Laplace Transform Integration
(ACM 40520)

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Outline

Basic Theory

Residue Theorem

Ordinary Differential Equations

General Vector NWP Equation

Lagrangian Formulation
The Laplace Transform: Definition

For a function of time $f(t)$, the LT is defined as

$$\hat{f}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$ 

Here, $s$ is complex and $\hat{f}(s)$ is a complex function of $s$. 
The Laplace Transform: Definition

For a function of time \( f(t) \), the LT is defined as

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Here, \( s \) is complex and \( \hat{f}(s) \) is a complex function of \( s \).

The inversion from \( \hat{f}(s) \) back to \( f(t) \) is

\[
f(t) = \frac{1}{2\pi i} \oint_{C_1} e^{st} \hat{f}(s) \, ds.
\]

where \( C_1 \) is a contour in the \( s \)-plane, parallel to the imaginary axis, to the right of all singularities of \( \hat{f}(s) \).
Contour for inversion of Laplace Transform
Integral Transforms

The LT is one of a large family of integral transforms. They can be defined as

\[ F(s) = \int_{\mathcal{R}} K(s, t) f(t) \, dt \]

where \( \mathcal{R} \) is the range of \( f(t) \) and \( K(s, t) \) is called the kernel of the transform.
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where $\mathcal{R}$ is the range of $f(t)$ and $K(s, t)$ is called the kernel of the transform.

For example, the Fourier transform is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+i\omega t} \tilde{f}(\omega) \, d\omega$$
Integral Transforms

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\[ \tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \, dt \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \tilde{f}(\omega) \, d\omega \]

The Hilbert transform is another \ldots and many more.
The LT is a **linear operator**

\[
\mathcal{L}\{f(t)\} = \hat{f}(s) \equiv \int_{0}^{\infty} e^{-st} f(t) \, dt.
\]

Therefore

\[
\mathcal{L}\{\alpha f(t)\} = \int_{0}^{\infty} e^{-st} \alpha f(t) \, dt = \alpha \int_{0}^{\infty} e^{-st} f(t) \, dt = \alpha \mathcal{L}\{f(t)\}.
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Also

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\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st} [f(t)+g(t)] \, dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.
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\[ \mathcal{L}\{f(t) + g(t)\} = \int_0^\infty e^{-st} [f(t) + g(t)] \, dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}. \]

Therefore

\[ \mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}. \]
Basic Properties of the LT

- $\mathcal{L}\{a\} = a/s$
- $\mathcal{L}\{\exp(at)\} = 1/(s - a)$

All these results can be demonstrated immediately by using the definition of the Laplace transform $\mathcal{L}\{f(t)\}$. 

Basic Theory Residues ODEs NWP Equation Lagrange
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A Simple Oscillation

Let $f(t)$ have a single harmonic component

$$f(t) = \alpha \exp(i\omega t)$$
A Simple Oscillation

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Exercise: Show that the LT of \( f(t) \) is

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\hat{f}(s) = \frac{\alpha}{s - i\omega},
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a holomorphic function with a simple pole at \( s = i\omega \).
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A pure (monochrome) oscillation in time transforms to a function with a single pole.
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a holomorphic function with a simple pole at \( s = i\omega \).

A pure (monochrome) oscillation in time transforms to a function with a single pole.

The position of the pole is determined by the frequency of the oscillation.
Again

\[ \hat{f}(s) = \mathcal{L}\{\alpha \exp(i \omega t)\} = \frac{\alpha}{s - i\omega}. \]
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The inverse transform of \( \hat{f}(s) \) is

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f(t) = \frac{1}{2\pi i} \int_{C_1} e^{st} \hat{f}(s) \, ds = \frac{1}{2\pi i} \int_{C_1} \frac{\alpha \exp(st)}{s - i\omega} \, ds.
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We augment \( C_1 \) by a semi-circular arc \( C_2 \) in the left half-plane. Denote the resulting closed contour by \( C_0 = C_1 \cup C_2 \).
Again

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The inverse transform of \( \hat{f}(s) \) is

\[ f(t) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{st} \hat{f}(s) \, ds = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\alpha \exp(st)}{s - i\omega} \, ds. \]

We augment \( \Gamma_1 \) by a semi-circular arc \( \Gamma_2 \) in the left half-plane. Denote the resulting closed contour by \( \Gamma_0 = \Gamma_1 \cup \Gamma_2 \).

