2

The fundamental equations

These equations contain the laws according to which subsequent atmospheric conditions develop from those that preceded them. (Bjerknes, 1914a)

The behaviour of the atmosphere is governed by the fundamental principles of conservation of mass, energy and momentum. Conservation of mass ensures that matter cannot be created or destroyed. Conservation of energy implies that internal energy can be altered only by performance of work or by adding or removing heat. The law of motion states that the momentum can be changed only by a force. These principles, expressed in quantitative form, provide the framework for the study of atmospheric dynamics. The physical principles may be expressed mathematically in terms of differential equations. The prediction of the future development of atmospheric motion systems—the basis of weather forecasting—amounts to calculating the solution of these equations, given the state of the atmosphere at some initial time.

The idea of solving the equations to calculate future weather was propounded by Bjerknes in his famous 1904 manifesto but, although he outlined in principle how this might be done, he did not construct a detailed plan for implementing his programme or attempt to carry it through to a practical realization. The first attempt to put this idea into practice was that of Richardson. In this chapter we will develop the equations used by Richardson, and begin to look at the way in which they might be solved.

Richardson was careful not to make any unnecessary approximations, and he took account of several physical processes that had the most marginal effect on his forecast. As we have seen, approximately half his book and more than half the computing forms were devoted to thermodynamic and hydrological processes and calculations that had only a minor influence on the outcome of his forecast.

He included in his equations many terms that are negligible; with the benefit of hindsight, we can omit most of these.

2.1 Richardson's general circulation model

Since we are concerned primarily with the reasons for Richardson's catastrophic forecast failure, which can be treated from a purely dynamical viewpoint, it is feasible and convenient to disregard all diabatic processes. In the sequel, we will ignore all the effects of moisture and thermal forcing, and consider the adiabatic evolution of a dry atmosphere. However, a full appreciation of Richardson's work requires at least a brief examination of the wide variety of physical processes that he discussed. Many of the physical phenomena relevant to the atmosphere were considered for the first time by Richardson. He constructed the basis for what was, in effect, a comprehensive physics parameterization package. If all the factors treated by him were included, one would have a model of the general circulation of the atmopshere, or, in modern terms, a General Circulation Model (GCM).

Richardson's description of physical processes was comprehensive and he contributed much that was original. His quantitative formulations of physical processes were based on the best field data available to him. Indeed, he carried out several innovative field experiments himself to measure a range of physical parameters. Ch. 4 of WPNP, entitled *The Fundamental Equations*, is 94 pages long and, in addition to the basic dynamical equations, contains detailed discussions of all major physical processes. The role of water in all its phases, and the thermodynamic consequences of phase changes, are discussed. Clouds and precipitation processes are considered: 'In order to save labour I have supposed there to be a sharp distinction between rain which falls and clouds which float. Actually there is a gradual transition' (WPNP, page 44). Short-wave solar radiation and long-wave terrestrial radiation and their interaction with the atmosphere are treated. Richardson notes the lack of observational data at wave-lengths exceeding 15 μm, remarking that, until such data is available, meteorologists 'must carry on business on premises which are, so to speak, in the hands of the builders' (WPNP, page 49). He introduces the concept of a 'parcel' of air (WPNP, page 50), an idea that has gained great popularity and utility.

The longest section of Ch. 4 deals with eddy motions. In this section, Richardson presents his famous rhyme, 'Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity', that beautifully encapsulates the essence of the turbulent energy cascade. He made fundamental contributions to turbulence theory and his writings here can still be read with profit. The dense style is occasionally lightened by a whimsical touch as, when discussing the tendency of turbulence to increase diversity, he writes 'This one can believe without the aid of mathematics, after watching the process of stirring together water and lime-juice' (WPNP, page 101). Finally, Richardson discusses the interaction between the atmosphere and the sea and land surfaces beneath it. He suggests that climatological sea temperatures may suffice, but also discusses how the sea surface temperature might be predicted. He considers heat and moisture transports within the soil and discusses at some length the influence of vegetation: 'Leaves, when present, exert a paramount influence on the interchanges of moisture and heat' (WPNP, page 111).[1] Clearly, Richardson is thinking far beyond short-range forecasting here, and has entered the realm of climate modelling.

The scope of Richardson's treatment of physical processes may be appreciated by examining the running headers of his Ch. 4; these are presented in Table 2.1. A full appraisal of his work in this area would entail a more intensive investigation than can be undertaken here:

> We will not have a true appreciation of Richardson's achievement in atmospheric modelling until his suite of physical parameterizations is implemented in his own or some other dynamical framework, and its performance validated by comparison with the best current formulations of physical processes (Hollingsworth, 1994).

Let us hope that, before too long, someone with the requisite expertise, energy and enthusiasm will undertake this task.

