Fundamentals of Atmospheric Modelling
Peter Lynch, Met Éireann

Mathematical Computation Laboratory (Opp. Room 30)
Dept. of Maths. Physics, UCD, Belfield.
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Lecture 9

Mixed Rossby & Gravity Waves
We continue our investigation of the linear solutions of the shallow water equations.

First, we compare the properties of the simple wave solutions already found.

Then we consider the case where gravity waves and Rossby waves occur simultaneously as solutions.

Finally, we introduce the concept of filtering the equations.
Comparison of Wave Speeds

We can estimate the relative sizes of the phase speeds of the two types of pure wave solutions for parameter values typical of the atmosphere. In the absence of a mean flow, the phase speeds are

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c_G = \sqrt{\Phi} = \sqrt{gH}; \quad c_R = -\frac{\beta}{k^2} = -\frac{\beta L^2}{4\pi^2}.
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Taking approximate values

\[ g = 10 \text{ m s}^2, \quad H = 10^4 \text{ m}, \]
\[ \beta = 10^{-11} \text{ m}^{-1} \text{s}^{-1}, \quad L = 10^6 \text{ m} \]

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Thus the gravity waves travel much faster than the Rossby or planetary waves:

\[ |c_R| \ll |c_G| \]
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The other crucial distinction between the two types of waves:

- **For gravity waves, the divergence is vital for the dynamics.** The vorticity vanishes.
- **For Rossby waves, the vorticity is the important quantity.** The divergence vanishes.
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Then the divergence, vorticity and continuity equations for the linearized perturbation variables are:

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\begin{align*}
\alpha \left( \frac{\partial \delta}{\partial t} + \bar{u} \frac{\partial \delta}{\partial x} \right) - f \zeta + \beta u + \Phi_{xx} &= 0 \\
\left( \frac{\partial \zeta}{\partial t} + \bar{u} \frac{\partial \zeta}{\partial x} \right) + f \delta + \beta v &= 0 \\
\left( \frac{\partial \Phi}{\partial t} + \bar{u} \frac{\partial \Phi}{\partial x} \right) + v \frac{\partial \Phi}{\partial y} + \Phi \frac{\partial u}{\partial x} &= 0.
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Here \(\delta = u_x\), \(\zeta = v_x\) and the basic state satisfies \(f \bar{u} = -\Phi_y\).
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Here $\delta = u_x$, $\zeta = v_x$ and the basic state satisfies $f \bar{u} = -\bar{\Phi}_y$.

**Exercise:** Check that these equations are correct.
The coefficient $\alpha$ normally has the value 1. It is a tracer, which is carried through the analysis as a parameter, and allows us to examine the effect of omitting the term $d\delta/dt$ in the divergence equation by giving it the value zero in the final result. (This is just a handy trick to avoid repetition).
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We are interested in wave-like solutions of the form:

$$\begin{pmatrix} u \\ v \\ \Phi \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \Phi_0 \end{pmatrix} \exp[ik(x - ct)]$$

so that $u = \delta/ik$, $v = \zeta/ik$ and the differential operators are:

$$\frac{\partial}{\partial x} \sim ik; \quad \frac{\partial^2}{\partial x^2} \sim -k^2; \quad \frac{\partial}{\partial t} \sim -ikc.$$
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The form of the basic state allows us to write

$$
v\bar{\Phi}_y = \frac{\zeta}{ik}(-f\bar{u}) = -\left(\frac{f\bar{u}}{ik}\right)\zeta.
$$
We can now substitute the exponential expression for the dependent variables into the equations, which may then be written in matrix form:

\[
\begin{pmatrix}
-f & \alpha [ik(\bar{u} - c)] + \frac{\beta}{ik} & -k^2 \\

[ik(u - c) + \frac{\beta}{ik}] & +f & 0 \\

-f\bar{u} & -\bar{\Phi} & ik(\bar{u} - c)
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Since this is a homogeneous system, there is a solution iff the determinant of the coefficient matrix vanishes. Expanding out this gives us the dispersion equation:

\[
\left[\alpha(\bar{u} - c) - \frac{\beta}{k^2}\right]^2 (\bar{u} - c) - \left[\frac{f^2}{k^2} + \bar{\Phi}\right] (\bar{u} - c) + \left[\frac{f^2}{k^2}\bar{u} + \bar{\Phi}\frac{\beta}{k^2}\right] = 0
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This is a cubic equation for the phase speed $c$. We can solve the cubic analytically or numerically. However, a simpler way is to note the relative sizes of the terms and estimate the roots approximately.
First, suppose the magnitude of the phase speed is large. Specifically, let us suppose

\[ |\bar{u} - c| \gg |\bar{u}| \quad \text{and} \quad |\bar{u} - c| \gg |\beta/k^2|. \]
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Then we may neglect \( f^2\bar{u}/k^2 \) compared to \( (f^2/k^2)(\bar{u} - c) \), and also the terms involving \( \beta \). The cubic then reduces to

