3.2.6. Semi-Lagrangian Advection

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In numerical weather prediction (NWP), \textit{timeliness} of the forecast is of the essence.

In this lecture, we study an alternative approach to time integration, which is \textit{unconditionally stable} and so, free from the shackles of the CFL condition.
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In a fully Lagrangian scheme, the trajectories of actual physical parcels of fluid would be followed throughout the motion.

The problem with this approach, is that the distribution of representative parcels rapidly becomes highly non-uniform. In the semi-Lagrangian scheme the individual parcels are followed only for a single time-step. After each step, we revert to a uniform grid.
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The first operational implementation of a semi-Lagrangian scheme was in 1982 at the Irish Meteorological Service.

Semi-Lagrangian advection schemes are now in widespread use in all the main Numerical Weather Prediction centres.
Multiply-Upstream, Semi-Lagrangian Adveotive Schemes: Analysis and Application to a Multi-Level Primitive Equation Model

J. R. Bates and A. McDonald

Irish Meteorological Service, Dublin, Ireland

(Manuscript received 12 April 1982, in final form 16 September 1982)

ABSTRACT

The stability properties of some simple semi-Lagrangian advective schemes, based on a multiply-upstream interpolation, are examined. In these schemes, the interpolation points are chosen to surround the departure points of the fluid particles at the beginning of a time step. It is shown that the schemes, though explicit, are unconditionally stable for a constant wind field.

Application of the schemes to a multi-level split explicit model shows that they enable full advantage to be taken of the splitting method by allowing a long time step for advection. It is shown that they can thus lead to a considerable saving of computer time compared to Eulerian schemes, while giving comparable accuracy.

We consider the *linear advection equation* which describes the conservation of a quantity $Y(x, t)$ following the motion of a fluid flow in one dimension with constant velocity $c$. 
Eulerian and Lagrangian Approach

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This may be written in either of two alternative forms:

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\frac{\partial Y}{\partial t} + c \frac{\partial Y}{\partial x} = 0 \quad \Leftarrow \quad \text{Eulerian Form}
\]

\[
\frac{dY}{dt} = 0 \quad \Leftarrow \quad \text{Lagrangian Form}
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The general solution is $Y = Y(x - ct)$.
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The general solution is $Y = Y(x - ct)$.

To develop numerical solution methods, we may start from either the Eulerian or the Lagrangian form of the equation. For the semi-Lagrangian scheme, we choose the latter.
Since the advection equation is linear, we can construct a general solution from Fourier components

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This expression may be separated into the product of a function of space and a function of time:

\[ Y = a \times \exp(-i\omega t) \times \exp(ikx) ; \quad \omega = kc . \]
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Therefore, in analysing the properties of numerical schemes, we seek a solution of the form

\[ Y_{nm}^n = a \times \exp(-i\omega n\Delta t) \times \exp(ikm\Delta x) = aA^n\exp(ikm\Delta x) \]

where \( A = \exp(-i\omega \Delta t) \).
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where \( A = \exp(-i\omega \Delta t) \).

The character of the solution depends on the modulus of \( A \):

- If \( |A| < 1 \), the solution \textit{decays} with time.
- If \( |A| = 1 \), the solution is \textit{neutral} with time.
- If \( |A| > 1 \), the solution \textit{grows} with time.
Since the advection equation is linear, we can construct a general solution from Fourier components

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- If \( |A| > 1 \), the solution grows with time.

In the third case (growing solution), the scheme is unstable.
Numerical Domain of Dependence.  

Space axis horizontal  

Time axis vertical  

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m-3  m-2  m-1  m  m+1  m+2  m+3
Numerical Domain of Dependence.

For the Eulerian Leap from Scheme, the value $Y^n_m$ at time $n \Delta t$ and position $m \Delta x$ depends on values within the area depicted by asterisks. Values outside this region have no influence on $Y^n_m$. 

**Space axis horizontal**

**Time axis vertical**
Numerical Domain of Dependence

Each computed value $Y^n_m$ depends on previously computed values and on the initial conditions. The set of points which influence the value $Y^n_m$ is called the *numerical domain of dependence* of $Y^n_m$. 
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Each computed value $Y_{nm}^n$ depends on previously computed values and on the initial conditions. The set of points which influence the value $Y_{nm}^n$ is called the *numerical domain of dependence* of $Y_{nm}^n$.

It is clear on physical grounds that if the parcel of fluid arriving at point $m\Delta x$ at time $n\Delta t$ originates *outside the numerical domain of dependence*, the numerical scheme cannot yield an accurate result: the necessary information is not available to the scheme.
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Worse again, the numerical solution may bear absolutely no relationship to the physical solution and may grow exponentially with time even when the true solution is bounded.
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Worse again, the numerical solution may bear absolutely no relationship to the physical solution and may grow exponentially with time even when the true solution is bounded.