If it can be shown that this leaves the value of the integral unchanged, the inverse is an integral around a closed contour.
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For an integral around a closed contour,

\[ f(t) = \frac{1}{2\pi i} \oint_{C_0} \frac{\alpha \exp(st)}{s - i\omega} \, ds, \]

we can apply the residue theorem:
For an integral around a closed contour,

\[ f(t) = \frac{1}{2\pi i} \oint_{C_0} \frac{\alpha \exp(st)}{s - i\omega} \, ds, \]

we can apply the residue theorem:

\[ f(t) = \sum_{C_0} \left[ \text{Residues of } \left( \frac{\alpha \exp(st)}{s - i\omega} \right) \right] \]

so \( f(t) \) is the sum of the residues of the integrand within \( C_0 \).
Residue Theorem

\[ \oint_C f(z) \, dz = \left[ \text{Sum of Residues of } f(z) \right] \text{ at poles within } C \]
Again

\[ f(t) = \sum_{c_0} \left[ \text{Residues of } \left( \frac{\alpha \exp(st)}{s - i\omega} \right) \right] \]
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There is just one pole, at \( s = i\omega \). The residue is

\[ \lim_{s \to i\omega} (s - i\omega) \left( \frac{\alpha \exp(st)}{s - i\omega} \right) = \alpha \exp(i\omega t) \]
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So we recover the input function:

\[ f(t) = \alpha \exp(i\omega t) \]
A Two-Component Oscillation

Let $f(t)$ have two harmonic components

$$f(t) = a \exp(i \omega t) + A \exp(i \Omega t) \quad |\omega| \ll |\Omega|$$
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The LT is a linear operator, so the transform of \( f(t) \) is

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\hat{f}(s) = \frac{a}{s - i \omega} + \frac{A}{s - i \Omega},
\]

which has two simple poles, at \( s = i \omega \) and \( s = i \Omega \).
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which has two simple poles, at \( s = i\omega \) and \( s = i\Omega \).

The LF pole, at \( s = i\omega \), is close to the origin.

The HF pole, at \( s = i\Omega \), is far from the origin.
Again

\[ \hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}. \]
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The inverse transform of \( \hat{f}(s) \) is

\[
\begin{align*}
    f(t) &= \frac{1}{2\pi i} \oint_{C_0} \frac{a \exp(st)}{s - i\omega} \, ds + \frac{1}{2\pi i} \oint_{C_0} \frac{A \exp(st)}{s - i\Omega} \, ds \\
    &= a \exp(i\omega t) + A \exp(i\Omega t).
\end{align*}
\]
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\[ = a \exp(i\omega t) + A \exp(i\Omega t). \]

We now replace \( C_0 \) by a circular contour \( C^* \) centred at the origin, with radius \( \gamma \) such that \( |\omega| < \gamma < |\Omega| \).
S-plane

\[ S \text{-plane} \]

[Diagram showing a complex plane with labeled points and arrows]

Basic Theory  Residues  ODEs  NWP Equation  Lagrange
Again: We replace \( C_0 \) by \( C^\star \) with \(|\omega| < \gamma < |\Omega|\).
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$.

We denote the modified operator by $L^*$.
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We denote the modified operator by \( L^* \).

Since the pole \( s = i\omega \) falls \textit{within} the contour \( C^* \), it contributes to the integral.

Since the pole \( s = i\Omega \) falls \textit{outside} the contour \( C^* \), it makes \textit{no contribution}. 

We have filtered \( f(t) \): the function \( f^*(t) \) is the LF component of \( f(t) \). The HF component is gone.
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$.

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Since the pole $s = i\omega$ falls within the contour $C^*$, it contributes to the integral.

Since the pole $s = i\Omega$ falls outside the contour $C^*$, it makes no contribution.

Therefore,

$$f^*(t) \equiv L^*\{\hat{f}(s)\} = \frac{1}{2\pi i} \int_{C^*} \frac{a \exp(st)}{s - i\omega} \, ds = a \exp(i\omega t).$$
Again: We replace $C_0$ by $C^*$ with $|\omega| < \gamma < |\Omega|$.

We denote the modified operator by $L^*$.

Since the pole $s = i\omega$ falls *within* the contour $C^*$, it contributes to the integral.

Since the pole $s = i\Omega$ falls *outside* the contour $C^*$, it makes *no contribution*.

