Fundamentals of Atmospheric Modelling

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Lecture 8

Linear Wave Solutions

Introduction

In this lecture we will investigate the *wave solutions* of the *linearized* Shallow Water Equations (SWE).

The equations are linearized by the *perturbation method*:

- **The dependent variables are separated into mean and perturbation components**
- **The mean state is assumed to be known and constant in time**
- **The perturbations are assumed to be of small amplitude, so that quadratic and higher terms in them can be ignored**
- **The system then becomes a linear system in the perturbation variables.**

The Basic State

Recall from previous work that the Shallow Water Equations are

$$\begin{aligned} \frac{du}{dt} - fv + \frac{\partial \Phi}{\partial x} &= 0 \\ \frac{dv}{dt} + fu + \frac{\partial \Phi}{\partial y} &= 0 \\ \frac{d\Phi}{dt} + \Phi \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 \end{aligned}$$

The total time derivative is:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \, .$$

The independent variables are

- x: **Zonal (eastward) coordinate**
- y: **Meridional (northward) coordinate**
- t: **Time.**

The dependent variables are

- u: **Zonal (west-to-east) component of the wind**
- v: **Meridional (south-to-north) component of the wind**
- Φ: **Geopotential,** $\Phi = gh$ **where** h **is depth/height.**

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The Basic State

We consider a basic state of constant zonal (east-west) flow $\bar{u}$, independent of time and of the space coordinates, and a corresponding basic geopotential field $\bar{\Phi}$

The shallow water equations for this state reduce to

$$f\bar{u} + \frac{\partial \bar{\Phi}}{\partial y} = 0 ; \qquad v = 0 ,$$

That is, the basic state is in *geostrophic balance.*

The Perturbation Equations

Consider a small perturbation of the basic state

$$
\begin{aligned}
u &= \bar{u} + u'(x,y,t) \\
v &= \phantom{\bar{u} + {}} v'(x,y,t) \\
\Phi &= \bar{\Phi}(y) + \Phi'(x,y,t)
\end{aligned}
$$

We assume that

$$|u'| \ll \bar{u}\,, \qquad |v'| \ll \bar{u}\,, \qquad |\Phi'| \ll \bar{\Phi}$$

Thus, squares and higher powers in the perturbations or primed quantities can be neglected.
The equations for the primed quantities can then be linearized.
This is essential for the progress of our analysis. In general, linear equations may be solved analytically, whereas *nonlinear* equations must be treated *numerically*.

Substituting the assumed form of the solution into the SWE, The perturbation equations take the following form:

$$\left(\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x}\right) - fv' + \frac{\partial \Phi'}{\partial x} = 0$$

$$\left(\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x}\right) + fu' + \frac{\partial \Phi'}{\partial y} = 0$$

$$\left(\frac{\partial \Phi'}{\partial t} + \bar{u}\frac{\partial \Phi'}{\partial x}\right) + \bar{\Phi}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

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In the special case of vanishing mean zonal flow ($\bar{u} = 0$):

$$\frac{\partial u'}{\partial t} - fv' + \frac{\partial \Phi'}{\partial x} = 0$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{\partial \Phi'}{\partial y} = 0$$

$$\frac{\partial \Phi'}{\partial t} + \bar{\Phi}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

These are called the *Laplace Tidal Equations*.

Divergence and Vorticity

It is straightforward to derive divergence and vorticity equations for the perturbation variables.

Recall the definitions of divergence and vorticity:

$$\delta = \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) \; ; \qquad \zeta = \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) \, .$$

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Adding the x-derivative of the u'-equation to the y-derivative of the v'-equation gives an equation for the divergence:

$$\frac{\partial \delta}{\partial t} + \bar{u}\frac{\partial \delta}{\partial x} - f\zeta + \beta u' + \nabla^2 \Phi = 0 .$$

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$$\frac{\partial \delta}{\partial t} + \bar{u}\frac{\partial \delta}{\partial x} - f\zeta + \beta u' + \nabla^2\Phi = 0 \, .$$

Subtracting the x-derivative of the v'-equation from the y-derivative of the u'-equation gives an equation for the vorticity:

$$\frac{\partial \zeta}{\partial t} + \bar{u}\frac{\partial \zeta}{\partial x} + f\delta + \beta v' = 0 \, .$$

The vorticity equation may also be written in the form:

$$\frac{d}{dt}(\zeta + f) + f\delta = 0\,.$$

This equation expresses the *Conservation of absolute vorticity.*

Exercise:

Fill in the missing steps in the derivation of the vorticity and divergence equations.

$\star \quad \star \quad \star \quad \star \quad \star$

Note: from now on, the primes on u', etc., are omitted.

General Remarks on Wave Motion

To have wave motion it is necessary to have some restoring mechanism, so that a particle which is disturbed from an equilibrium position will be induced to return there.

Examples of restoring forces are the elastic force in springs, gravity acting on a pendulum, electrostatic and electromagnetic forces in oscillating circuits, negative feedbacks in biological systems, etc.