A necessary condition for avoidance of this phenomenon is that the numerical domain of dependence should include the physical trajectory. This condition is fulfilled by the semi-Lagrangian scheme.
Parcel coming from Outside Domain of Dependence

\[
\begin{align*}
\text{m-5} & \quad \text{m-4} & \quad \text{m-3} & \quad \text{m-2} & \quad \text{m-1} & \quad \text{m} & \quad \text{m+1}
\end{align*}
\]
Parcel coming from Outside Domain of Dependence

The line of bullets (●) represents a parcel trajectory ($\mu = \frac{5}{3}$). The value everywhere on the trajectory is $Y^m_n$. ($c = 5\Delta x/3\Delta t$).
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The value everywhere on the trajectory is $Y^m_m$. ($c = 5\Delta x/3\Delta t$).
Since the parcel originates outside the numerical domain of dependence, the Eulerian scheme cannot model it correctly.
The central idea of the Lagrangian scheme is to represent the physical trajectory of the fluid parcel.
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We consider a parcel arriving at gridpoint \( m \Delta x \) at the new time \( (n + 1) \Delta t \) and ask: Where has it come from?
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We consider a parcel *arriving* at gridpoint $m\Delta x$ at the new time $(n + 1)\Delta t$ and ask: *Where has it come from?*

The *departure point* will not normally be a grid point. Therefore, the value at the departure point must be calculated by *interpolation from surrounding points*.
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But this interpolation ensures that the trajectory falls within the numerical domain of dependence.

We will show that this leads to a *numerically stable scheme*. 
Interpolation using Surrounding Points

The line of circles (○) represents a parcel trajectory \( (c = \frac{5\Delta x}{3\Delta t}) \)
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At time \(n\Delta t\) the parcel is at (●), which is not a grid-point.

The value at the departure point is obtained by interpolation from surrounding points.

Thus we ensure that, even though \(\mu = \frac{5}{3} > 1\), the physical trajectory is within the domain of numerical dependence.
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In physical terms, this equation says that the value of \( Y \) is constant for a fluid parcel.
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Applying the equation over the time interval \([n\Delta t, (n+1)\Delta t]\), we get

\[
\begin{pmatrix}
\text{Value of } Y \text{ at point } m\Delta x \\
\text{at time } (n+1)\Delta t
\end{pmatrix}
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\end{pmatrix}
\]

In a more compact form, we may write

\[ Y_{m}^{n+1} = Y_{\bullet}^{n} \]

where \( Y_{\bullet}^{n} \) represents the value at the departure point, which is normally not a grid point.
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The Courant Number is $\mu = \frac{c \Delta t}{\Delta x}$. Here, $\mu = \frac{5}{3}$. We define:

$$p = \lfloor \mu \rfloor = \text{Integral part of } \mu$$

$$\alpha = \mu - p = \text{Fractional part of } \mu$$

Note that, by definition, $0 \leq \alpha < 1$ (here, $p = 1$ and $\alpha = \frac{2}{3}$).
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A linear interpolation gives

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Y^n_\bullet = \alpha Y^n_{m-p-1} + (1 - \alpha)Y^n_{m-p}.
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A linear interpolation gives

$$Y^n_\bullet = \alpha Y^n_{m-p-1} + (1 - \alpha)Y^n_{m-p}.$$ 

Check: Show what this implies in the limits $\alpha = 0$ and $\alpha \to 1$. 
Break here
The discrete equation may be written

\[ Y_{m}^{n+1} = \alpha Y_{m-p}^{n} + (1 - \alpha) Y_{m-p}^{n}. \]
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Substituting into the equation we get

\[ aA^{n+1} \exp(ikm\Delta x) = \alpha \cdot aA^n \exp[ik(m - p - 1)\Delta x] \]
\[ + (1 - \alpha) \cdot aA^n \exp[ik(m - p)\Delta x] \]
Numerical Stability of the Scheme

