Fundamentals of Atmospheric Modelling

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Lecture 6

Vorticity and Divergence

Introduction

In this Lecture we will continue our investigation of the properties of the Shallow Water Equations (SWE).

We introduce *vorticity* and *divergence* and derive equations for them.

We show that an arbitrary velocity field may be partitioned into curl-free and divergence-free components.

Also, we show that the velocity may be reconstructed from knowledge of vorticity and divergence.

The most important result we derive is the *Conservation of Potential Vorticity.*

Recall the form of the SWE:

$$\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-fv+\frac{\partial \Phi}{\partial x}=0 \tag{1}$$

$$\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+fu+\frac{\partial \Phi}{\partial y}=0 \tag{2}$$

$$\left(\frac{\partial \Phi}{\partial t}+u\frac{\partial \Phi}{\partial x}+v\frac{\partial \Phi}{\partial y}\right)+\Phi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \tag{3}$$

The geopotential is $\Phi = gh$ and the Coriolis parameter is $f = 2\Omega\sin\phi$. Recall that we have neglected all effects of spherical geometry except in the Coriolis term.

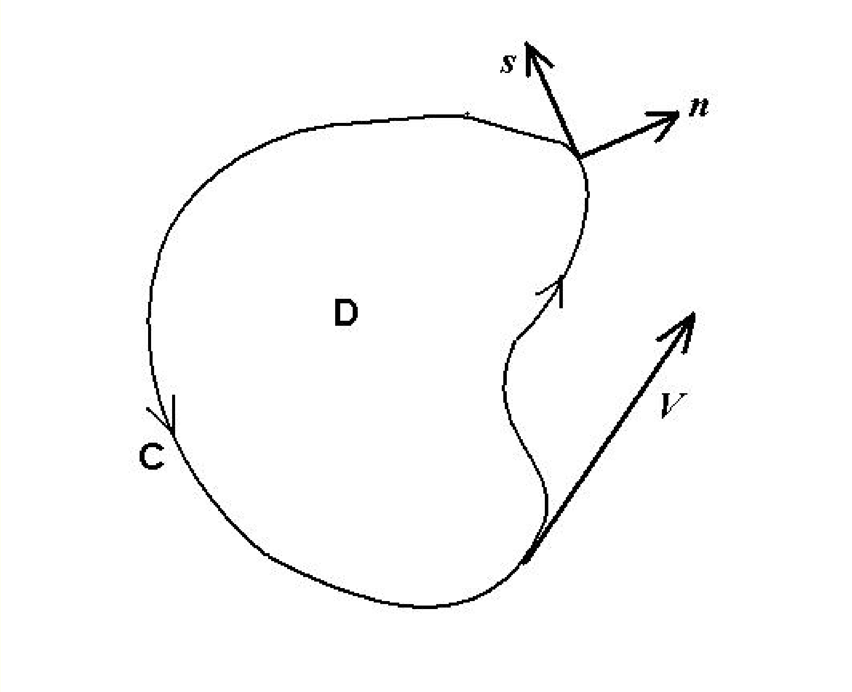
We define the *beta parameter*:

$$\beta = \frac{df}{dy} = \frac{2\Omega\cos\phi}{a}\,.$$

For latitudes ϕ not too far from a central value ϕ_0, we may assume that

$$f = 2\Omega\sin\phi \approx 2\Omega\sin\phi_0 \qquad \text{and} \qquad \beta = \frac{2\Omega\cos\phi}{a} \approx \frac{2\Omega\cos\phi_0}{a}$$

are both constant, unless differentiated w.r.t. y.



Tangent and normal unit vectors s *and* n.

"Spin" and "Spread"

The extent to which the fluid is *rotating* may be measured by calculating the *circulation* around a small circle C and taking the limit as the area A goes to zero:

$$\zeta = \lim_{A\to 0} \frac{1}{A} \oint_C \mathbf{V}\cdot\mathbf{s}\,ds\,.$$

We may call this the ***Spin*** or, more usually, the ***Vorticity***.

The extent to which the fluid is *spreading* may be measured by calculating the *outward flux* from a small circle C and taking the limit as the area A goes to zero:

$$\delta = \lim_{A\to 0} \frac{1}{A} \oint_C \mathbf{V}\cdot\mathbf{n}\,ds\,.$$

We may call this the ***Spread*** or, more usually, the ***Divergence***.