We will consider some restoring forces important in the atmosphere.

***Compressibility*: If a fluid is compressible, the increase in pressure of a parcel of fluid which is compressed will tend to make it expand again. This effect can result in compression or sound waves propagating through the fluid. In the present context (for the Shallow Water Equations) we have assumed that the fluid is incompressible. Thus, sound waves have been eliminated as possible solutions.**

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Gravity: If the fluid surface is not horizontal, there will be pressure forces due to differing masses of fluid above different points, *i.e.,* there will be a pressure gradient from deep to shallow parts of the fluid. This will tend to push fluid away from places where the surface is elevated towards regions where it is depressed. This tendency to even out disturbances in the height field can lead to gravity waves.

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The beta-effect: The variation in the planetary vorticity f, which we have denoted by β, combined with the conservation of absolute vorticity, amounts to a restoring force and results in the Rossby waves which we will study below.

Longitudinal v. Transverse waves:

We have eliminated the longitudinal waves by the assumption of incompressibility. Transverse waves may be primarily vertical or horizontal. However, there is something of a paradox: How can pure transverse waves (*i.e.*, waves having no velocity component in the direction of travel of the waves) exist in an inviscid fluid?

Since there is no shearing force, there seems to be no way of communicating the wave motion orthogonally to the flow. Yet such waves do exist in the atmosphere! (The argument of their non-existence is used in seismology to deduce the liquid nature of the earth's core).

Exercise:

Try to resolve this paradox.

Note: A *gadankenexperiment* may be productive even if it does not yield the result that is sought.

Pure Types of Wave Solution

I: Vertical Transverse Solutions: Gravity Waves

Let us assume that the mean flow $\bar{u}$ is zero, and consider motions in an east-west plane. Moreover, we neglect the effect of rotation. Thus

$$f \equiv 0\,; \qquad v \equiv 0\,; \qquad \frac{\partial}{\partial y} \equiv 0$$

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We can immediately eliminate u and derive an equation for Φ:

$$\frac{\partial^2 \Phi}{\partial t^2} - \bar{\Phi}\frac{\partial^2 \Phi}{\partial x^2} = 0$$

(Exercise: Show that the same equation governs u.)

This is the familiar wave equation, usually appearing as

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$$

and it has solutions of the general form

$$\Phi = \Phi_1(x - ct) + \Phi_2(x + ct)\,.$$

The wave *phase speed* c is given by

$$c = \pm\sqrt{\bar{\Phi}}$$

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Exercise: **Repeat the analysis with a mean flow $\bar{u}$, and show that the wave speed is now given by $c = \bar{u} \pm \sqrt{\bar{\Phi}}$.**

Exercise: **Take the scale-height of the atmosphere, $H \approx 10\,$km as the mean depth. Calculate the gravity-wave speed in this case.**

Exercise: **If the duck-pond in St. Stephen's Green has depth $h = 1\,$m, at what speed do the waves on the pond travel? Check by observation!**

To examine the structure of the gravity wave solutions, let us consider a single sinusoidal component:

$$\begin{Bmatrix} u \\ \Phi \end{Bmatrix} = \begin{Bmatrix} u_0 \\ \Phi_0 \end{Bmatrix} \exp[ik(x - ct)] \, .$$

Note: It is analytically convenient to consider a complex exponential solution, but the physical solution is the *real part* of this expression.

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The differential operators now have the following expression:

$$\frac{\partial}{\partial x} \sim ik \, ; \qquad \frac{\partial}{\partial t} \sim -ikc$$

and the wave equations for u and Φ become

$$\left(\frac{\partial^2}{\partial t^2} - \bar{\Phi} \frac{\partial^2}{\partial x^2} \right) \begin{Bmatrix} u \\ \Phi \end{Bmatrix} = -[k^2(c^2 - \bar{\Phi})] \begin{Bmatrix} u_0 \\ \Phi_0 \end{Bmatrix} e^{ik(x-ct)} = 0 \, .$$

Clearly, this implies $c = \pm\sqrt{\bar{\Phi}}$.

The divergence and vorticity reduce to

$$\delta = (u_x + v_y) = iku$$
$$\zeta = (v_x - u_y) \equiv 0$$

(recall that $\partial/\partial y \equiv 0$.)
Note: Pure gravity waves are *irrotational*, with vanishing vorticity.

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The momentum equation gives the relationship between u and Φ:

$$\frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} = 0 \qquad \Rightarrow \qquad -ikcu + ik\phi = 0 \qquad \Rightarrow \qquad u = \frac{\Phi}{c}.$$

Thus, for $c > 0$, *i.e.*, eastward-moving waves, the perturbation wind u and geopotential (or height) are positive and negative together:

<u>Where the fluid is elevated, the movement is eastward.</u>

Recall that we consider only the *real part* of the complex expressions for u and Φ to be physically significant.