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Removing the common term \( a A^{n} \exp(ikm\Delta x) \), we get

\[ A = \alpha \exp[ik(-p - 1)\Delta x] + (1 - \alpha) \exp[ik(-p)\Delta x] \]
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We can write this as

\[ A = \exp(-ikp\Delta x) \cdot [(1 - \alpha) + \alpha \exp(-ik\Delta x)] \]
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Now consider the squared modulus of \( A \):

\[
|A|^2 = |\exp(-ikp\Delta x)|^2 \cdot |(1 - \alpha) + \alpha \exp(-ik\Delta x)|^2 \\
= |(1 - \alpha) + \alpha \cos k\Delta x - i\alpha \sin k\Delta x|^2 \\
= [(1 - \alpha) + \alpha \cos k\Delta x]^2 + \alpha^2 \sin^2 k\Delta x \\
= (1 - \alpha)^2 + 2(1 - \alpha)\alpha \cos k\Delta x + \alpha^2 \cos^2 k\Delta x + \alpha^2 \sin^2 k\Delta x \\
= (1 - 2\alpha + \alpha^2) + 2\alpha(1 - \alpha) \cos k\Delta x + \alpha^2 \\
= 1 - 2\alpha(1 - \alpha)[1 - \cos k\Delta x].
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= |(1 - \alpha) + \alpha \cos k\Delta x - i\alpha \sin k\Delta x|^2
\]

\[
= [(1 - \alpha) + \alpha \cos k\Delta x]^2 + \alpha [\sin k\Delta x]^2
\]

\[
= (1 - \alpha)^2 + 2(1 - \alpha)\alpha \cos k\Delta x + \alpha^2 \cos^2 k\Delta x + \alpha^2 \sin^2 k\Delta x
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We note that, for all \( \theta \), we have \( 0 \leq (1 - \cos \theta) \leq 2 \).
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Taking the largest value of \( 1 - \cos k\Delta x \) gives

\[
|A|^2 = 1 - 4\alpha(1 - \alpha) = (1 - 2\alpha)^2 \leq 1.
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= |(1 - \alpha) + \alpha \cos k\Delta x - i\alpha \sin k\Delta x|^2 \\
= [(1 - \alpha) + \alpha \cos k\Delta x]^2 + \alpha [\sin k\Delta x]^2 \\
= (1 - \alpha)^2 + 2(1 - \alpha)\alpha \cos k\Delta x + \alpha^2 \cos^2 k\Delta x + \alpha^2 \sin^2 k\Delta x \\
= (1 - 2\alpha + \alpha^2) + 2\alpha(1 - \alpha) \cos k\Delta x + \alpha^2 \\
= 1 - 2\alpha(1 - \alpha)[1 - \cos k\Delta x].
\]

We note that, for all \( \theta \), we have \( 0 \leq (1 - \cos \theta) \leq 2 \).

Taking the largest value of \( 1 - \cos k\Delta x \) gives

\[
|A|^2 = 1 - 4\alpha(1 - \alpha) = (1 - 2\alpha)^2 \leq 1.
\]

Taking the smallest value of \( 1 - \cos k\Delta x \) gives

\[
|A|^2 = 1.
\]

In either case, \( |A|^2 \leq 1 \), so there is numerical stability.
We have determined the departure point by linear interpolation.
We have determined the departure point by *linear interpolation*.

This ensures that $0 \leq \alpha < 1$. 
Discussion and Conclusion

- We have determined the departure point by *linear interpolation*.
- This ensures that $0 \leq \alpha < 1$.
- This in turn ensures that $|A| \leq 1$. 
We have determined the departure point by linear interpolation.

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In other words, we have unconditional numerical stability.
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The implication is that the time step is unlimited.
We have determined the departure point by *linear interpolation*.

*This ensures that* $0 \leq \alpha < 1$.

*This in turn ensures that* $|A| \leq 1$.

*In other words, we have unconditional numerical stability.*

*The implication is that the time step is unlimited.*

*In contradistinction to the Eulerian scheme there is no CFL criterion.*
Of course, we must consider **accuracy** as well as **stability**.
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The time step $\Delta t$ is chosen to ensure **sufficient accuracy**, but can be much larger than for an Eulerian scheme.
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The time step $\Delta t$ is chosen to ensure **sufficient accuracy**, but can be much larger than for an Eulerian scheme.

Typically, $\Delta t$ is about **six times larger** for a semi-Lagrangian scheme than for an Eulerian scheme.

This is a **substantial gain** in computational efficiency.
Miscellaneous Issues
Miscellaneous Issues

- Calculation of departure point
Miscellaneous Issues

- Calculation of departure point
- Higher order interpolation
Miscellaneous Issues

- **Calculation of departure point**
- **Higher order interpolation**
- **Interpolation in two dimensions**
Miscellaneous Issues

- Calculation of departure point
- Higher order interpolation
- Interpolation in two dimensions
- Interpolation in the vertical
Miscellaneous Issues

- Calculation of departure point
- Higher order interpolation
- Interpolation in two dimensions
- Interpolation in the vertical
- Coriolis terms: Pseudo-implicit scheme
Miscellaneous Issues

- Calculation of departure point
- Higher order interpolation
- Interpolation in two dimensions
- Interpolation in the vertical
- Coriolis terms: Pseudo-implicit scheme
- Inclusion of Physics
End of §3.2.6