Therefore,

$$f^*(t) \equiv L^*\{\hat{f}(s)\} = \frac{1}{2\pi i} \int_{C^*} \frac{a \exp(st)}{s - i\omega} \, ds = a \exp(i\omega t).$$

We have filtered $f(t)$: the function $f^*(t)$ is the LF component of $f(t)$. The HF component is gone.
Exercise

Consider the test function

\[ f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \quad |\omega_1| < |\omega_2| \]
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Consider the test function

\[ f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \quad |\omega_1| < |\omega_2| \]

Show that the LT is

\[ \hat{f}(s) = \frac{\alpha_1}{2} \left[ \frac{e^{-i\psi_1}}{s - i\omega_1} + \frac{e^{i\psi_1}}{s + i\omega_1} \right] + \frac{\alpha_2}{2} \left[ \frac{e^{-i\psi_2}}{s - i\omega_2} + \frac{e^{i\psi_2}}{s + i\omega_2} \right] \]
Exercise

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\[ f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \quad |\omega_1| < |\omega_2| \]

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Show how, by choosing \( C^* \) with \( |\omega_1| < \gamma < |\omega_2| \), the HF component can be eliminated.
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General Vector NWP Equation

Lagrangian Formulation
Applying LT to an ODE

We consider a nonlinear ordinary differential equation

$$\frac{dw}{dt} + i\omega w + n(w) = 0 \quad w(0) = w_0$$

The LT of the equation is

$$\left(\hat{s}w - w_0\right) + i\omega \hat{w} + n_0 \hat{s} = 0.$$

We have frozen $n(w)$ at its initial value $n_0 = n(w_0)$.

We can immediately solve for the transform solution:

$$\hat{w}(s) = \frac{1}{s + i\omega} \left[w_0 - n_0 s\right].$$
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Applying LT to an ODE

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$$\frac{dw}{dt} + i\omega w + n(w) = 0 \quad \text{with} \quad w(0) = w_0$$

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We can immediately solve for the transform solution:

$$\hat{w}(s) = \frac{1}{s + i\omega} \left[ w_0 - \frac{n_0}{s} \right]$$
Using partial fractions, we write the transform as

\[ \hat{w}(s) = \left( \frac{w_0}{s + i\omega} \right) + \frac{n_0}{i\omega} \left( \frac{1}{s} - \frac{1}{s + i\omega} \right) \]

There are two poles, at \( s = -i\omega \) and at \( s = 0 \).
Using partial fractions, we write the transform as

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There are two poles, at \( s = -i\omega \) and at \( s = 0 \).

The pole at \( s = 0 \) always falls within the contour \( C^\star \).
The pole at \( s = -i\omega \) may or may not fall within \( C^\star \).
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There are two poles, at \( s = -i\omega \) and at \( s = 0 \).

The pole at \( s = 0 \) always falls within the contour \( \mathcal{C}^* \). The pole at \( s = -i\omega \) may or may not fall within \( \mathcal{C}^* \).

Thus, the solution is

\[ w^*(t) = \begin{cases} 
(w_0 - \frac{n_0}{i\omega}) \exp(-i\omega t) + \frac{n_0}{i\omega} & : \ |\omega| < \gamma \\
\frac{n_0}{i\omega} & : \ |\omega| > \gamma 
\end{cases} \]
Again,

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So we see that, for a LF oscillation (|\omega| < \gamma), the solution \( w^*(t) \) is the same as the full solution \( w(t) \) of the ODE.
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So we see that, for a LF oscillation (\(|\omega| < \gamma\)), the solution \(w^*(t)\) is the same as the full solution \(w(t)\) of the ODE.

For a HF oscillation (\(|\omega| > \gamma\)), the solution contains only a constant term.
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\end{cases} \]

So we see that, for a LF oscillation (|\omega| < \gamma), the solution \(w^*(t)\) is the same as the full solution \(w(t)\) of the ODE.

For a HF oscillation (|\omega| > \gamma), the solution contains only a constant term.

Thus, high frequencies are filtered out.
Outline

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Residue Theorem

Ordinary Differential Equations

General Vector NWP Equation

Lagrangian Formulation
A General Vector Equation

We write the general NWP equations symbolically as

\[
\frac{dX}{dt} + iLX + N(X) = 0
\]

where \(X(t)\) is the state vector at time \(t\).
A General Vector Equation

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where $X(t)$ is the state vector at time $t$.

We apply the Laplace transform to get

$$(s \hat{X} - X_0) + iL\hat{X} + \frac{1}{s}N_0 = 0$$

where $X_0$ is the initial value of $X$ and $N_0 = N(X_0)$ is held constant at its initial value.
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where \(X_0\) is the initial value of \(X\) and \(N_0 = N(X_0)\) is held constant at its initial value.

The frequencies are entangled. How do we proceed?
Eigenanalysis

\[ \dot{X} + i L X + N(X) = 0 \]

Assume the eigenanalysis of \( L \) is \( L E = E \Lambda \) where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and \( E = (e_1, \ldots, e_N) \).