Thus, the physical solutions for geopotential, zonal wind and divergence are

$$\Phi = \Re\{\Phi_0 \exp[ik(x-ct)]\} = \Phi_0 \quad \cos[k(x-ct)]$$
$$u = \Re\{u_0 \exp[ik(x-ct)]\} = (\Phi_0/c) \quad \cos[k(x-ct)]$$

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For $c > 0$, they are in phase. The divergence is

$$\delta = \Re\{iku_0 \exp[ik(x-ct)]\} = (-k\Phi_0/c)\sin[k(x-ct)]$$

It is 90° *out of phase* with u and Φ. Thus, maxima and minima in the divergence are a quarter wavelength west of peaks and troughs in the height field. That is, *maximum divergence is west of a peak*, and maximum convergence to the east. Therefore, the mass flux will cause the peak to move eastward (see Figure).

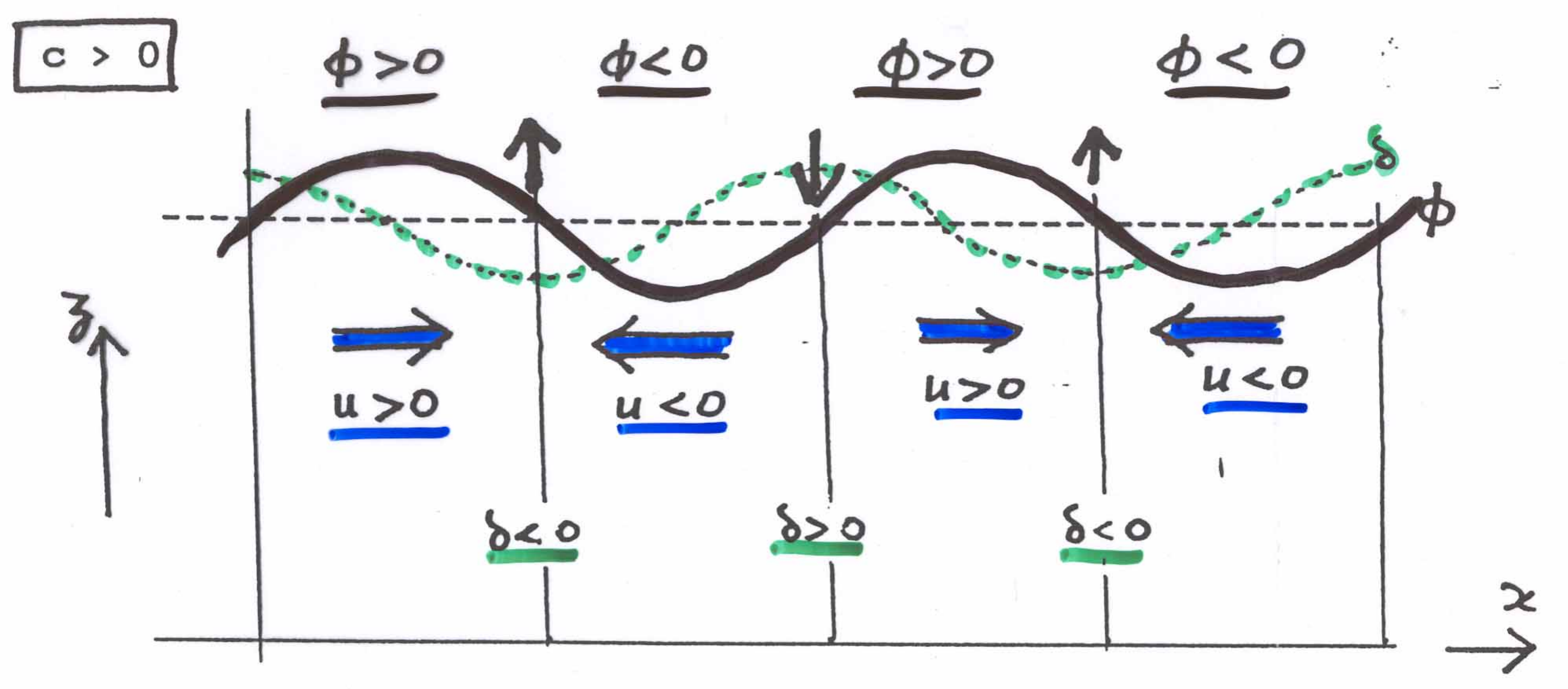

Fig. 1: Structure of an eastward propagating gravity wave.

The essential mechanism for the propagation of gravity waves is the change of pressure due to varying weight of fluid above a given horizontal surface.

Exercises:

(1) Repeat the above description of the relative phases of the Φ, u and δ fields in the case where $c < 0$, *i.e.*, for westward wave propagation.

(2) Derive the vertical velocity, and combine it with the zonal velocity u to describe the particle trajectories.

II: Horizontal Transverse Solutions: Rossby waves

These solutions are of *central importance in meteorology and oceanography*. They are also found in other rotating 'fluid' systems, for example, galaxies.

