -m_3}_{n_1 n_3 n}$ and finally table 4 for $B^{m_2 -m_3}_{n_2 n_3 n}$.

Table 2. Coupling constants $B^{m_1 m_2}_{n_1 n_2 n}$ and $b^{m_1 m_2}_{n_1 n_2 n}$.

n	$B^{m_1 m_2}_{n_1 n_2 n}$	$b^{m_1 m_2}_{n_1 n_2 n}$
11	17.2301i	−60i
13	12.4548i	−27.3913i
15	8.29596i	−16.8i
17	4.96375i	−11.6667i
19	2.34338i	−8.68966i

Table 3. Coupling constants $B^{m_1 -m_3}_{n_1 n_3 n}$ and $b^{m_1 -m_3}_{n_1 n_3 n}$.

n	$B^{m_1 -m_3}_{n_1 n_3 n}$	$b^{m_1 -m_3}_{n_1 n_3 n}$
4	−8.84799i	−13.2632i
6	−9.22981i	−15i
8	−6.00544i	−18.2609i
10	1.35831i	−25.2i
12	11.9378i	−46.6667i

Table 4. Coupling constants $B^{m_2 -m_3}_{n_2 n_3 n}$ and $b^{m_2 -m_3}_{n_2 n_3 n}$.

n	$B^{m_2 -m_3}_{n_2 n_3 n}$	$b^{m_2 -m_3}_{n_2 n_3 n}$
8	14.5228i	−84i
10	0.162338i	−37.0588i
12	−15.9295i	−22.1053i
14	−27.7451i	−15i
16	−29.5888i	−10.9565i
18	−17.2662i	−8.4i
20	9.12981i	−6.66667i
22	37.8582i	−5.43103i

With the coupling constants in ψ_1 for the triad determined we must examine the $O(\delta^2)$ equation to see what zonal flow terms are created. It is found that this example produces zonal flow at $O(\delta^2)$ for n ranging between 3 and 27, n odd. Therefore the solution which we must consider to examine the equation up to $O(\delta^2)$ is

$$\psi_0 = \sum_{j=1}^{16} \psi_0^j,$$

where

$$\psi_0^j = (A_j(\tau_1, \tau_2)e^{i(m_j\lambda - \sigma_j t)} + \bar{A}_j(\tau_1, \tau_2)e^{-i(m_j\lambda - \sigma_j t)})P_{n_j}^{m_j}(\mu),$$

$$\sigma_j = \frac{-2m_j}{n_j(n_j+1)},$$

$$m_j = 0, \quad A_j = \bar{A}_j, \quad j = 4, 5, \ldots, 16,$$

and

$$n_4 = 3, \quad n_5 = 5, \quad n_6 = 7, \ldots n_{16} = 27.$$

Using this new leading order solution we calculate the additional terms in ψ_1. The numbers produced for the $j = 3$ case are given in table 5. Band structures are similarly produced for both $B_{n_1 n_k n}^{m_1 0}$ and $B_{n_2 n_k n}^{m_2 0}$.

Using all these numbers we can calculate $\partial A_k / \partial \tau_2$ from the equation

$$\frac{\partial A_k}{\partial \tau_2} = \Lambda_k (A_1 A_2 \bar{A}_3 - \bar{A}_1 \bar{A}_2 A_3), \qquad k = 4, 5, \ldots, 16, \tag{33}$$

where Λ_k, $k = 4, 5, \ldots, 16$, are given in table 6.

Table 5. Coupling constant $B_{n_3 n_k n}^{m_3 0}$.

	n				
k	11	13	15	17	19
4	9.68024i	0	0	0	0
5	15.9894i	15.0291i	0	0	0
6	15.6375i	20.0493i	20.0394i	0	0
7	6.97454i	15.8943i	23.4323i	24.8466i	0
8	−7.92121i	3.19092i	15.6298i	26.4245i	29.5218i
9	−22.422i	−14.0255i	−0.611462i	15.0951i	29.17i
10	−26.0408i	−27.9296i	−19.6557i	−4.36602i	14.4062i
11	−8.30671i	−28.1223i	−32.8054i	−24.9489i	−8.05991i
12	28.6664i	−5.85785i	−29.8404i	−37.2756i	−29.9979i
13	0	33.4367i	−3.47309i	−31.3437i	−41.4713i
14	0	0	38.0873i	−1.14539i	−32.7115i
15	0	0	0	42.6545i	1.13458i
16	0	0	0	0	47.1613i

Table 6. The τ_2 derivative values for the zonal flow amplitudes.

k	Λ_k
4	7249.72i
5	14514.5i
6	20640.9i
7	24040.2i
8	23165.6i
9	15626.6i
10	8386.19i
11	−618.13i
12	1009.79i
13	−1009.82i
14	−2570.45i
15	−3481.92i
16	−3315.51i

6. Conclusions

This paper describes the generation of zonal flow through the resonant interactions of Rossby-Haurwitz waves. We have proved that this generation cannot take place at $O(\delta)$. Furthermore, if we start out with two nonresonantly interacting Rossby-Haurwitz waves then zonal flow will not be generated up to $O(\delta^2)$. However, two resonantly interacting Rossby-Haurwitz waves will generate zonal flow at order $O(\delta^2)$. This effect disappears in the β-plane approximation, i.e. in the limit of large wavenumbers for the triad. In this limit, a quartet mechanism is required.

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