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If you fully understand the toy model, you should find the more realistic application straightforward.
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As an introduction to statistical estimation, we consider the simple problem, that we call the two temperatures problem: Given two independent observations $T_1$ and $T_2$, determine the best estimate of the true temperature $T_t$. 
Simple (toy) Example

Let the two observations of temperature be

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\begin{align*}
T_1 &= T_t + \varepsilon_1 \\
T_2 &= T_t + \varepsilon_2
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[For example, we might have two *iffy* thermometers].
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The observations have errors \( \varepsilon_i \), which we don’t know.

Let \( E(\ ) \) represent the expected value, i.e., the average of many similar measurements.

We assume that the measurements \( T_1 \) and \( T_2 \) are unbiased:

\[
E(T_1 - T_t) = 0, \quad E(T_2 - T_t) = 0
\]

or equivalently,

\[
E(\varepsilon_1) = E(\varepsilon_2) = 0
\]
We also assume that we know the variances of the observational errors:

\[ E(\varepsilon^2_1) = \sigma^2_1 \quad E(\varepsilon^2_2) = \sigma^2_2 \]
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The above equations represent the statistical information that we need about the actual observations.
We estimate $T_t$ as a **linear combination** of the observations:

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$T_a$ will be the best estimate of $T_t$ if the coefficients are chosen to minimize the mean squared error of $T_a$:

$$\sigma_a^2 = E[(T_a - T_t)^2] = E\{[a_1(T_1 - T_t) + a_2(T_2 - T_t)]^2\}$$

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This may be written

$$\sigma_a^2 = E[(a_1\varepsilon_1 + a_2\varepsilon_2)^2]$$
Expanding this expression for $\sigma_a^2$, we get

$$\sigma_a^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2$$

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Naïve solution: \( \partial \sigma_a^2 / \partial a_1 = 2a_1 \sigma_1^2 = 0 \), so $a_1 = 0$.
Similarly, $\partial \sigma_a^2 / \partial a_2 = 0$ implies $a_2 = 0$. 
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**Equating the derivative w.r.t.** $a_1$ **to zero**, $\partial \sigma_a^2 / \partial a_1 = 0$, **gives**

$$a_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad a_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$
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$$a_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad a_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

Thus, we have expressions for the weights $a_1$ and $a_2$ in terms of the variances (which are assumed to be known).
We define the **precision** to be the inverse of the variance. It is a measure of the accuracy of the observations.
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Substituting the coefficients in \( \sigma_a^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 \), we obtain

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This can be written in the alternative form:

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\frac{1}{\sigma_a^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}
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Thus, **if the coefficients are optimal, the precision of the analysis is the sum of the precisions of the measurements.**
Variational approach

We can also obtain the same best estimate of $T_t$ by minimizing a *cost function*. 
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The cost function is defined as the sum of the squares of the distances of $T$ to the two observations, weighted by their observational error precisions:

$$J(T) = \frac{1}{2} \left[ \frac{(T - T_1)^2}{\sigma_1^2} + \frac{(T - T_2)^2}{\sigma_2^2} \right]$$
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Exercise: Prove that $\partial J/\partial T = 0$ gives the same value for $T_a$ as the least squares method.
The control variable for the minimization of $J$ (i.e., the variable with respect to which we are minimizing the cost function) is the temperature.

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The equivalence between the minimization of the analysis error variance and the variational cost function approach is important.
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The equivalence between the minimization of the analysis error variance and the variational cost function approach is important.

This equivalence also holds true for multidimensional problems, in which case we use the covariance matrix rather than the scalar variance.

It indicates that OI and 3D-Var are solving the same problem by different means.

*   *   *   *
Example: Suppose $T_1 = 2 \quad \sigma_1 = 2 \quad T_2 = 0 \quad \sigma_2 = 1$. 
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Show that $T_a = 0.4$ and $\sigma_a = \sqrt{0.8}$.

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a_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{1}{5} \quad a_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \frac{4}{5}
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This solution is illustrated in the next figure.
The probability distribution for a simple case.

The analysis has a pdf with a maximum closer to $T_2$, and a smaller standard deviation than either observation.
Conclusion of the foregoing.
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The requirement that the analysis be unbiassed led to
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Assume that one of the two temperatures, say $T_1 = T_b$, is not an observation, but a background value, such as a forecast.

Assume that the other value is an observation, $T_2 = T_o$. 

\[ \text{Simple Sequential Assimilation} \]
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Assume that the other value is an observation, \( T_2 = T_o \).