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We will use the vorticity and divergence equations, in which the β-term occurs explicitly. We consider the simplest case:

- **We *ignore all variations with latitude* (*i.e.*, in the y direction) except for the β-term.**
- **We seek solutions for which the motion is *purely horizontal***
- **We will also assume that the *zonal velocity is constant*, $\bar{u}$, and that the perturbation zonal velocity u' vanishes identically**
- **This implies *purely transverse wave motion* (i.e., no component of perturbation velocity in the direction of travel of the wavefronts)**

We have

$$u \equiv 0\,; \qquad \frac{\partial}{\partial y} \equiv 0\,; \qquad w \equiv 0\,;$$

The divergence and vorticity become

$$\delta = (u_x + v_y) \equiv 0\,; \quad \zeta = (v_x - u_y) = v_x\,.$$

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The divergence and vorticity become

$$\delta = (u_x + v_y) \equiv 0\,; \quad \zeta = (v_x - u_y) = v_x\,.$$

Recall the perturbation divergence and vorticity equations:

$$\frac{\partial \delta}{\partial t} + \bar{u}\frac{\partial \delta}{\partial x} - f\zeta + \beta u + \nabla^2 \Phi = 0\,.$$

$$\frac{\partial \zeta}{\partial t} + \bar{u}\frac{\partial \zeta}{\partial x} + f\delta + \beta v = 0\,.$$

The divergence equation reduces to a diagnostic relationship:

$$f\zeta = \nabla^2 \Phi\,.$$

This equivalent to geostrophic balance:

$$fv = \Phi_x \qquad \Rightarrow \qquad fv_x = \Phi_{xx}\,.$$

The vorticity equation becomes

$$\frac{\partial \zeta}{\partial t} + \bar{u}\frac{\partial \zeta}{\partial x} + \beta v = 0\,,$$

This may be written in terms of the meridional velocity v:

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \bar{u}\frac{\partial^2 v}{\partial x^2} + \beta v = 0\,.$$

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$$\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \bar{u}\frac{\partial^2 v}{\partial x^2} + \beta v = 0\,.$$

Now let us assume a sinusoidal variation of v with time and the x coordinate:

$$v = v_0 \exp[ik(x - ct)]\,.$$

Substituting this into the vorticity equation, we get

$$\left[k^2(c - \bar{u}) + \beta\right] v = 0$$

which can hold only if the quantity in square brackets vanishes. This gives us an expression for the phase speed:

$$c = \bar{u} - \frac{\beta}{k^2}\,.$$

The dispersion relationship

$$c = \bar{u} - \frac{\beta}{k^2}\,.$$

is the celebrated *Rossby wave formula*.

The *wavenumber* k **and *wavelength*** L **are related by** $k = 2\pi/L$**. Thus the phase speed can also be written**

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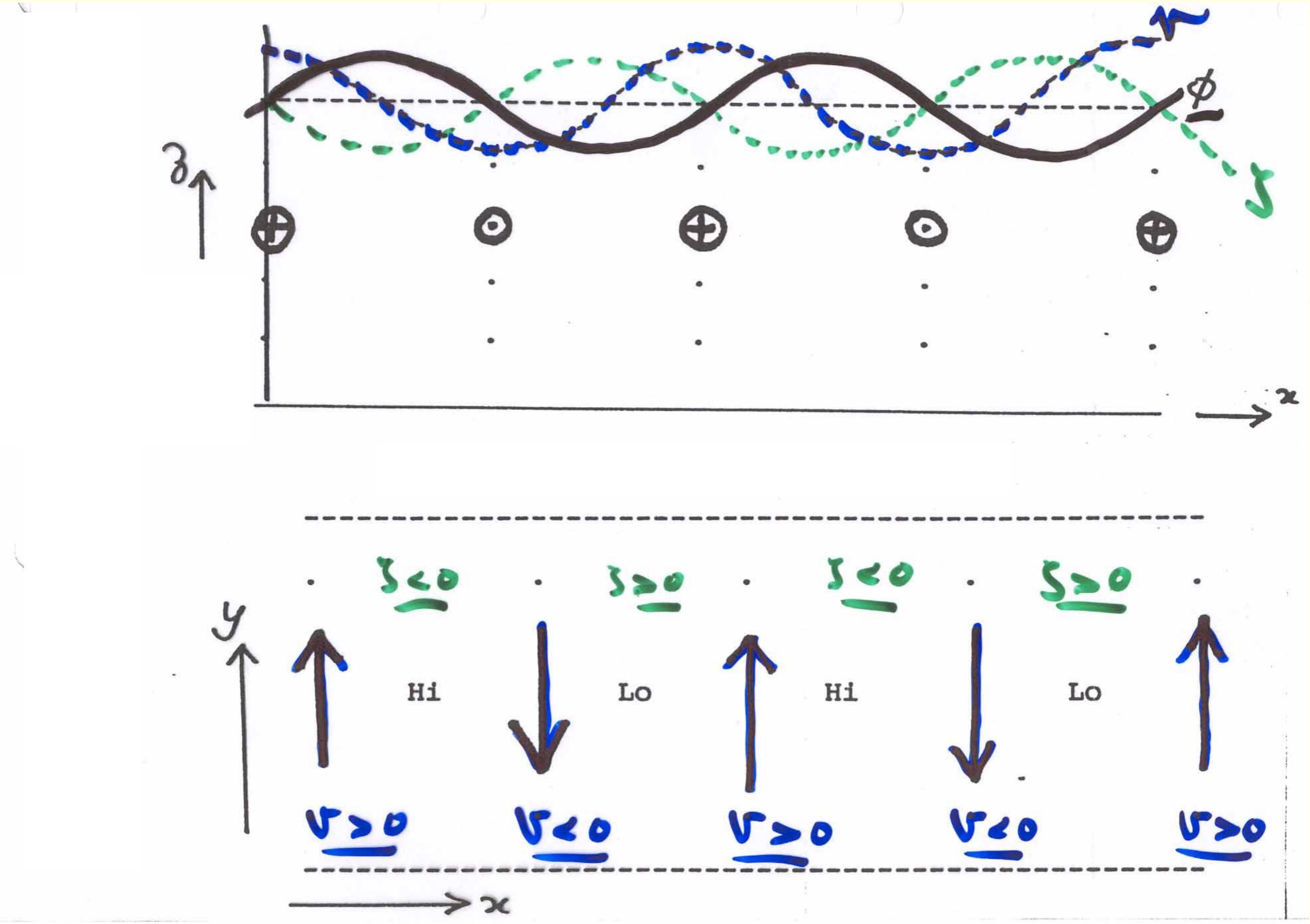
The geostrophic balance gives a relationship between v and Φ:

$$fv = \Phi_x = ik\Phi ; \qquad \Phi = -(if/k)v$$

So, assuming v is real and considering the real parts of the solution, we get

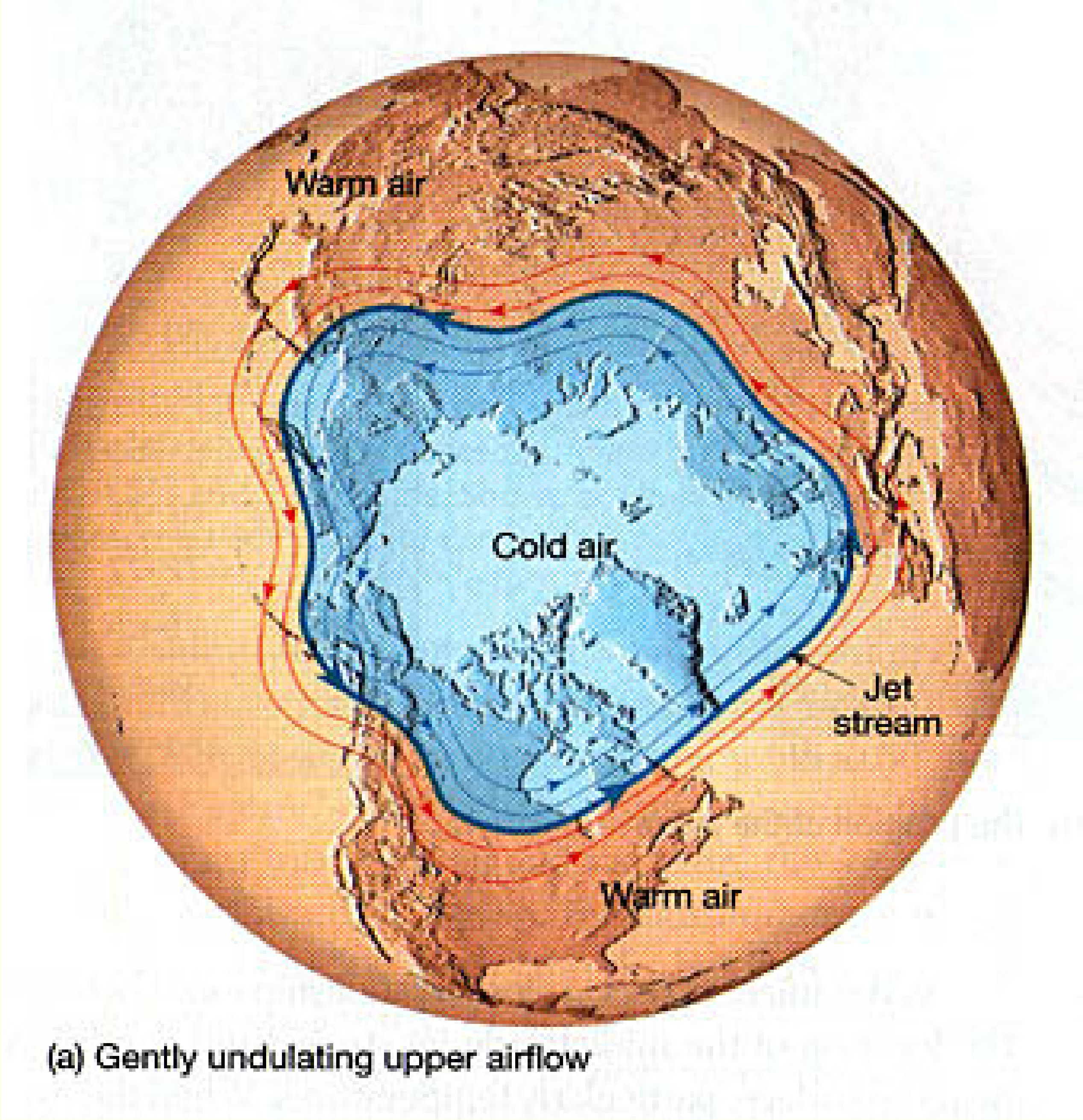