2.2 The basic equations

We will set out the basic equations as commonly used today, and then convert them to the form used by Richardson. Some of Richardson's notation is archaic and the modern equivalents will be used (a Table of notation used in this book appears in Appendix A; a full list of Richardson's notation may be found in Ch. XII of WPNP).

2.2.1 The exact equations

The basic principles of motion are embodied in Newton's second law: the rate of change of momentum is equal to the applied force. Euler formulated the equations of motion as they apply to the flow of a continuous fluid. Laplace, in his study of tides, took account of the dynamical effects of rotation, and derived the equations of motion in a frame of reference spinning with the Earth. Later, Gustave Gaspard Coriolis again showed how the equations must be modified to account for a rotating frame of reference. Coriolis was studying rotating hydraulic machinery; he did not consider the geophysical consequences of rotation, but he elucidated its dynamical

[1] Richardson even allows (on Form P_{VIII}) for the insolation due to a layer of dead leaves on the ground — in May!

Table 2.1. *Running headers,* Weather Prediction by Numerical Process, *Ch. 4.*

Page 23	The equation of continuity of mass
Page 25	The conveyance of water
Page 27	Conveyance of water, when W is given
Page 29	Conveyance of water, when μ is given
Page 31	Dynamics
Page 33	Dynamics of strata
Page 35	Forms of energy of air
Page 37	The flow of energy
Page 39	Adiabatic law independent of gravity and motion
Page 41	Entropy and potential temperature
Page 43	Temperature of air supply
Page 45	Uniform cloud and precipitation
Page 47	Radiation and absorption of air for long waves
Page 49	Need for data beyond 15 microns
Page 51	Distribution of long-wave radiation
Page 53	Estimate of absorptivity for long waves
Page 55	Water vapour and long waves
Page 57	Solar radiation
Page 59	Simplifying approximations for solar radiation
Page 61	Absorption and scattering of solar radiation
Page 63	Numerical constants for solar radiation
Page 65	Turbulence
Page 67	The equation for eddy-diffusion
Page 69	Diffusion of momentum by eddies
Page 71	The eddy-flux of heat
Page 73	Eddy-viscosity
Page 75	Diffusion of water and of potential temperature
Page 77	Criterion of turbulence
Page 79	c as a function of height and of stability
Page 81	Diffusion treated by upper conventional strata
Page 83	Surface friction
Page 85	Surface friction
Page 87	Flux of heat at the surface
Page 89	Partition of heat formed from radiation at surface
Page 91	Flux of water treated by thick lower stratum
Page 93	Thick strata *versus* thin
Page 95	Irregularities of temperature
Page 97	Pressure of heterogeneity
Page 99	Exact form of eddy-stresses
Page 101	Production of heterogeneity by eddies
Page 103	Dissipation of heterogeneity by molecular diffusion
Page 105	Temperature of sea
Page 107	Evaporation from fallow land
Page 109	Creeping and distillation of water in soil
Page 111	Three ways by which heat moves through soil
Page 113	Water transpired by foliage

effects.[2] Navier and Stokes derived the form of the additional terms required to allow for the effects of frictional forces. The full equations of motion are now called the Navier-Stokes equations.

The forces influencing the motion of the atmosphere are those due to pressure gradients, gravity and friction and the apparent deflective forces due to rotation. If we consider a unit mass of air, the equation of motion may be written

$$\frac{d\mathbf{U}}{dt} = -2\boldsymbol{\Omega} \times \mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{F} + \mathbf{g}^* - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) . \tag{2.1}$$

The dependent variables here are the velocity $\mathbf{U}$ relative to the rotating Earth, the pressure p and the density ρ. The independent variables are the radius vector $\mathbf{r}$ from the Earth's centre and the time t. The frictional force is denoted $\mathbf{F}$. This equation is valid for a frame of reference fixed to the Earth and rotating with angular velocity $\boldsymbol{\Omega}$. The effects of rotation are accounted for by the Coriolis force $-2\boldsymbol{\Omega} \times \mathbf{U}$ and a centrifugal term $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$. Since the latter depends only on position, we may combine it with the true gravity $\mathbf{g}^*$ to produce an apparent gravitational acceleration

$$\mathbf{g} = \mathbf{g}^* - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) .$$

This composite force is the quantity actually measured at the Earth's surface. The magnitude g of $\mathbf{g}$ varies by only a few percent below 100 km (although, in his meticulous fashion, Richardson allowed for variations in g; see §2.2.3 below).

