Introduction

- We have studied the Eulerian *leapfrog scheme* and found it to be *conditionally stable*.
- The criterion for stability was the CFL condition
  \[ \frac{c \Delta t}{\Delta x} \leq 1. \]
- For high spatial resolution (small $\Delta x$) this severely limits the *maximum time step* $\Delta t$ that is allowed.
- In numerical weather prediction (NWP), *timeliness* of the forecast is of the essence.
- In this lecture, we study an alternative approach to time integration, which is *unconditionally stable* and so, free from restrictions of the CFL condition.

The Basic Idea

The semi-Lagrangian scheme for advection is based on the idea of approximating the Lagrangian time derivative. It is so formulated that *the numerical domain of dependence always includes the physical domain of dependence*. This necessary condition for stability is satisfied automatically by the scheme.

In a *fully Lagrangian* scheme, the trajectories of actual physical parcels of fluid would be followed throughout the motion.

The problem with this aproach, is that the distribution of representative parcels rapidly becomes *highly non-uniform*.

In the semi-Lagrangian scheme the individual parcels are followed only for a single time-step. After each step, we revert to a uniform grid.
The semi-Lagrangian algorithm has enabled us to integrate the primitive equations using a time step of 15 minutes. This can be compared to a typical timestep of 2.5 minutes for conventional schemes.

The consequential saving of computation time means that the operational numerical guidance is available to the forecasters much earlier than would otherwise be the case.

The semi-Lagrangian method was pioneered by André Robert, the renowned Canadian meteorologist.

The first operational implementation of such a scheme was in 1982 at the Irish Meteorological Service.

Semi-Lagrangian advection schemes are now in widespread use in all the main Numerical Weather Prediction centres.

Since the advection equation is linear, we can construct a general solution from Fourier components

\[ Y = a \exp[ik(x - ct)]; \quad k = 2\pi/L. \]

This expression may be separated into the product of a function of space and a function of time:

\[ Y = a \times \exp(-i\omega t) \times \exp(ikx); \quad \omega = kc. \]

Therefore, in analysing the properties of numerical schemes, we seek a solution of the form

\[ Y_m^n = a \times \exp(-i\omega n\Delta t) \times \exp(ikm\Delta x) = A^n \exp(ikm\Delta x) \]

where \( A = \exp(-i\omega \Delta t) \).

The character of the solution depends on the modulus of \( A \):

- If \(|A| < 1\), the solution decays with time.
- If \(|A| = 1\), the solution is neutral with time.
- If \(|A| > 1\), the solution grows with time.

In the third case (growing solution), the scheme is unstable.

We consider the linear advection equation which describes the conservation of a quantity \( Y(x, t) \) following the motion of a fluid flow in one space dimension with constant advecting velocity \( c \).

This may be written in either of two alternative forms:

\[ \frac{\partial Y}{\partial t} + c \frac{\partial Y}{\partial x} = 0 \quad \Leftarrow \quad \text{Eulerian Form} \]
\[ \frac{dY}{dt} = 0 \quad \Leftarrow \quad \text{Lagrangian Form} \]

The general solution is \( Y = Y(x - ct) \).

To develop numerical solution methods, we may start from either the Eulerian or the Lagrangian form of the equation.

For the semi-Lagrangian scheme, we choose the latter.

For the Eulerian Leapfrom Scheme, the value \( Y_m^n \) at time \( n\Delta t \) and position \( m\Delta x \) depends on values within the area depicted by asterisks. Values outside this region have no influence on \( Y_m^n \).
Each computed value $Y_{nm}^n$ depends on previously computed values and on the initial conditions. The set of points which influence the value $Y_{nm}^n$ is called the \textit{numerical domain of dependence} of $Y_{nm}^n$.

It is clear on physical grounds that if the parcel of fluid arriving at point $m \Delta x$ at time $n \Delta t$ originates \textit{outside the numerical domain of dependence}, the numerical scheme cannot yield an accurate result: the necessary information is not available to the scheme.

Worse again, the numerical solution may bear absolutely no relationship to the physical solution and may grow exponentially with time even when the true solution is bounded.

A \textit{necessary condition} for avoidance of this phenomenon is that the numerical domain of dependence should include the physical trajectory.

This condition is fulfilled by the semi-Lagrangian scheme.

\textbf{The central idea of the Lagrangian scheme is to represent the physical trajectory of the fluid parcel.}

We consider a parcel arriving at gridpoint $m \Delta x$ at the new time $(n + 1) \Delta t$ and ask whence it has come.