$$v = v_0 \cos[k(x - ct)]$$

$$\Phi = (fv_0/k) \sin[k(x - ct)]$$

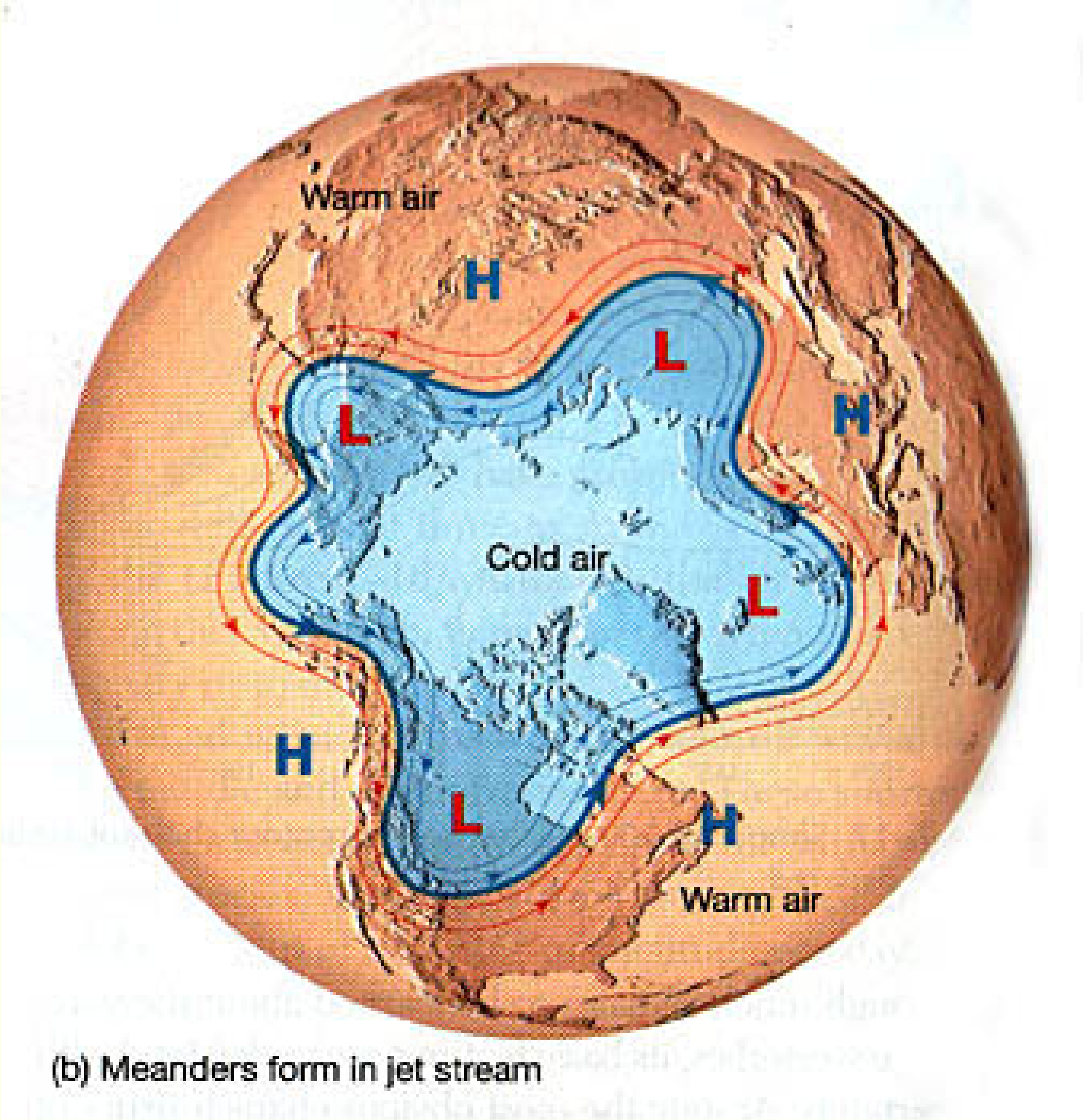


Schematic figure of Rossby wave structure.