Equation (2.1) is effectively in the form set down by the American meteorologist William Ferrel in about 1860 (of course, his notation was different). Ferrel's derivation of the equations as they apply in a rotating frame fixed to the Earth was independent of the work of Coriolis, and sprang directly from the *Méchanique Céleste* of Laplace. Ferrel (1859) was the first to present a comprehensive treatment of the geophysical implications of the deflecting force due to rotation, and to deduce its consequences for the general circulation of the atmosphere and oceans (Kutzbach, 1979). He gave a quantitative description of the geostrophic wind and an account of the thermal wind relation. Ferrel considered the possibility of deriving a mathematical expression for the general circulation but felt that this could not be done because the frictional terms were inadequately known. Lorenz (1967) observed that

> It was a great loss to nineteenth-century meteorology that the man who introduced the equations of motion never saw fit to seek a complete solution of them.

The indestructability of mass is expressed in terms of the continuity equation

$$\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{U} = 0 .$$

[2] 'The term "Coriolis acceleration" . . . is frequently used by oceanographers and meteorologists without appreciation that it was first introduced by Laplace before Coriolis was born.' (Cartwright, 1999)

Here, as in (2.1), the time derivative is the material derivative following the flow:

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \mathbf{U}\cdot\nabla(\ )\,.$$

Using this, the continuity equation may be written in *flux form*:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{U} = 0\,. \tag{2.2}$$

The principle of conservation of energy, which is the basis of thermodynamics, was formulated in the nineteenth century by Clausius, Helmholtz, Joule and Kelvin amongst others. In the context of atmospheric dynamics, it states that the heat energy added to a parcel of air may increase its internal energy or induce it to do work by expansion, and that the sum of these is equal to the energy supplied. Thus, the first law of thermodynamics gives the change of the internal energy (c_vT) of a unit mass of air, in terms of work done and heat energy supplied:

$$c_v\frac{dT}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right) = \dot{Q}\,, \tag{2.3}$$

where T is the temperature, c_v the specific heat at constant volume and $\dot{Q}$ the heating rate.

Finally, we treat the atmosphere as an ideal gas, obeying Boyle's Law and Charles' Law, so that the equation of state is

$$p = \Re\rho T \tag{2.4}$$

where $\Re$ is the gas constant for dry air ($\Re = c_p - c_v$ is the difference of the specific heats at constant pressure and volume).

Equations (2.1)–(2.4) comprise a complete set of equations for the dependent variables $(\mathbf{U}, p, \rho, T)$, provided we can specify the energy sources and sinks represented by $\dot{Q}$ and **F**. The ultimate source of all atmospheric motion is the energy radiated by the Sun. The source term $\dot{Q}$ is thus the dominant factor in determining the dynamics and climate of the atmosphere. However, for time-scales of a day or so, a reasonable approximation to reality is obtained when diabatic forcing is ignored. As this hugely simplifies the problem, we will disregard all radiative and other diabatic processes by setting $\dot{Q} = 0$. We may also ignore the frictional drag **F**, which has a relatively small impact in the free atmosphere. Finally, we have omitted all consideration of moisture, which is of paramount importance in the real atmosphere. Richardson devoted considerable attention to such matters but it transpired that they did not affect his final results in any major way. With these simplifying assumptions, the system of equations is complete.

2.2.2 *The primitive equations*

A predominant feature of the atmosphere is that the gravitational force is almost exactly cancelled by the vertical pressure gradient force. The assumption that this balance holds exactly is called the hydrostatic approximation, expressed as

$$\frac{\partial p}{\partial z} + g\rho = 0\,. \tag{2.5}$$

This fundamental approximation was made by Richardson. It was an essential step, necessitated by the lack of observations of vertical velocity, w. Since there is no longer an equation for dw/dt, another means of deducing the vertical velocity is required. We return to this in §2.3 below. Richardson's adoption of the hydrostatic approximation was influenced by the work of Bjerknes (WPNP, p. *viii*; Dover Edition, p. *xii*).

Following Richardson, we now simplify the exact equations by introducing the hydrostatic approximation, together with an assumption that the atmosphere is a thin layer on the Earth's surface (the shallow atmosphere assumption). We also make the 'traditional approximation' of neglecting the Coriolis terms involving the vertical velocity. Richardson included the Coriolis terms proportional to $\cos\phi$. However, it is now known that, for a shallow atmosphere, it is dynamically inconsistent to include these terms. They must be omitted in order to maintain the conservation of angular momentum if the shallowness assumption is adopted (Phillips, 1966).

The result of the above approximations is the set of equations known as the primitive equations. They may be found in standard works on dynamic meteorology (*e.g.*, Lorenz, 1967; Phillips, 1973; Holton, 2004). Lorenz presents a detailed derivation of the primitive equations, carefully clarifying each approximation that he makes. He stresses the requirement that appropriate energy and angular momentum principles should hold for any approximate system.

