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However, it has a number of limitations. In particular, it is not straightforward to apply NNMI in *limited geographical domains*.

Recently, an alternative method of initialization, called *digital filter initialization* (DFI), was introduced.

In this lecture we review DFI, and describe how the method is applied in operational NWP.
The Notion of Filtering

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A filter is any device or contrivance designed to carry out such a selection.

It may be represented by a simple system diagram, having an input with both desired and undesired components, and an output comprising only the former.

\[
\text{Good/Bad/Ugly} \quad \implies \quad \text{Filter} \quad \rightarrow \quad \text{Good}
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Other ideal filters can be discussed:

- High-pass filters
- Band-pass filters
- Notch filters

But the Low-Pass Filter is the one needed for initialization.
Frequency response of ideal low-pass filter.
Given a discrete function of time, \( \{ x_n \} \), a nonrecursive digital filter is defined by

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Nonrecursive and Recursive Filters

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y_n = \sum_{k=K}^{N} a_k x_{n-k} + \sum_{k=1}^{L} b_k y_{n-k}
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where \( L \) and \( N \) are positive integers. Usually, \( K = 0 \).
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To find the frequency response of a recursive filter, let

$$x_n = \exp(in\theta)$$

and assume an output of the form

$$y_n = H(\theta) \exp(in\theta)$$
Substitute $y_n = H(\theta) \exp(i n \theta)$ into the defining formula

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$$H(\theta) = \frac{\sum_{k=K}^{N} a_k e^{-i k \theta}}{1 - \sum_{k=1}^{L} b_k e^{-i k \theta}}.$$
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For nonrecursive filters the denominator reduces to unity:

$$H(\theta) = \sum_{k=-N}^{N} a_k e^{-ik\theta}$$
Response function of a FIR:

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The entire area of filter design is concerned with finding filters having desired properties.

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Typically, \( H_c(\omega) \) is a step function

\[
H_c(\omega) = \begin{cases} 
1, & |\omega| \leq |\omega_c|; \\
0, & |\omega| > |\omega_c|, 
\end{cases}
\]

where \( \omega_c \) is a cutoff frequency.
Equivalence of filtering and convolution.

\[(h \ast f)(t) = \mathbf{F}^{-1}\{\mathbf{F}\{h\} \cdot \mathbf{F}\{f\}\}\]
These three steps are equivalent to a convolution of $f(t)$ with the inverse Fourier transform of $H_c(\omega)$ ($h(t) = \sin(\omega_c t)/\pi t$).

This follows from the convolution theorem

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Suppose now that $f$ is known only at discrete moments $t_n = n\Delta t$, so that the sequence $\{\cdots, f_{-2}, f_{-1}, f_0, f_1, f_2, \cdots\}$ is given.
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For example, \( f_n \) could be the value of some model variable at a particular grid point at time \( t_n \).
The shortest period component which can be represented with a time step $\Delta t$ is $\tau_{Ny} = 2\Delta t$, corresponding to a maximum frequency, the so-called Nyquist frequency, $\omega_{Ny} = \pi/\Delta t$. 
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The sequence $\{f_n\}$ may be regarded as the Fourier coefficients of a function $F(\theta)$:

$$F(\theta) = \sum_{n=-\infty}^{\infty} f_ne^{-in\theta},$$

where $\theta = \omega\Delta t$ is the digital frequency and $F(\theta)$ is periodic with period $2\pi$: $F(\theta) = F(\theta + 2\pi)$. [Note: $\theta_{Ny} = \omega_{Ny}\Delta t = \pi$]
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High frequency components of the sequence may be eliminated by multiplying $F(\theta)$ by a function $H_d(\theta)$ defined by

