Lecture 6

Vorticity and Divergence

Recall the form of the SWE:

\[
\begin{align*}
\left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) - fv + \frac{\partial \Phi}{\partial x} &= 0 \quad (1) \\
\left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + fu + \frac{\partial \Phi}{\partial y} &= 0 \quad (2) \\
\left( \frac{\partial \Phi}{\partial t} + u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} \right) + \Phi \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 \quad (3)
\end{align*}
\]

The geopotential is \( \Phi = gh \) and the Coriolis parameter is \( f = 2\Omega \sin \phi \). Recall that we have neglected all effects of spherical geometry except in the Coriolis term.

We define the beta parameter:

\[
\beta = \frac{df}{dy} = \frac{2\Omega \cos \phi}{a}.
\]

For latitudes \( \phi \) not too far from a central value \( \phi_0 \), we may assume that

\[
f = 2\Omega \sin \phi \approx 2\Omega \sin \phi_0 \quad \text{and} \quad \beta = \frac{2\Omega \cos \phi}{a} \approx \frac{2\Omega \cos \phi_0}{a}
\]

are both constant, unless differentiated w.r.t. \( y \).
“Spin” and “Spread”

The extent to which the fluid is rotating may be measured by calculating the circulation around a small circle $C$ and taking the limit as the area $A$ goes to zero:

$$\zeta = \lim_{A \to 0} \frac{1}{A} \oint_{C} \mathbf{V} \cdot \mathbf{s} \, ds.$$  

We may call this the Spin or, more usually, the Vorticity.

The extent to which the fluid is spreading may be measured by calculating the outward flux from a small circle $C$ and taking the limit as the area $A$ goes to zero:

$$\delta = \lim_{A \to 0} \frac{1}{A} \oint_{C} \mathbf{V} \cdot \mathbf{n} \, ds.$$  

We may call this the Spread or, more usually, the Divergence.

Using Stokes’ and Gauss’s Theorems, we will obtain differential forms of the vorticity and divergence.

First, consider Stokes’ Theorem:

$$\oint_{C} \mathbf{V} \cdot \mathbf{s} \, ds = \iint_{A} \mathbf{k} \cdot \nabla \times \mathbf{V} \, da.$$  

Assuming the area $A$ of the circle is small, we get

$$\frac{1}{A} \oint_{C} \mathbf{V} \cdot \mathbf{s} \, ds \approx \mathbf{k} \cdot \nabla \times \mathbf{V}.$$  

Taking the limit $A \to 0$, we define the vorticity as

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V}$$

Now recall Gauss’s Theorem

$$\oint_{C} \mathbf{V} \cdot \mathbf{n} \, ds = \iint_{A} \nabla \cdot \mathbf{V} \, da.$$  

Assuming the area $A$ of the circle is small, we get

$$\frac{1}{A} \oint_{C} \mathbf{V} \cdot \mathbf{n} \, ds \approx \nabla \cdot \mathbf{V}.$$  

Taking the limit $A \to 0$, we define the divergence as

$$\delta = \nabla \cdot \mathbf{V}$$

We define the vorticity and divergence as follows:

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\delta = \nabla \cdot \mathbf{V} = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Note that $\zeta$ is the vertical component of the vorticity and $\delta$ is the horizontal divergence. However, we use the words divergence and vorticity to mean $\delta$ and $\zeta$.

We will derive equations for the vorticity and divergence by differentiating and combining the momentum equations.

**Exercise:**

Show that the ratio of the vertical to horizontal component of the (3-D) vorticity is of the order $w/V$ so that, with the assumptions we have made, the vertical component dominates.

If we relax the assumption $\partial V/\partial z = 0$, how does this affect the conclusion?
Exercise: Geostrophic Divergence

Suppose the wind is geostrophic. Derive expressions for vorticity and divergence in terms of geopotential.

\[ \star \star \star \]

The geostrophic velocity is given by

\[ u = -\frac{1}{f} \frac{\partial \Phi}{\partial y}, \quad v = +\frac{1}{f} \frac{\partial \Phi}{\partial x}. \]

Calculating vorticity directly, we get

\[ \zeta = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( +\frac{1}{f} \frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y} \left( -\frac{1}{f} \frac{\partial \Phi}{\partial y} \right) = \frac{1}{f} \nabla^2 \Phi + \frac{\beta}{f} u. \]

For constant \( f \) the stream function is \( \psi = \Phi/f \).

Calculating divergence directly, we get

\[ \delta = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \frac{\partial}{\partial x} \left( -\frac{1}{f} \frac{\partial \Phi}{\partial y} \right) + \frac{\partial}{\partial y} \left( +\frac{1}{f} \frac{\partial \Phi}{\partial x} \right) = -\frac{\beta}{f} v. \]

For \( f \) constant, it follows immediately that \( \delta = 0 \). Therefore, the geostrophic divergence depends on the variation of \( f \), the beta-effect.