Using Stokes' and Gauss's Theorems, we will obtain differential forms of the vorticity and divergence.

First, consider Stokes' Theorem:

$$\oint_C \mathbf{V}\cdot\mathbf{s}\,ds = \iint_A \mathbf{k}\cdot\nabla\times\mathbf{V}\,da\,.$$

Assuming the area A of the circle is small, we get

$$\frac{1}{A}\oint_C \mathbf{V}\cdot\mathbf{s}\,ds \approx \mathbf{k}\cdot\nabla\times\mathbf{V}\,.$$

Taking the limit $A \longrightarrow 0$, we define the vorticity as

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V}$$

Now recall Gauss's Theorem

$$\oint_C \mathbf{V}\cdot\mathbf{n}\,ds = \iint_A \nabla\cdot\mathbf{V}\,da\,.$$

Assuming the area A of the circle is small, we get

$$\frac{1}{A}\oint_C \mathbf{V}\cdot\mathbf{n}\,ds \approx \nabla\cdot\mathbf{V}\,.$$

Taking the limit $A \longrightarrow 0$, we define the divergence as

$$\delta = \nabla\cdot\mathbf{V}$$

We define the *vorticity* and *divergence* as follows:

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\delta = \nabla\cdot\mathbf{V} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Note that ζ is the *vertical component* of the vorticity and δ is the *horizontal divergence*. However, we use the words *divergence* and *vorticity* to mean δ and ζ.

We will derive equations for the vorticity and divergence by differentiating and combining the momentum equations.

Exercise:

Show that the ratio of the vertical to horizontal component of the (3-D) vorticity is of the order w/V so that, with the assumptions we have made, the vertical component dominates.

If we relax the assumption $\partial\mathbf{V}/\partial z = 0$, how does this affect the conclusion?

Exercise: Geostrophic Divergence

Suppose the wind is geostrophic. Derive expressions for vorticity and divergence in terms of geopotential.

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The geostrophic velocity is given by

$$u = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \qquad v = +\frac{1}{f}\frac{\partial \Phi}{\partial x},$$

Calculating vorticity directly, we get

$$\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(+\frac{1}{f}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) = \frac{1}{f}\nabla^2\Phi + \frac{\beta}{f}u\,.$$

For constant f the stream function is $\psi = \Phi/f$.

Calculating divergence directly, we get

$$\delta = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(+\frac{1}{f}\frac{\partial \Phi}{\partial x}\right) = -\frac{\beta}{f}v\,.$$

For f constant, it follows immediately that $\delta = 0$. Therefore, the geostrophic divergence depends on the variation of f, the beta-effect.

Assume that the scale of motion is synoptic, $\mathsf{L} = 10^6\,\mathrm{m}$.
Since $a = 6371\,\mathrm{km} \approx 10^7\,\mathrm{m}$, we may assume

$$\frac{\mathsf{L}}{a} \sim \mathsf{Ro} \ll 1\,.$$

Let us assume that the flow is approximately geostrophic. Then

$$\zeta \sim \frac{\mathsf{V}}{\mathsf{L}} \qquad \delta \sim \frac{\mathsf{V}}{a} \sim \frac{\mathsf{L}}{a}\zeta\,.$$

Thus, for typical synoptic motions, the divergence is *an order of magnitude smaller* than the vorticity

$$\delta \sim \mathsf{Ro}\,\zeta$$

The smallness of the divergence is due to approximate cancellation between influx and outflow. The terms $\partial u/\partial x$ and $\partial v/\partial y$ are roughly equal in magnitude but opposite in sign. *This makes accurate calculation of divergence very difficult.*

The Helmholtz Theorem (2D-Form)

A fundamental theorem due to Stokes (1849) states that a velocity field can be decomposed into the sum of an irrotational (curl-free) and a non-divergent part.

$$\mathbf{V} = \mathbf{V}_{\mathrm{D}} + \mathbf{V}_{\mathrm{R}} = \nabla\chi + \nabla\times\mathbf{A}\,,$$

where χ and $\mathbf{A}$ are called the scalar and vector potentials. It is possible to impose an additional constraint, $\nabla\cdot A = 0$.