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\left[ \alpha(\bar{u} - c)^2 - \left( \frac{f^2}{k^2} + \Phi \right) \right] (\bar{u} - c) = 0.
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High Frequency Roots

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\]

The solution \( c = \bar{u} \) cannot be admitted, as we have assumed that \( c \) is large. Thus, the quadratic term must vanish, giving the two roots:

\[
c = \bar{u} \pm \sqrt{\bar{\Phi} + \frac{f^2}{k^2}} .
\]

These are the \textit{inertia-gravity wave} solutions.
In the case of no rotation, the inertia-gravity wave phase speed

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reduces to that for pure gravity wave solutions with

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In this case the phase speed is independent of the wavelength (or wavenumber \( k \)). More generally, the phase speed is modified by the effect of rotation and depends on \( k \), making the waves *dispersive*. 
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Taking typical parameter values, we find that the phase speed of these solutions is of the order \( c = 300 \text{ m/s} \). Thus, our assumptions are justified \textit{a posteriori}. 

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**Exercise:** Calculate the group velocity of the inertia-gravity waves.

Group velocity is defined two frames below
Next, suppose the phase speed is small, specifically that it is much smaller than the pure gravity wave speed:

\[ c^2 \ll \bar{\Phi} \quad \text{and also} \quad |\bar{u} - c|^2 \ll \bar{\Phi}. \]
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We also recall that the pure Rossby speed \( (\beta/k^2) \) is very much smaller that the gravity wave speed. Thus, the cubic term in the dispersion equation can be neglected compared to the linear term and the equation becomes

\[
- \left[ \frac{f^2}{k^2} + \bar{\Phi} \right] (\bar{u} - c) + \left[ \frac{f^2}{k^2} \bar{u} + \bar{\Phi} \frac{\beta}{k^2} \right] = 0.
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Low Frequency Roots

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- \left[ \frac{f^2}{k^2} + \Phi \right] (\bar{u} - c) + \left[ \frac{f^2}{k^2} \bar{u} + \Phi \frac{\beta}{k^2} \right] = 0. 
\]

This has the following solution

\[ c = \bar{u} - \left( \frac{\beta + f^2 \bar{u}}{k^2 + f^2/\Phi} \right). \]

For typical values of \( \Phi \), say \( 10^5 \text{ m}^2\text{s}^{-2} \), we find that \( c \) is quite small, thereby justifying our assumptions.
This solution is, of course, the Rossby wave solution, which always travels westward relative to the mean flow:

\[ c = \bar{u} - \left( \frac{\beta + f^2 \bar{u}/\bar{\Phi}}{k^2 + f^2/\bar{\Phi}} \right). \]

For large values of \( \bar{\Phi} \), it is approximated by the previously obtained simple formula

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**Exercise:** Calculate the group velocity of the Rossby waves.

**Definition:** The group velocity \( c_g \) for a wave in the \( x \)-direction is defined by

\[ c_g = \frac{\partial \omega}{\partial k} = \frac{\partial k}{\partial k}. \]
Filtering

In the above analysis, we used a tracer $\alpha$ to mark the term

$$\frac{d\delta}{dt} = \left( \frac{\partial\delta}{\partial t} + \bar{u} \frac{\partial\delta}{\partial x} \right)$$

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If we now set $\alpha = 0$, the dispersion equation becomes linear and only the Rossby wave solution remains.
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To rephrase, omission of the term corresponding to changes in the divergence is sufficient (in the present case) to eliminate solutions corresponding to gravity waves.

This process of modifying the governing equations in such a way as to exclude certain solutions is called \textit{filtering}. 
The Quasi-geostrophic Equations

The question arises: How can we generalise this idea? How can the general nonlinear equations be modified so that the high frequency gravity wave solutions are eliminated, while the low frequency Rossby waves, which are the important solutions, are preserved?
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This question will not be addressed in this lecture series, but an affirmative answer can be obtained. The equations resulting from the filtering procedure are called the Quasi-geostrophic Equations.
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This question will not be addressed in this lecture series, but an affirmative answer can be obtained. The equations resulting from the filtering procedure are called the Quasi-geostrophic Equations.

The quasi-geostrophic equations provide a powerful basis for the elucidation of the dynamics of the atmosphere in middle latitudes.