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The cutoff frequency \( \theta_c = \omega_c \Delta t \) is assumed to fall in the Nyquist range \( (-\pi, \pi) \) and \( H_d(\theta) \) has period \( 2\pi \).
The function $H_d(\theta)$ may be expanded:

$$H_d(\theta) = \sum_{n=-\infty}^{\infty} h_n e^{-in\theta} ; \quad h_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\theta) e^{in\theta} d\theta.$$
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Clearly,

$$H_d(\theta) \cdot F(\theta) = \sum_{n=-\infty}^{\infty} f_n^* e^{-in\theta}.$$
The convolution theorem now implies that $H_d(\theta) \cdot F(\theta)$ is the transform of the convolution of $\{h_n\}$ with $\{f_n\}$:

$$f_n^* = (h * f)_n = \sum_{k=-\infty}^{\infty} h_k f_{n-k}.$$
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\* \* \* \*

In practice the summation must be **truncated**.

Thus, an approximation to the LF part of $\{f_n\}$ is given by

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We see that the finite approximation to the discrete convolution is identical to a nonrecursive digital filter.
Gibbs oscillations

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The response of the filter is improved if $h_n$ is multiplied by the Lanczos window:

$$w_n = \frac{\sin\left(n\pi/(N + 1)\right)}{n\pi/(N + 1)}.$$
Truncation of a Fourier series gives rise to **Gibbs oscillations**. These may be greatly reduced by means of an appropriately defined “window” function.

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$$w_n = \frac{\sin \left( \frac{n\pi}{N+1} \right)}{\frac{n\pi}{N+1}}.$$

**Exercise:** Write a **MATLAB** program to compute the FFT of a step function with various truncations. Thus investigate the Gibbs phenomenon.

*   *   *   *

The truncated Fourier analysis of a square wave is shown in the following figures.
Original Square wave function.
Truncation: $N = 11 \ (N_{\text{max}} = 50)$
Truncation: $N = 21 \ (N_{\text{max}} = 50)$
\textbf{Truncation: } \( N = 31 \ (N_{\text{max}} = 50) \)
Truncation: $N = 41 \ (N_{\text{max}} = 50)$
Original Square wave function.
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With the time step \( \Delta t = 6 \) minutes, this corresponds to a (digital) cutoff frequency \( \theta_c = \pi/30 \).
An initialization scheme using a nonrecursive digital filter was developed by Lynch and Huang (1992) for HiRLAM.

The value chosen for the cutoff frequency corresponded to a period $\tau_c = 6$ hours.

With the time step $\Delta t = 6$ minutes, this corresponds to a (digital) cutoff frequency $\theta_c = \pi/30$.

The coefficients were derived by Fourier expansion of a step-function, truncated at $N = 30$, with a Lanczos window:

$$h_n = \left[ \frac{\sin(n\pi/(N+1))}{n\pi/(N+1)} \right] \left( \frac{\sin(n\theta_c)}{n\pi} \right).$$
The use of the window decreases the Gibbs oscillations in the stop-band $|\theta| > |\theta_c|$.

However, it also has the effect of widening the pass-band beyond the nominal cutoff.

For a fuller discussion of windowing see e.g. Hamming (1989) or Oppenheim and Schafer (1989).
The central lobe of the coefficient function spans a period of six hours, from $t = -3\, \text{h}$ to $t = +3\, \text{h}$: $T_{\text{Span}} = 6\, \text{hours}$.

The filter summation was calculated over this range, with the coefficients normalized to have unit sum over the span.
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\[
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The uninitialized fields of surface pressure, temperature, humidity and winds were first integrated forward for three hours, and running sums of the form

\[
f_F^*(0) = \frac{1}{2}h_0f_0 + \sum_{n=1}^{N} h_{-n}f_n,
\]

where \( f_n = f(n\Delta t) \), were calculated for each field at each gridpoint and on each model level.
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These were stored at the end of the three hour forecast.
Repeat:

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The original fields were then used to make a three hour ‘hindcast’, during which running sums

\[ f_B^*(0) = \frac{1}{2} h_0 f_0 + \sum_{n=-1}^{-N} h_{-n} f_n \]

were accumulated for each field, and stored as before.
Repeat:

\[ f_\star^F(0) = \frac{1}{2} h_0 f_0 + \sum_{n=1}^{N} h_{-n} f_n, \]

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\[ f^\star(0) = f_\star^F(0) + f_\star^B(0) = \sum_{n=-N}^{-N} h_{-n} f_n. \]
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\[ f^*_B(0) = \frac{1}{2} h_0 f_0 + \sum_{n=-1}^{-N} h_{-n} f_n \]

were accumulated for each field, and stored as before. The two sums were then combined to give

\[ f^*(0) = f^*_F(0) + f^*_B(0) = \sum_{n=-N}^{-N} h_{-n} f_n. \]

These fields correspond to the application of the digital filter to the original data. They are the filtered data.
Phase Errors

In the foregoing, only the amplitudes of the transfer functions have been discussed.

Since these functions are complex, there is also a phase change induced by the filters.
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It is salutary to recall that phase-errors are amongst the most prevalent and pernicious problems in forecasting.
Break here
The Dolph-Chebyshev Filter

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We use the **Chebyshev polynomials**, defined by

\[
T_n(x) = \begin{cases} 
\cos(n \cos^{-1} x), & |x| \leq 1; \\
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In the interval \(|x| \leq 1\), \( T_n(x) \) oscillates between +1 and −1.
Now consider the function defined in the frequency domain:

\[ H(\theta) = \frac{T_{2M}(x_0 \cos(\theta/2))}{T_{2M}(x_0)}, \]

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- As \( \theta \) varies from 0 to \( \theta_s \), \( H(\theta) \) falls from 1 to \( r = 1/T_{2M}(x_0) \).
- For \( \theta_s \leq \theta \leq \pi \), \( H(\theta) \) oscillates in the range \( \pm r \).
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The form of \( H(\theta) \) is that of a low-pass filter with a cut-off at \( \theta = \theta_s \).

By means of the definition of \( T_n(x) \) and basic trigonometric identities, \( H(\theta) \) can be written as a finite expansion

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The coefficients \( \{h_n\} \) may be evaluated from the inverse Fourier transform

\[ h_n = \frac{1}{N} \left[ 1 + 2r \sum_{m=1}^{M} T_{2M} \left( x_0 \cos \frac{\theta_m}{2} \right) \cos m\theta_n \right], \]

where \(|n| \leq M\), \( N = 2M + 1 \) and \( \theta_m = 2\pi m/N \).
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The weights \( \{h_n : -M \leq n \leq +M\} \) define the **Dolph-Chebyshev** or, for short, **Dolph filter**.
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The desired frequency cut-off is specified by choosing a value for the cut-off period, \( \tau_s \).

Then \( \theta_s = 2\pi \Delta t / \tau_s \) and the parameters \( x_0 \) and \( r \) are given by

\[
\frac{1}{x_0} = \cos \frac{\theta_s}{2}, \quad \frac{1}{r} = \cosh \left( 2M \cosh^{-1} x_0 \right).
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The ripple ratio $r$ is a measure of the maximum amplitude in the stop-band $[\theta_s, \pi]$:

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The Dolph filter has minimum ripple-ratio for a given main-lobe width and filter order.
Example of Dolph Filter

Suppose components with period less than three hours are to be eliminated ($\tau_s = 3 \text{ h}$) and the time step is $\Delta t = \frac{1}{8} \text{ h}$. 
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The parameters chosen for the DFI are:

- **Span** \(T_S = 2 \text{ h}\)
- **Cut-off period** \(\tau_s = 3 \text{ h}\)
- **Time step** \(\Delta t = 450 \text{ s} = \frac{1}{8} \text{ h}\)
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So, \(M = 8,\, N = 17\) and \(\theta_s = 2\pi \Delta t / \tau_s \approx 0.26\).
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So, $M = 8$, $N = 17$ and $\theta_s = 2\pi \Delta t / \tau_s \approx 0.26$.

