Richardson Extrapolation: The Power of the 2-gon

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Richardson's extrapolation procedure yields a significant increase in the accuracy of numerical solutions of differential equations. We consider his elegant illustration of the technique, the evaluation of π, and show how the estimates improve dramatically with higher order extrapolation.

The Deferred Approach to the Limit

The editorial of the April issue of *Mathematics Today* drew attention to the fiftieth anniversary of the death of Lewis Fry Richardson. Richardson (1881–1953), a most original thinker, made significant advances in several branches of mathematics and science. A collection of his scientific papers has been published [1] and a biography of this eccentric and uniquely talented scientist has been written [2]. Modern weather forecasting follows essentially the lines mapped out in his magnum opus *Weather Prediction by Numerical Process* [3]. The Richardson number recalls his fundamental work on turbulence. He spent some years researching quantitative psychology and made notable contributions to the field. His final years were devoted to 'peace studies' and his work is the basis of modern conflict resolution research. In the course of his peace studies, he discovered the scaling properties of curves such as coastlines, inspiring Mandelbrot's theory of fractals. Most relevant in the current context is Richardson's introduction of finite difference methods for the solution of partial differential equations.

The finite difference method was developed by Richardson in his study of stresses in masonry dams [1]. The derivative dy/dx of a function $y(x)$ exists if the ratio $[(y(x+h)) - y(x)]/h$ tends to a definite limit as the increment h tends to zero. Differential calculus depends upon justifying the limiting process $h \to 0$. In approximating a differential equation by finite differences, we reverse the procedure, and replace derivatives by corresponding ratios of increments of the dependent and independent variables. Richardson [4] described the procedure thus:

> Although the infinitesimal calculus has been a splendid success, yet there remain problems in which it is cumbrous or unworkable. When such difficulties are encountered it may be well to return to the manner in which they did things before the calculus was invented, postponing the passage to the limit until after the problem has been solved for a moderate number of moderately small differences.

Richardson called this postponement of the process $h \to 0$ 'the deferred approach to the limit'. The accuracy of the finite difference approximation depends on the grid size h. Richardson was one of the first to analyse the behaviour of the errors. He was careful to use centered differences wherever possible, so the derivative dy/dx would be replaced by $[y(x+h) - y(x-h)]/2h$. For centered differences, the accuracy is typically of the second order; that is, the error diminishes like h^2 as $h \to 0$.

The quadratic behaviour of the errors provides a universal means of checking and correcting them. If the quantity $f(x)$ to be evaluated is estimated by $F(x, h)$, we assume

$$F(x,h) = f(x) + k_2(x)h^2 + k_4(x)h^4 + k_6(x)h^6 + \dots, \qquad (1)$$

where the functions $k_n(x)$ are generally unknown and where odd powers of h are absent. For small h it may be sufficient to retain only the first two terms, $F(x,h) = f(x) + k_2(x)h^2$. Then if F is computed for two different grid sizes, h_1 and h_2, the solutions being denoted F_1 and F_2, we can eliminate k_2 and obtain

$$f(x) = F_2 + \frac{\rho^2}{1-\rho^2}(F_2 - F_1) \qquad (2)$$

where $\rho = h_2/h_1$. This is what Richardson calls h^2-extrapolation. He notes that the method may be refined: if the solution is calculated for three values of h, the k_4 term may also be eliminated; this is h^4-extrapolation. The extension to higher orders is obvious. The h^{2n}-extrapolation requires the solution of a system of $n+1$ simultaneous linear equations

$$F_k = f(x) + k_2 h_k^2 + k_4 h_k^4 + \dots + k_{2n} h_k^{2n}, \qquad k = 1, 2, \dots, n+1.$$

It is an arithmetic burden, but an elementary task, to solve for $f(x)$.

Richardson extrapolation is the basis for a number of algorithms currently used for solving differential equations. For example, the Bulirsch-Stoer method, described in [5] (subroutine BSSTEP) is a modern implementation of the method. Marchuk [6] devotes an entire chapter of his book to the deferred approach to the limit. Körner [7] gives a good elementary description of the technique, while Roache [8] provides references to a large number of applications.

Estimating π by Extrapolation

To illustrate the power of his extrapolation method, Richardson considered the ancient problem of evaluating π. The approximation of π by inscribing and circumscribing polygons in a unit circle goes back at least to Archimedes. By an N-gon, we mean a regular polygon having N sides. In *Measurement of a Circle*, Archimedes squeezed the unit circle between two 96-gons, bounding within the interval [3.141,3.143]. A huge advance was made by the Persian mathematician Jamshīd al-Kāshī who in 1424 gave π correct to sixteen figures. Starting with an inscribed hexagon, he repeatedly doubled the number of sides, obtaining an exponentially converging set of approximations to π. It was two centuries before this was bettered by Ludolph van Cuelen. Al-Kāshī's value comes from a polygon with 805,306,368 sides [9].