For <u>two-dimensional flow</u>, the decomposition is as follows:

$$\mathbf{V} = \mathbf{V}_\chi + \mathbf{V}_\psi = \nabla\chi + \mathbf{k}\times\nabla\psi\,.$$

Here ψ and χ are the *stream function* and *velocity potential.*

We recall the vector identities ($\mathrm{curl}\,\mathrm{grad}\,\chi = 0$ and $\mathrm{div}\,\mathrm{curl}\,\mathbf{A} = 0$):

$$\nabla\times\nabla\chi = 0 \qquad \nabla\cdot(\nabla\times\mathbf{A}) = 0\,.$$

In 2-D, the latter implies $\nabla\cdot\mathbf{k}\times\nabla\psi = 0$.

Thus,

$$\delta = \nabla\cdot\mathbf{V} = \nabla^2\chi \qquad \zeta = \mathbf{k}\cdot\nabla\times\mathbf{V} = \nabla^2\psi\,.$$

Note the useful vector identity: $\mathbf{k}\cdot\nabla\times\mathbf{V} = \nabla\cdot(\mathbf{V}\times\mathbf{k})$.

Given the vorticity and divergence, we can recover the velocity field provided appropriate boundary conditions are specified.

Procedure:

(1) Solve the two *Poisson equations*

$$\nabla^2\chi = \delta\,, \qquad \nabla^2\psi = \zeta\,,$$

for the stream function and velocity potential.

(2) Calculate the wind from

$$\mathbf{V} = \nabla\chi + \mathbf{k}\times\nabla\psi\,.$$

or, in component form

$$u = \frac{\partial\chi}{\partial x} - \frac{\partial\psi}{\partial y}\,, \qquad v = \frac{\partial\chi}{\partial y} + \frac{\partial\psi}{\partial x}\,.$$

The Vorticity Equation

Recall the momentum equations

$$\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-fv+\frac{\partial \Phi}{\partial x}=0 \qquad (1)$$

$$\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+fu+\frac{\partial \Phi}{\partial y}=0 \qquad (2)$$

Taking the x-derivative of (2) and subtracting from it the y-derivative of (1), we get an equation for ζ:

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+(\zeta+f)\delta+\beta v=0\,.$$

Note that, since f is independent of time,

$$\frac{df}{dt}=v\frac{\partial f}{\partial y}=\beta v\,.$$

Thus the *vorticity equation* may also be written:

$$\frac{d}{dt}(\zeta+f)+(\zeta+f)\delta=0\,.$$

The *absolute vorticity* η is defined as the sum of *relative vorticity* and *planetary vorticity*:

$$\underbrace{\eta}_{\text{Absolute Vorticity}}=\underbrace{\zeta}_{\text{Relative Vorticity}}+\underbrace{f}_{\text{Planetary Vorticity}}\,.$$

The vorticity equation may now be written

$$\frac{1}{\eta}\frac{d\eta}{dt}+\delta=0\,.$$

The relative rate-of-change of absolute vorticity is equal to (minus) the divergence.

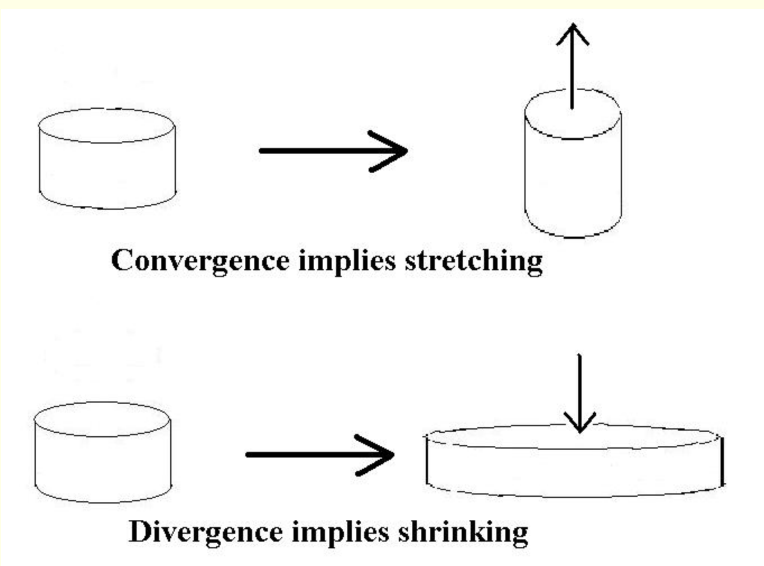
We note the formal similarity to the continuity equation:

$$\frac{1}{h}\frac{dh}{dt}+\delta=0\,.$$

The relative rate-of-change of depth is equal to (minus) the divergence.