The DFI procedure employed in the HiRLAM model involves a double application of the filter.

We examine the frequency response $H(\theta)$ and its square, $H(\theta)^2$ (a second pass squares the frequency response).
Frequency response for Dolph filter with span $T_S = 2h$, order $N = 2M + 1 = 17$ and cut-off $\tau_s = 3h$. Results for single and double application are shown.

Logarithmic response (dB) as a function of frequency.
Frequency response for Dolph filter with span $T_S = 2h$, order $N = 2M + 1 = 17$ and cut-off $\tau_s = 3h$. Results for single and double application are shown.

Amplitude response as a function of period.
The ripple ratio of the filter has the value $r = 0.241$. 
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A single pass attenuates high frequencies (components with \(|\theta| > |\theta_s|\)) by at least 12.4dB.

For a double pass, the minimum attenuation is about 25dB, more than adequate for elimination of HF noise.
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The amplitudes of components with periods less than two hours are reduced to less than 5% of their original value.

Components with periods greater than one day are left substantially unchanged.
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It is crucial that an initialization scheme does not distort the meteorologically significant components of the flow.

It can be proved (Lynch, 1997) that the Dolph window is an optimal filter whose pass-band edge, \( \theta_p \), is the solution of the equation \( H(\theta) = 1 - r \).
The digital filter initialization is performed by applying the filter to time series of model variables.
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The filter is applied in two stages:

In the first stage, a backward integration from \( t = 0 \) to \( t = -T_S \) is performed, with all irreversible physics switched off.
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The filter output is calculated by accumulating the sums

\[
\bar{x} = \sum_{n=0}^{n=-N} h_{N-n}x_n.
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The output \( \bar{x} \) is valid at time \( t = -\frac{1}{2}T_S \).
Implementation in HIRLAM

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$$\bar{x} = \sum_{n=-N}^{n=0} h_{N-n} x_n.$$ 

The output $\bar{x}$ is valid at time $t = -\frac{1}{2}T_S$.

In the second stage, a forward integration is made from $t = -\frac{1}{2}T_S$ to $t = +\frac{1}{2}T_S$, starting from the output $\bar{x}$. 
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In the second stage, a \textbf{forward integration} is made from $t = -\frac{1}{2}T_S$ to $t = +\frac{1}{2}T_S$, starting from the output $\bar{x}$.

Again, the filter is applied by accumulating sums formally identical to those of the first stage.
The output of the second stage is valid at the centre of the interval \([-\frac{1}{2}T_S, +\frac{1}{2}T_S]\), i.e., at \(t = 0\).
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The output of the second pass is the initialized data.
The basic measure of noise is the mean absolute value of the surface pressure tendency

\[ N_1 = \left( \frac{1}{\text{NGRID}} \right) \sum_{n=1}^{\text{NGRID}} \left| \frac{\partial p_s}{\partial t} \right|. \]
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For well balanced fields this quantity has a value of about 1 hPa per 3 hours.

For uninitialized fields it can be up to an order of magnitude larger.
DFI: Sample Results

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In the following figure, we plot the value of \( N_1 \) for three forecasts.
Mean absolute surface pressure tendency for three forecasts. Forecast with no initialization (NIL); normal mode initialization (NMI); digital filter initialization (DFI). Units are hPa/3 hours.
The measure $N_1$ indicates the noise in the vertically integrated divergence field.

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The uninitialized vertical velocity field is physically quite unrealistic.

The DFI vertical velocity is much smoother, and much more realistic.
Vertical velocity at 500 hPa for uninitialized analysis (NIL).
Vertical velocity at 500 hPa after digital filtering (DFI).
Root mean square divergence at each model level.
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8. Applicable to non-hydrostatic models.
End of §4.3