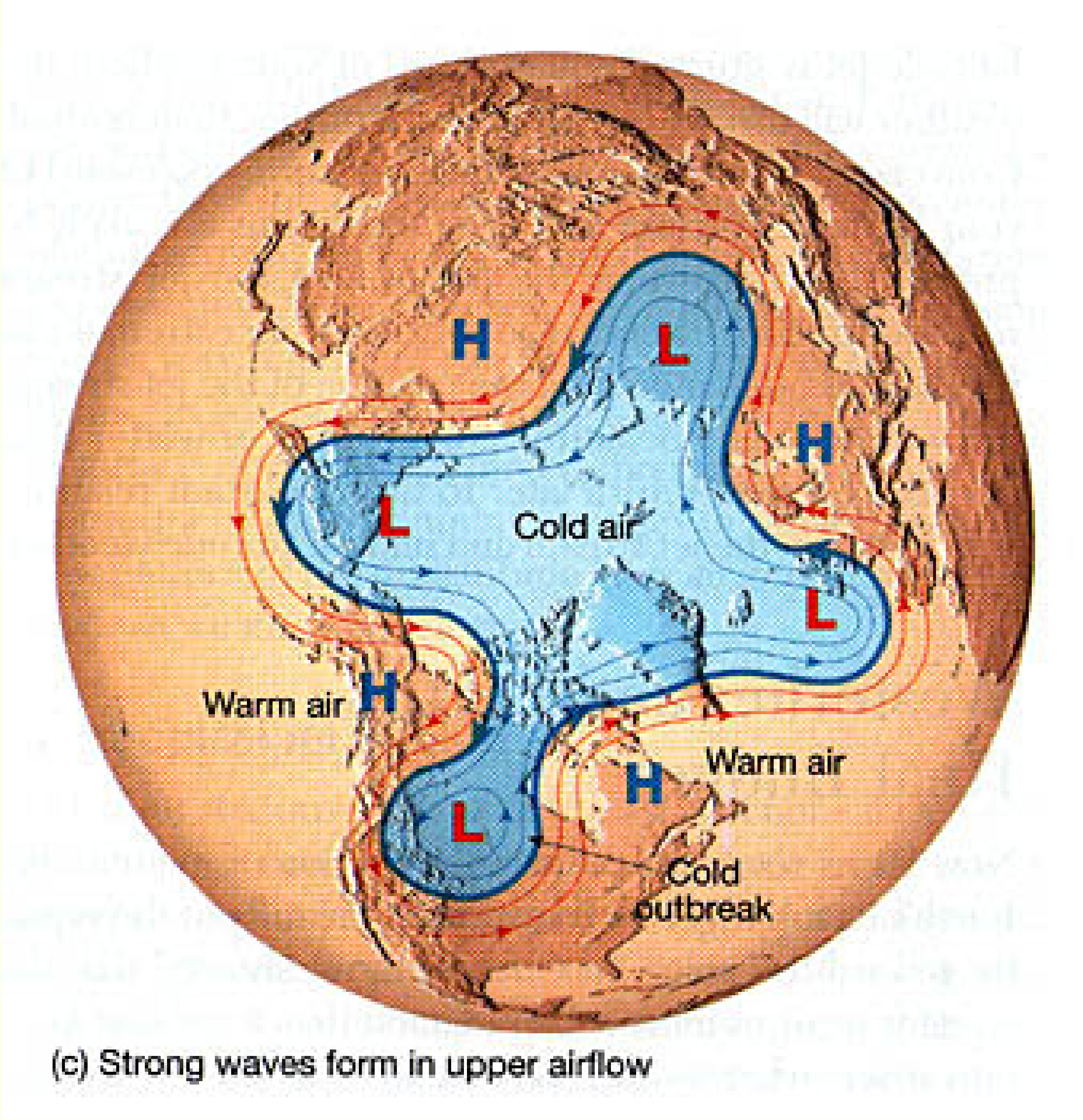
Top: Vertical section. **Bottom:** Horizontal section.