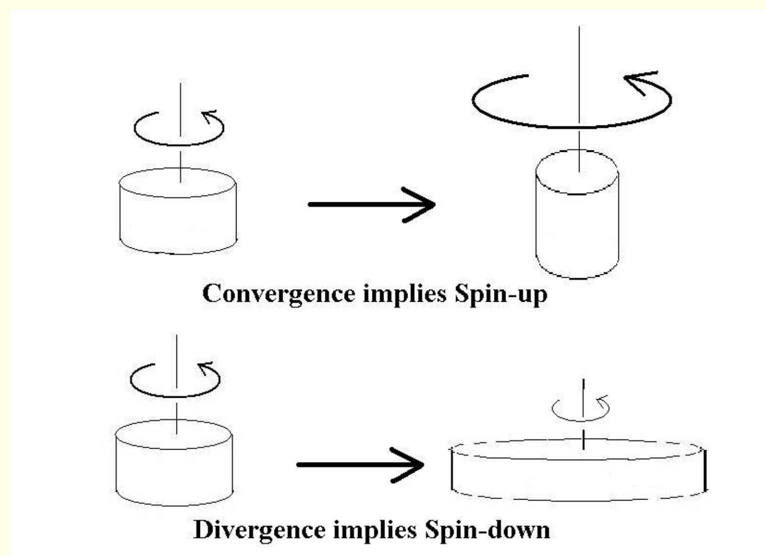
We may illustrate this by considering a column of fluid.

The *Continuity Equation* may be interpreted pictorially.



Convergence is associated with *stretching* of the column.
Divergence is associated with *shrinking* of the column.

The *Vorticity Equation* may be interpreted pictorially.



Convergence is associated with *spin-up* of the fluid column.
Divergence is associated with *spin-down* of the column.

The Divergence Equation

Recall again the momentum equations

$$\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-fv+\frac{\partial\Phi}{\partial x}=0 \qquad (1)$$

$$\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+fu+\frac{\partial\Phi}{\partial y}=0 \qquad (2)$$

Taking the x-derivative of (1) and adding it to the y-derivative of (2), we get an equation for δ:

$$\frac{\partial\delta}{\partial t}+u\frac{\partial\delta}{\partial x}+v\frac{\partial\delta}{\partial y}-\zeta f+\delta^2-2J(u,v)+\beta u+\nabla^2\Phi=0\,.$$

The Jacobian term is defined as

$$J(u,v)=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)\,.$$

Note: The derivation of the divergence equation in the above form is elementary, but it requires a page or two of algebraic manipulation.

Since we will not make explicit use of the full divergence equation, we need not consider it further.

Observation: for large-scale atmospheric flow in middle latitudes, the divergence is much smaller than the vorticity:

$$|\delta|\ll|\zeta|\,.$$

This allows us to make approximations to the equations.

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New Definitions:

The flow is ***cyclonic*** if $f\zeta>0$.
The flow is ***anticyclonic*** if $f\zeta<0$.

The Potential Vorticity Equation

The continuity equation may be written:

$$\frac{dh}{dt}+h\delta=0$$

The vorticity equation may be written:

$$\frac{d}{dt}(\zeta+f)+(\zeta+f)\delta=0\,.$$

Wo eliminate δ between the vorticity and continuity equations to get:

$$\frac{1}{\zeta+f}\frac{d(\zeta+f)}{dt}=\frac{1}{h}\frac{dh}{dt}\,.$$

This may also be put in the following form (take logs):

$$\frac{d}{dt}\left(\frac{\zeta+f}{h}\right)=0\,.$$

This is the equation of *conservation of potential vorticity.*

Exercise: Bottom Orography

We have assumed the bottom surface is flat. Now we will relax this.

Assume the height of the bottom boundary is $h_{\rm B}(x,y)$.

Show that the Conservation of Potential Vorticity takes the form:

$$\frac{d}{dt}\left(\frac{\zeta+f}{h-h_{\rm B}}\right)=0\,.$$

This states that the following ratio is conserved:

$$\frac{d}{dt}\left(\frac{\text{Absolute Vorticity}}{\text{Fluid Depth}}\right)=0\,.$$

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This is an important exercise. The solution will not be given. The proof is straightforward, requiring only a minor adjustment of the derivation in the case $h_{\rm B}(x,y)\equiv 0$.