The ENIAC Integrations

Numerical Solution of the BVE

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Outline

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The dynamical behaviour of planetary waves in the atmosphere is modelled by the barotropic vorticity equation (BVE):

$$\frac{d(\zeta + f)}{dt} = 0 .$$

Rossby (1939) used a simplified (linear) form of this equation for his study of atmospheric waves.

Charney, Fjørtoft & von Neumann (1950) integrated the BVE to produce the earliest numerical weather predictions on the ENIAC.

They integrated the equation on a rectangular domain, in planar geometry.

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$$\mathbf{V} = \mathbf{k} \times \nabla\psi \qquad \nabla\cdot\mathbf{V} = 0$$

$$u = -\frac{\partial\psi}{\partial y} \qquad v = +\frac{\partial\psi}{\partial x}$$

$$\begin{aligned} \frac{\mathrm{d}\bullet}{\mathrm{d}t} &= \frac{\partial\bullet}{\partial t} + u\frac{\partial\bullet}{\partial x} + v\frac{\partial\bullet}{\partial y} \\ &= \frac{\partial\bullet}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial\bullet}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\bullet}{\partial y} \\ &= \frac{\partial\bullet}{\partial t} + J(\psi,\bullet) \end{aligned}$$

$$\nabla\cdot\mathbf{V} = 0 \qquad \zeta = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

Since f does not vary with time, we have

$$\frac{\partial}{\partial t}(\zeta + f) = \frac{\partial\zeta}{\partial t} = \frac{\partial\nabla^2\psi}{\partial t}$$

Thus, the BVE may be written

$$\frac{\partial\nabla^2\psi}{\partial t} + J(\psi, \nabla^2\psi + f) = 0$$

This is a single partial differential equation with just one dependent variable, the streamfunction $\psi(x, y, t)$.

Once initial and boundary values are given, the equation can be solved for $\psi = \psi(x, y, t)$.

The Jacobian operator is defined as

$$J(\psi,\zeta) = \left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}\right)$$

The Jacobian operator represents advection:

$$\begin{aligned} \mathbf{V}\cdot\nabla\zeta &= u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} \\ &= -\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} \\ &= J(\psi,\zeta) \end{aligned}$$

It is essentially nonlinear. The BVE must be solved by numerical means. We come to this next.

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$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi, \nabla^2\psi + f)$$

Assume that $\psi(x, y) = \psi_0(x, y)$ at $t = 0$.

We write the system of equations

$$\zeta = \nabla^2\psi \quad (1)$$

$$\frac{\partial\zeta}{\partial t} = -J(\psi, \zeta + f) \quad (2)$$

$$\nabla^2\frac{\partial\psi}{\partial t} = \frac{\partial\zeta}{\partial t} \quad (3)$$

We assume that the values of $\psi(x, y)$ on the boundary remain unchanged during the integration.

ALGORITHM:

- **Given: $\psi^n(x, y)$ at time $t = n\Delta t$.**
- **Compute $\zeta^n(x, y)$ using (1).**
- **Solve (2) for $(\partial\zeta/\partial t)^n$.**
- **Solve (3) with homogeneous boundary conditions for $(\partial\psi/\partial t)^n$.**
- **Advance ψ to time $t = (n + 1)\Delta t$ using** $\psi^{n+1} = \psi^{n-1} + 2\Delta t(\partial\psi/\partial t)^n$.

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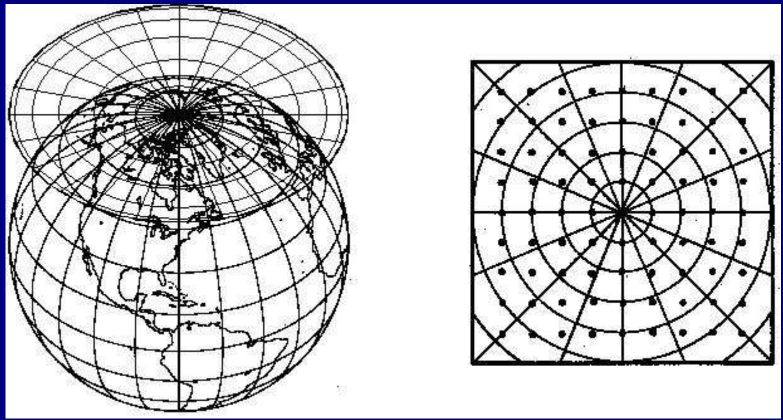

Figure: Polar Stereographic projection

Map Factor $\mu = \dfrac{1}{1 + \sin\phi}$

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We need to find the streamfunction by solving a Poisson equation of the form

$$\nabla^2\Phi = F \qquad \text{with} \qquad \Phi = 0 \ \text{ on the boundary}$$

on a rectangular domain.