Let us denote the estimate of π from an inscribed N-gon as $\Pi(N)$. A square inscribed in the unit circle has perimeter $4\sqrt{2}$, giving a crude estimate $\Pi(4) = 2\sqrt{2} \approx 2.828$. An inscribed hexagon has unit side, giving $\Pi(6) = 3$, better but still pretty poor. We define the resolution by $h = 1/N$ so that $\rho = h_2/h_1 = \frac{2}{3}$ Assuming the error is proportional to the square of the resolution, the errors in the two estimates are in the ratio $1/4^2$ to $1/6^2$ or 9:4. Richardson's h^2-extrapolation formula (2) then gives:

$$\Pi(4,6) = 3 + \tfrac{4}{5}(3 - 2\sqrt{2}) = 3.137$$

Table 1: Estimates of π for extrapolation using various combinations of polygons. The error of each estimate and the order of a single N-gon required to produce comparable accuracy are given.

N-gons used	Estimate Π	Error $\in_N$	Equivalent N-gon
2, 4	3.104	3.702×10^{-2}	12
2, 6	3.125	1.659×10^{-2}	18
4, 6	3.137	4.334×10^{-3}	35
2, 4, 6	3.14134	2.483×10^{-4}	145
3, 4, 6	3.14148	1.124×10^{-4}	215
3, 5, 6	3.1415204	7.219×10^{-5}	268
3, 4, 5, 6	3.1415920	6.203×10^{-7}	2887
2, 3, 4, 5, 6	3.14159264	1.378×10^{-8}	19364

which is correct to three figures. This is a dramatic improvement on the two estimates from which it is constructed. As Richardson noted, a single inscribed N-gon would require 35 sides to produce such accuracy.

We may wonder if even better estimates of π can be obtained without recourse to trigonometry, using Richardson's approach. Indeed this is the case. An inscribed 2-gon is the degenerate polygon comprising two coincident diameters. It yields the unimpressive estimate $\Pi(2)=2$. However, when used in conjunction with $\Pi(4)$ and $\Pi(6)$ in a h^4-extrapolation, it leads to the estimate $\Pi(2, 4, 6) = 3.14134$, correct to four digits. A single N-gon of order 145 is required to give similar precision. Spurred by this success, we consider higher order extrapolations. Retaining the simple spirit of Richardson, we 'return to the manner in which they did things before trigonometry was invented', and use only polygons whose sides can be computed by elementary geometry. An inscribed equilateral triangle has side $\sqrt{3}$. A pentagon requires more ingenuity, but can be drawn with ruler and compass and its edge length shown by elementary methods to be $\frac{1}{2}\sqrt{10-2\sqrt{5}}$ [10]. Table 1 gives estimates of π for extrapolation using various combinations of polygons. We see that using four polygons, an estimate $\Pi(3, 4, 5, 6) = 3.14159$ is obtained, correct to six figures and equivalent to a single N-gon with 2887 sides. However, the most surprising result is that obtained by adding in the 2-gon. The estimate then improves to $\Pi(2, 3, 4, 5, 6) = 3.14159264$ which has eight good digits and gives a result similar to an inscribed 19364-gon.

Discussion

Overcoming our earlier reluctance to use trigonometry, we note that the estimate of π from an inscribed N-gon may be expressed as $\Pi(N) = N\sin(\pi/N)$. By Taylor's theorem, the error in the estimate is

$$\in_N \equiv \pi - \Pi(N) = \frac{\pi^3}{6}\frac{1}{N^2} - \frac{\pi^5}{120}\frac{1}{N^4} + \cdots .$$

We see that the assumption (1) that $\in_N$ involves only even powers of $1/N$ is justified. Neglecting terms beyond second order, an N-gon required to yield an error $\in$ is of order

$$N = \sqrt{\frac{\pi^3}{6\in}}.$$

Values derived from this expression are included in Table 1. We may also note that the expression for the error using a circumscribed N-gon is obtained by replacing the sine in $\Pi(N)$ by a tangent and is

$$\varepsilon_N \equiv N\tan\left(\frac{\pi}{N}\right) - \pi = \frac{\pi^3}{3}\frac{1}{N^2} + O\left(\frac{1}{N^4}\right).$$

We see that the error at order h^2 is twice as large as in the case of inscribed N-gons. The tangent formula gives an appropriate value of ∞ for the circumscribed 2-gon.

The estimation of π is no longer a major concern of mainstream mathematics, although ever-higher accuracy is a challenge for computer scientists. A simple estimate such as $\frac{355}{113}$ gives seven correct digits, adequate for most problems. Moreover, Richardson's extrapolation process requires us to solve a linear system of equations, whose order increases with the order of the extrapolation. However, the important conclusion from the above results is that the extrapolation procedure can dramatically improve accuracy. Thus, combining the abysmally poor estimate $\Pi(2)$ with $\Pi(4,6)$ reduced the error by a factor of 17, and combining it with $\Pi(3, 4, 5, 6)$ resulted in a 45-fold reduction. The implication is that even a rough-and-ready estimate can enhance our knowledge if we have an understanding of the pattern of errors. Such a conclusion is of interest to practitioners such as meteorologists, who must solve systems of differential equations routinely, using initial data with significant errors.❑

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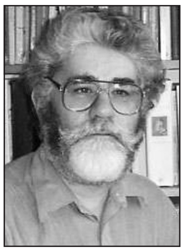

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