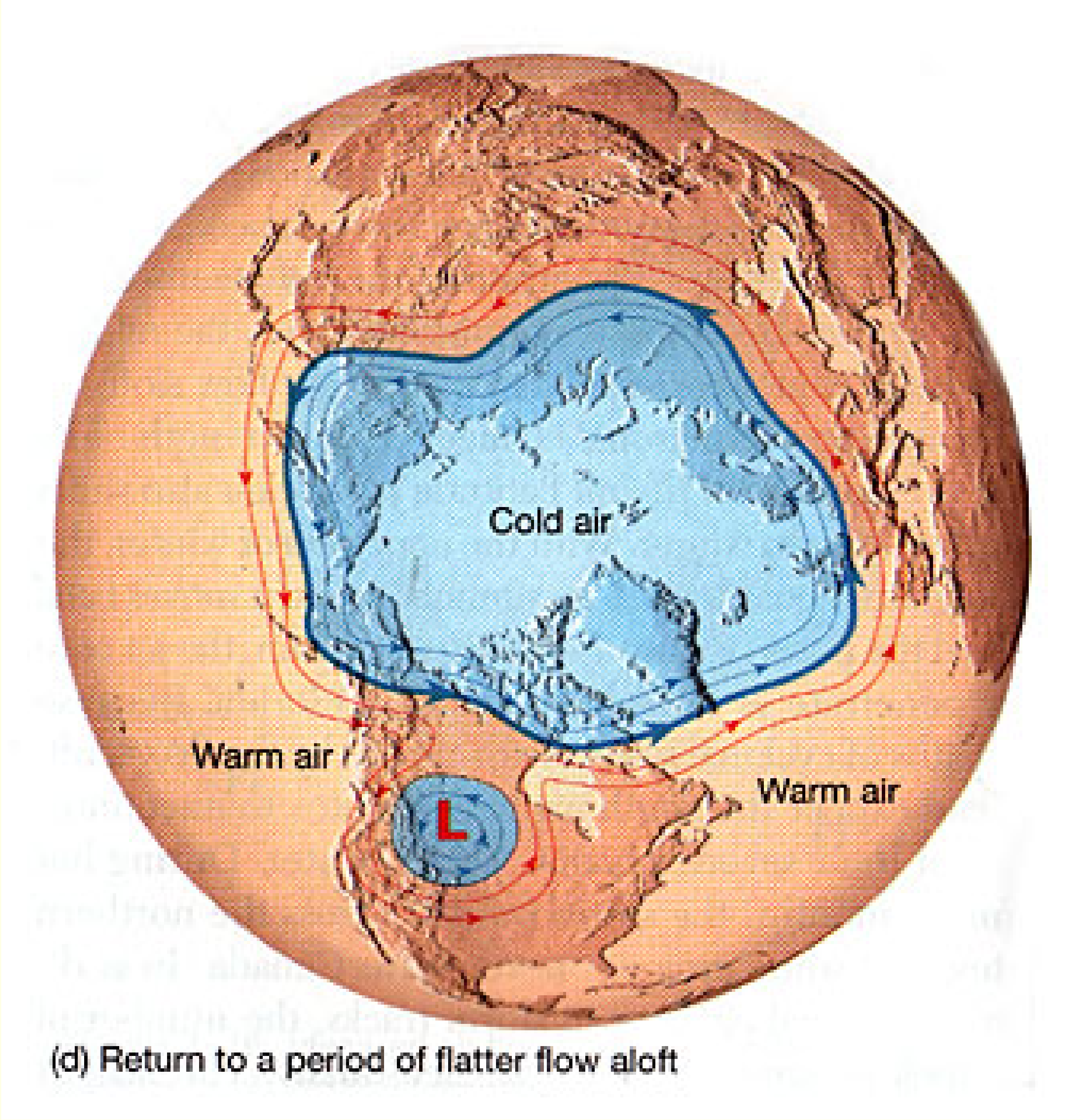
Predominantly Zonal Flow



Meandering Pattern Develops



Strongly Meridional Regime



Cut-off Low. Return to zonal Regime

Properties of Rossby Waves

Rossby waves always move westward relative to the flow. They are the prototype of the large scale wavelike disturbances in the atmosphere. The motion is horizontal and there is geostrophic balance between the pressure or height disturbance and the wind field. The divergence vanishes identically under the assumptions we have made.

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The dynamics of the planetary waves were first elucidated by the Swedish meteorologist Carl-Gustav Rossby, around 1940, using the principle of conservation of absolute vorticity.

Summary of Properties of Rossby Waves

- **The solutions derive from the principle of *conservation of absolute vorticity***
- **They owe their existence to the *rotation* Ω of the Earth**
- **The dynamics were first elucidated by Carl-Gustav Rossby**
- **They waves always *move westward* relative to $\bar{u}$**
- **The motion is *purely horizontal***
- **The meridional wind is in *geostrophic balance***
- **The *divergence vanishes* identically**

Exercises

Exercise: Consider the conservation absolute vorticity

$$\frac{d}{dt}(\zeta + f) = 0\,.$$

Linearize this equation about a constant mean flow $\bar{u}$ (don't forget the beta term!) and assume a wavelike solution with dependency proportional to $\exp[ik(x-ct)]$. Deduce the Rossby wave formula

$$c = \bar{u} - \frac{\beta}{k^2} = \bar{u} - \frac{\beta L^2}{4\pi^2}\,.$$

Exercise: Find the phase speed of a Rossby wave, assuming that $\beta = 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$, and that the wavelength is 3,000 km,

Exercise: Show that the vorticity equation

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \bar{u}\frac{\partial^2 v}{\partial x^2} + \beta v = 0\,.$$

is a *hyperbolic* partial differential equation (PDE).