We can write the analysis as

\[ T_a = T_b + W(T_o - T_b) \]

where \( W = a_2 \) can be expressed in terms of the variances.
The least squares method gave us the optimal weight:

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the quantity \((T_o - T_b)\) is called the *observational innovation*, i.e., the new information brought by the observation.
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It is also known as the observational increment (with respect to the background).
The analysis error variance is, as before, given by

\[ \frac{1}{\sigma_a^2} = \frac{1}{\sigma_b^2} + \frac{1}{\sigma_o^2} \quad \text{or} \quad \sigma_a^2 = \frac{\sigma_b^2 \sigma_o^2}{\sigma_b^2 + \sigma_o^2} \]
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Exercise: Verify all the foregoing formulæ.
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We have shown that the simple two-temperatures problem serves as a paradigm for the problem of objective analysis of the atmospheric state.
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\[ T_a = T_b + W(T_o - T_b) \]

\[ W = \frac{\sigma_b^2}{\sigma_b^2 + \sigma_o^2} \]

\[ \sigma_a^2 = \frac{\sigma_b^2 \sigma_o^2}{\sigma_b^2 + \sigma_o^2} = W \sigma_o^2 \]

\[ \sigma_a^2 = (1 - W) \sigma_b^2 \]
These four equations have been derived for the simplest scalar case . . .

. . . but they are important for the problem of data assimilation because they have exactly the same form as more general equations:
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The least squares sequential estimation method is used for real multidimensional problems (OI, interpolation, 3D-Var and even Kalman filtering).
These four equations have been derived for the simplest scalar case . . .

. . . but they are important for the problem of data assimilation because they have exactly the same form as more general equations:

The least squares sequential estimation method is used for real multidimensional problems (OI, interpolation, 3D-Var and even Kalman filtering).

Therefore we will interpret these four equations in detail.
The first equation

\[ T_a = T_b + W(T_o - T_b) \]
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This says:

The analysis is obtained by adding to the background value, or first guess, the innovation (the difference between the observation and first guess), weighted by the optimal weight.
The second equation

\[ W = \frac{\sigma_b^2}{\sigma_b^2 + \sigma_o^2} \]

This says:

The optimal weight is the background error variance multiplied by the inverse of the total error variance (the sum of the background and the observation error variances).
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* * *

Look at the limits: \( \sigma^2_o = 0 \); \( \sigma^2_b = 0 \).
The third equation

The variance of the analysis is

\[ \sigma_a^2 = \frac{\sigma_b^2 \sigma_o^2}{\sigma_b^2 + \sigma_o^2} \]
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This says:

The precision of the analysis (inverse of the analysis error variance) is the sum of the precisions of the background and the observation.
The fourth equation

\[ \sigma_a^2 = (1 - W) \sigma_b^2 \]

This says:

The error variance of the analysis is the error variance of the background, reduced by a factor equal to one minus the optimal weight.
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All the above statements are important because they also hold true for sequential data assimilation systems (OI and Kalman filtering) for multidimensional problems.
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In these problems, in which $T_b$ and $T_a$ are three-dimensional fields of size order $10^7$ and $T_o$ is a set of observations (typically of size $10^5$), we have to replace expressions as follows:
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- error variance \( \Rightarrow \) error covariance matrix
- optimal weight \( \Rightarrow \) optimal gain gain matrix.

Note that there is one essential tuning parameter in OI:

It is the ratio of the observational variance to the background error variance:

\[
\left( \frac{\sigma_o}{\sigma_b} \right)^2
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Assume that we have completed the analysis at time $t_i$ (e.g., at 06 UTC), and we want to proceed to the next cycle (time $t_{i+1}$, or 12 UTC).
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The analysis cycle has two phases, a forecast phase to update the background \( T_b \) and its error variance \( \sigma_b^2 \), and an analysis phase, to update the analysis \( T_a \) and its error variance \( \sigma_a^2 \).
Typical 6-hour analysis cycle.
In the **forecast phase of the analysis cycle**, the background is first obtained through a forecast:

\[ T_b(t_{i+1}) = M [T_a(t_i)] \]

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In OI, this is obtained by making a suitable simple assumption, such as that the model integration increases the initial error variance by a fixed amount, a factor \( a \) somewhat greater than 1:

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We also need the error variance of the background. In OI, this is obtained by making a suitable simple assumption, such as that the model integration increases the initial error variance by a fixed amount, a factor \( a \) somewhat greater than 1:

\[ \sigma^2_b(t_{i+1}) = a\sigma^2_a(t_i) \]

This allows the new weight \( W(t_{i+1}) \) to be estimated using

\[ W = \frac{\sigma^2_b}{\sigma^2_b + \sigma^2_o} \]
Analysis Phase

In the analysis phase of the cycle we get the new observation $T_o(t_{i+1})$, and we derive the new analysis $T_a(t_{i+1})$ using

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It is smaller than the background error.
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The new analysis error variance $\sigma^2_a(t_{i+1})$ comes from

$$\sigma^2_a = (1 - W)\sigma^2_b$$

It is smaller than the background error.

After the analysis, the cycle for time $t_{i+1}$ is completed, and we can proceed to the next cycle.
Study the Remarks in Kalnay, §5.3.1