The geostrophic velocity is given by

\[ V = \nabla \chi + k \times \nabla \psi. \]

The smallness of the divergence is due to approximate cancellation between influx and outflow. The terms \( \partial u/\partial x \) and \( \partial v/\partial y \) are roughly equal in magnitude but opposite in sign. This makes accurate calculation of divergence very difficult.

\[ \delta \sim Ro \zeta \]

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\[ \delta = \nabla \cdot V = \nabla^2 \chi \quad \zeta = k \cdot \nabla \times V = \nabla^2 \psi. \]

Note the useful vector identity: \( k \cdot \nabla \times V = \nabla \cdot (V \times k) \).

Given the vorticity and divergence, we can recover the velocity field.

Procedure:

1. Solve the two Poisson equations

\[ \nabla^2 \chi = \delta, \quad \nabla^2 \psi = \zeta, \]

for the stream function and velocity potential.

2. Calculate the wind from

\[ V = \nabla \chi + k \times \nabla \psi. \]

or, in component form

\[ u = \frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \chi}{\partial y} + \frac{\partial \psi}{\partial x}. \]
The Vorticity Equation

Recall the momentum equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv + \frac{\partial \Phi}{\partial x} = 0 \tag{1}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu + \frac{\partial \Phi}{\partial y} = 0 \tag{2}
\]

Taking the \(x\)-derivative of (2) and subtracting from it the \(y\)-derivative of (1), we get an equation for \(\zeta\):

\[
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (\zeta + f)\delta + \beta v = 0.
\]

Note that, since \(f\) is independent of time,

\[
\frac{df}{dt} = v \frac{\partial f}{\partial y} = \beta v.
\]

Thus the vorticity equation may also be written:

\[
\frac{d}{dt}(\zeta + f) + (\zeta + f)\delta = 0.
\]

The absolute vorticity \(\eta\) is defined as the sum of relative vorticity and planetary vorticity:

\[
\eta_{\text{Absolute Vorticity}} = \zeta + f_{\text{Planetary Vorticity}}.
\]

The vorticity equation may now be written

\[
\frac{1}{\eta} \frac{d\eta}{dt} + \delta = 0.
\]

The relative rate-of-change of absolute vorticity is equal to (minus) the divergence.

We note the formal similarity to the continuity equation:

\[
\frac{1}{h} \frac{dh}{dt} + \delta = 0.
\]

The relative rate-of-change of depth is equal to (minus) the divergence.

We may illustrate this by considering a column of fluid.

The Continuity Equation may be interpreted pictorially.

Convergence is associated with stretching of the column. Divergence is associated with shrinking of the column.

The Vorticity Equation may be interpreted pictorially.

Convergence is associated with spin-up of the fluid column. Divergence is associated with spin-down of the column.
The Divergence Equation

Recall again the momentum equations

\[
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial \Phi}{\partial x} = 0 \quad (1)
\]

\[
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial \Phi}{\partial y} = 0 \quad (2)
\]

Taking the \(x\)-derivative of (1) and adding it to the \(y\)-derivative of (2), we get an equation for \(\delta\):

\[
\frac{\partial \delta}{\partial t} + u\frac{\partial \delta}{\partial x} + v\frac{\partial \delta}{\partial y} - \zeta f + \delta^2 - 2J(u, v) + \beta u + \nabla^2 \Phi = 0 .
\]

The Jacobian term is defined as

\[ J(u, v) = \left( \frac{\partial u \partial v}{\partial x \partial y} - \frac{\partial v \partial u}{\partial x \partial y} \right) . \]

Note: The derivation of the divergence equation in the above form is elementary, but it requires a page or two of algebraic manipulation.

Since we will not make explicit use of the full divergence equation, we need not consider it further.

Observation: for large-scale atmospheric flow in middle latitutes, the divergence is much smaller than the vorticity:

\[ |\delta| \ll |\zeta| . \]

This allows us to make approximations to the equations.

New Definitions:

The flow is **cyclonic** if \( f\zeta > 0 \).

The flow is **anticyclonic** if \( f\zeta < 0 \).

The Potential Vorticity Equation

The continuity equation may be written:

\[ \frac{dh}{dt} + h\delta = 0 \]

The vorticity equation may be written:

\[ \frac{d}{dt}(\zeta + f) + (\zeta + f)\delta = 0 . \]

We eliminate \(\delta\) between the vorticity and continuity equations to get:

\[ \frac{1}{\zeta + f} \frac{d(\zeta + f)}{dt} = \frac{1}{h} \frac{dh}{dt} . \]

This may also be put in the following form (take logs):

\[ \frac{d}{dt} \left( \frac{\zeta + f}{h} \right) = 0 . \]

This is the equation of **conservation of potential vorticity**.

Exercise: Bottom Orography

We have assumed the bottom surface is flat. Now we will relax this.

Assume the height of the bottom boundary is \( h_B(x, y) \).

Show that the Conservation of Potential Vorticity takes the form:

\[ \frac{d}{dt} \left( \frac{\zeta + f}{h - h_B} \right) = 0 . \]

This states that the following ratio is conserved:

\[ \frac{d}{dt} \left( \frac{\text{Absolute Vorticity}}{\text{Fluid Depth}} \right) = 0 . \]

This is an important exercise. The solution will not be given.

The proof is straightforward, requiring only a minor adjustment of the derivation in the case \( h_B(x, y) \equiv 0 \).