We introduce a discrete grid

$$\begin{aligned} x &\longrightarrow \{x_0, x_1, x_2, \dots, x_M = M\Delta x\} \\ y &\longrightarrow \{y_0, y_1, y_2, \dots, y_N = N\Delta y\} \end{aligned}$$

For simplicity, we assume

$$\Delta x = \Delta y = \Delta s\,.$$

We use a spectral method that was devised by John von Neumann for the ENIAC integrations.

We recall some properties of the Fourier expansion:

$$\Phi_{mn} = \sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1} \tilde{\Phi}_{k\ell} \sin\left(\frac{km\pi}{M}\right) \sin\left(\frac{\ell n\pi}{N}\right)$$

The inverse transform is

$$\tilde{\Phi}_{k\ell} = \left(\frac{2}{M}\right)\left(\frac{2}{N}\right) \sum_{i=1}^{M-1}\sum_{j=1}^{N-1} \Phi_{ij} \sin\left(\frac{ik\pi}{M}\right) \sin\left(\frac{j\ell\pi}{N}\right)$$

We note that

$$\sum_{i=1}^{M-1}\sum_{j=1}^{N-1} \sin\left(\frac{im\pi}{M}\right) \sin\left(\frac{jn\pi}{N}\right) \sin\left(\frac{km\pi}{M}\right) \sin\left(\frac{\ell n\pi}{N}\right) = \delta_{ik}\delta_{j\ell}\left(\frac{M}{2}\right)\left(\frac{N}{2}\right)$$

The usual five-point approximation to $\nabla^2\Phi$ is

$$(\nabla^2\Phi)_{mn} \approx \left(\frac{\Phi_{m+1,n}+\Phi_{m-1,n}+\Phi_{m,n+1}+\Phi_{m,n-1}-4\Phi_{m,n}}{\Delta s^2}\right)$$

We expand Φ in a double Fourier series

$$\Phi_{mn} = \sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\tilde{\Phi}_{k\ell}\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

We use approximations like the following:

$$\frac{\partial^2}{\partial x^2}\sin\left(\frac{km\pi}{M}\right) \approx -4\sin^2\left(\frac{k\pi}{2M}\right)\sin\left(\frac{km\pi}{M}\right)$$

[Exercise: confirm the details.]

Thus:

$$\nabla^2\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right) \approx$$
$$-\frac{4}{\Delta s^2}\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

The Laplacian is applied term-by-term to Φ:

$$\nabla^2\Phi_{mn} \approx -\frac{4}{\Delta s^2}\sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]\tilde{\Phi}_{k\ell}\times \sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

We now expand the right hand side function:

$$F_{mn} = \sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\tilde{F}_{k\ell}\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

Now we equate the coefficients of $\nabla^2\Phi$ and F:

$$\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]\tilde{\Phi}_{k\ell} = (-\Delta s^2/4)\tilde{F}_{k\ell}$$

or

$$\tilde{\Phi}_{k\ell} = \frac{(-\Delta s^2/4)\tilde{F}_{k\ell}}{\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)}$$

Now $\tilde{\Phi}_{k\ell}$ is known, and we can invert it:

$$\Phi_{mn} = \frac{\Delta s^2}{MN}\sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\tilde{\Phi}_{k\ell}\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

We can compute the inverse transform in one go:

$$\Phi_{mn} = -\frac{\Delta s^2}{MN}\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}\sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]^{-1}\times F_{ij}\sin\left(\frac{im\pi}{M}\right)\sin\left(\frac{jn\pi}{N}\right)\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

We now substitute

$$F_{ij} \longrightarrow \left(\frac{\partial\zeta}{\partial t}\right)_{ij}.$$

Then

$$\Phi_{mn} = \left(\frac{\partial\psi}{\partial t}\right)_{mn}$$

and we have the solution for Φ.

The equation

$$\frac{d(\zeta + f)}{dt} = 0 \,.$$

was used for the four integrations on the ENIAC.

Charney, Fjørtoft and von Neumann (*Tellus*, 1950) used z rather than ψ. This necessitates an approximation involving the β-term.

Lynch (*BAMS*, 2008) showed that the ψ-form yields forecasts that are slightly more accurate.

This confirmed a hypothesis advanced earlier by Norman Phillips.

Charney et al. used the 500mb analyses of the National Weather Service, discretized and digitized by hand.

The computation grid was 19×16 points, with a resolution of about 600 km.

The ENIAC forecasts had an "electrifying effect" on the meteorological community, and led ultimately to operational NWP.

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