The \textit{departure point} will not normally be a grid point. Therefore, the value at the departure point must be calculated by interpolation from \textit{surrounding points}.

But this interpolation ensures that the trajectory falls within the numerical domain of dependence. We will show that this leads to a \textit{numerically stable scheme}.

The value everywhere on the trajectory is $Y_{nm}^n$.

Since the parcel originates \textit{outside} the numerical domain of dependence, the Eulerian scheme \textbf{cannot} model it correctly.

\textbf{The line of bullets (●) represents a parcel trajectory.}

\textbf{The line of circles (○) represents a parcel trajectory.}

At time $(n - 1) \Delta t$ the parcel is at (●), which is not a gridpoint.

The value at the \textit{departure point} is obtained by interpolation from \textit{surrounding points}.

Thus we ensure that the physical trajectory is \textit{within} the domain of numerical dependence.
The advection equation in Lagrangian form may be written
\[ \frac{dY}{dt} = 0. \]
In physical terms, this equation says that the value of \( Y \) is constant for a fluid parcel.
Applying the equation over the time interval \([n\Delta t, (n+1)\Delta t]\), we get
\[
\begin{pmatrix}
\text{Value of } Y \text{ at point } m\Delta x \text{ at time } (n+1)\Delta t \\
\text{Value of } Y \text{ at departure point at time } n\Delta t
\end{pmatrix}
= \begin{pmatrix}
\text{Value of } Y \text{ at point } m\Delta x \text{ at time } n\Delta t
\end{pmatrix}
\]
or, in symbolic form,
\[ Y_{m+1}^n = Y_m^n \]
where \( Y_m^n \) represents the value at the departure point, normally not a grid point.

**Numerical Stability of the Scheme**

The discrete equation may be written
\[ Y_{m+1}^n = \alpha Y_{m-p-1}^n + (1 - \alpha) Y_{m-p}^n. \]
Let us look for a solution of the form
\[ Y_m^n = a A^n \exp(ikm\Delta x). \]
Substituting into the equation we get
\[
a A^{n+1} \exp(ikm\Delta x) = \alpha a A^n \exp[ik(m-p-1)\Delta x] + (1 - \alpha) a A^n \exp[ik(m-p)\Delta x]
\]
or, removing common terms,
\[ A = \alpha \exp[ik(-p-1)\Delta x] + (1 - \alpha) \exp[ik(-p)\Delta x]. \]
Again,
\[ A = \alpha \exp[ik(-p-1)\Delta x] + (1 - \alpha) \exp[ik(-p)\Delta x] \]
\[ = \exp(-ikp\Delta x) \cdot [(1 - \alpha) + \alpha \exp(-ik\Delta x)]. \]
Now consider the squared modulus of \( A \):
\[ |A|^2 = |\exp(-ikp\Delta x)|^2 \cdot |(1 - \alpha) + \alpha \exp(-ik\Delta x)|^2 \]
\[ = [(1 - \alpha) + \alpha \cos k\Delta x - \alpha \sin k\Delta x]^2 \]
\[ = [(1 - \alpha) + \alpha \cos k\Delta x]^2 + [\sin k\Delta x]^2 \]
\[ = (1 - \alpha)^2 + 2(1 - \alpha)\alpha \cos k\Delta x + \alpha^2 \cos^2 k\Delta x + \alpha^2 \sin^2 k\Delta x \]
\[ = 1 - 2\alpha(1 - \alpha)[1 - \cos k\Delta x]. \]
We note that \( 0 \leq (1 - \cos \theta) \leq 2. \) Taking the largest value of \( 1 - \cos k\Delta x \) gives
\[ |A|^2 = 1 - 4\alpha(1 - \alpha) = (1 - 2\alpha)^2 < 1. \]
Taking the smallest value of \( 1 - \cos k\Delta x \) gives
\[ |A|^2 = 1. \]
In either case, \(|A|^2 \leq 1\), so there is numerical stability.
We have determined the departure point by interpolation.

This ensures that $0 \leq \alpha < 1$.

This in turn ensures that $|A| \leq 1$.

In other words, we have unconditional numerical stability.

The implication is that the time step is unlimited.

In contradistinction to the Eulerian scheme there is no CFL criterion.

Of course, we must consider accuracy as well as stability.

The time step $\Delta t$ is chosen to ensure sufficient accuracy, but can be much larger than for an Eulerian scheme.

Typically, $\Delta t$ is about six times larger for a semi-Lagrangian scheme than for an Eulerian scheme.

This is a substantial gain in computational efficiency.