More explicitly, assume that the eigenfrequencies split in two:

\[ \Lambda = \begin{bmatrix} \Lambda_Y & 0 \\ 0 & \Lambda_Z \end{bmatrix} \]

\( \Lambda_Y \): Frequencies of rotational modes (LF)

\( \Lambda_Z \): Frequencies of gravity-inertia modes (HF)
Eigenanalysis

\[ \dot{X} + i L X + N(X) = 0 \]

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\[ LE = E \Lambda \]

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where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and \( E = (e_1, \ldots, e_N) \).

More explicitly, assume that the eigenfrequencies split in two:

\[ \Lambda = \begin{bmatrix} \Lambda_Y & 0 \\ 0 & \Lambda_Z \end{bmatrix} \]

\( \Lambda_Y \): Frequencies of rotational modes (LF)

\( \Lambda_Z \): Frequencies of gravity-inertia modes (HF)
We define a new set of variables: $W = E^{-1}X$. Multiplying the equation by $E^{-1}$ we get

$$E^{-1} \dot{X} + iE^{-1}L(E^{-1}X) + E^{-1}N(X) = 0$$

This is just

$$\dot{W} + i\Lambda W + E^{-1}N(X) = 0$$

This equation separates into two sub-systems:

$$\dot{Y} + i\Lambda Y + N(Y,Z) = 0$$

$$\dot{Z} + i\Lambda Z + N(Y,Z) = 0$$

where $W = (Y, Z)^T$. The variables $Y$ and $Z$ are all coupled through the nonlinear terms $N(Y,Z)$ and $N(Z,Y)$. Basic Theory Residues ODEs NWP Equation Lagrange
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We first consider a single component $w$:

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Holding the nonlinear term constant, we get

$$(s\dot{w} - w_0) + i\lambda \hat{w} + \frac{n_0}{s} = 0$$

or

$$\hat{w}(s) = \frac{1}{s + i\lambda} \left[ w_0 - \frac{n_0}{s} \right]$$
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This has two poles, at $s = 0$ and $s = -i\lambda$:

$$\hat{w}(s) = \frac{w_0 + n_0/i\lambda}{s + i\lambda} - \frac{n_0/i\lambda}{s}$$
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If \(|\lambda|\) is small, both poles are within \(C^*\), so

\[ w^*(t) = \mathcal{L}^*\{\hat{w}\} = \left( w_0 + \frac{n_0}{i\lambda} \right) e^{-i\lambda t} - \left( \frac{n_0}{i\lambda} \right), \]

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$$w^*(t) = L^*\{\hat{w}\} = -\left( \frac{n_0}{i\lambda} \right).$$

This corresponds to putting $\dot{w} = 0$ in the equation:

$$i\lambda w^* + n_0 = 0$$
General Solution Method

We recall that the Laplace transform of the equation is

\[(s \hat{X} - X_0) + i L \hat{X} + \frac{1}{s}N_0 = 0\]

where \(X_0\) is the initial value of \(X\) and \(N_0 = N(X_0)\) is held constant at its initial value.
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The solution can be written formally:

\[\hat{X}(s) = (s I + i L)^{-1} \left[ X^n - \frac{1}{s} N^n \right]\]
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We recover the filtered solution by applying \( \mathcal{L}^* \) at time \((n + 1)\Delta t\).

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The procedure may now be iterated to produce a forecast of any length.
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Further details are given in Clancy and Lynch, 2011a,b
Outline

Basic Theory

Residue Theorem

Ordinary Differential Equations

General Vector NWP Equation

Lagrangian Formulation
Lagrangian Formulation

We now consider how to combine the Laplace transform approach with Lagrangian advection.

\[ \frac{D}{Dt}X + iLX + N(X) = 0 \]

where advection is now included in the time derivative.

We re-define the Laplace transform to be the integral in time along the trajectory of a fluid parcel:

\[ \hat{X}(s) \equiv \int_T e^{-st} X(t) \, dt \]
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The equations thus transform to

\[
(s \dot{X} - X^n_D) + i L \dot{X} + \frac{1}{s} N^{n+\frac{1}{2}}_M = 0
\]

where we evaluate nonlinear terms at a mid-point, interpolated in space and extrapolated in time.
The solution can be written formally:

\[ \hat{X}(s) = (sI + iL)^{-1} \left[ X_D^n - \frac{1}{s} N^{n+\frac{1}{2}}_M \right] \]

The values at the departure point and mid-point are computed by interpolation. We recover the filtered solution by applying \( L^\star \) at time \((n+1)\Delta t\), or \( \Delta t \) after the initial time. The procedure may now be iterated to produce a forecast of any length.

Further details are given in Clancy and Lynch, 2011a,b.
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