The horizontal equations of motion, in co-ordinate form, become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \left(f + \frac{u\tan\phi}{a}\right)v + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2.6}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \left(f + \frac{u\tan\phi}{a}\right)u + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0\,. \tag{2.7}$$

The Earth's radius is a, its angular velocity is Ω and $f = 2\Omega\sin\phi$ is the Coriolis parameter. Distances eastward and northward on the globe are represented by x and y so that

$$\frac{\partial}{\partial x} = \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{1}{a}\frac{\partial}{\partial\phi}\,,$$

where λ and ϕ are the longitude and latitude. The momentum equation in vector form is

$$\left(\frac{d\mathbf{V}}{dt}\right)_H + f\mathbf{k}\times\mathbf{V} + \frac{1}{\rho}\nabla p = 0\,, \tag{2.8}$$

where $\mathbf{V} = (u, v)$ is the horizontal velocity, ∇ is the horizontal gradient operator with $1/a$ instead of $1/r$, $(d/dt)_H$ is the horizontal component of the total time-derivative and $\mathbf{k}$ is a vertical unit vector.

The continuity equation (2.2) becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} - \frac{\rho v\tan\phi}{a} + \frac{\partial\rho w}{\partial z} = 0 \tag{2.9}$$

(a negligibly small term $2\rho w/a$ has been dropped; naturally, Richardson retained it). In combination with this equation, the horizontal equations of motion may be written in flux form:

$$\begin{aligned}
\frac{\partial\rho u}{\partial t} + \frac{\partial\rho u^2}{\partial x} + \frac{\partial\rho uv}{\partial y} + \frac{\partial\rho uw}{\partial z} - \left(f + \frac{2u\tan\phi}{a}\right)\rho v + \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial\rho v}{\partial t} + \frac{\partial\rho vu}{\partial x} + \frac{\partial\rho v^2}{\partial y} + \frac{\partial\rho vw}{\partial z} + f\rho u + \frac{(\rho u^2 - \rho v^2)\tan\phi}{a} + \frac{\partial p}{\partial y} &= 0\,.
\end{aligned}$$

The adiabatic thermodynamic equation in co-ordinate form is

$$c_v\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) - \frac{p}{\rho^2}\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right) = 0\,. \tag{2.10}$$

Using the equation of state (2.4), this equation may be written

$$\frac{1}{\gamma p}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) - \frac{1}{\rho}\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right) = 0 \tag{2.11}$$

where $\gamma = c_p/c_v$ is the ratio of specific heats. The potential temperature is defined by $\theta = T(p/p_0)^{-\kappa}$, where $\kappa = \Re/c_p$ and $p_0 = 1000\,\mathrm{hPa}$. The adiabatic thermodynamic equation expresses conservation of potential temperature

$$\frac{d\theta}{dt} = \left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z}\right) = 0\,. \tag{2.12}$$

Defining the entropy per unit mass by $S = c_p\log\theta$, we see that entropy is conserved in adiabatic flow.

The complete system of equations

For convenience, we assemble the complete system of equations:

$$\frac{\partial\rho u}{\partial t} + \frac{\partial\rho u^2}{\partial x} + \frac{\partial\rho uv}{\partial y} + \frac{\partial\rho uw}{\partial z} - \left(f + \frac{2u\tan\phi}{a}\right)\rho v + \frac{\partial p}{\partial x} = 0 \qquad \text{(Ch. 4/4\#3)}$$

(Ch. 4/4#4) $$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho v u}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial \rho v w}{\partial z} + f\rho u + \frac{(\rho u^2 - \rho v^2)\tan\phi}{a} + \frac{\partial p}{\partial y} = 0$$

(Ch. 4/4#6) $$\frac{\partial p}{\partial z} + g\rho = 0(\text{Ch. } 4/2\#2)\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} - \frac{\rho v \tan\phi}{a} + \frac{\partial \rho w}{\partial z} = 0$$

(Ch. 4/5/0#18) $$c_v\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) = 0$$

(Ch. 4/1#1) $$p = \Re\rho T$$

Here, for example, the marginal number (Ch. 4/5/0#18) indicates a correspondence with equation (18) in §5/0 of Chapter 4 of WPNP. As we noted, Richardson made the shallow atmosphere assumption, replacing $r = a + z$ by a, but not the 'traditional approximation' of omitting the horizontal component of the Coriolis force and some small metric terms. Otherwise the equations assembled here are equivalent to his, with one curious exception, variations in the gravitational attraction, which we examine next.

2.2.3 Variations of gravity with height and latitude

The Earth is very close to a perfect sphere. The centrifugal force due to rotation has distorted the planet into an oblate spheroid, but the eccentricity is very small and deviations from sphericity are generally ignored; the difference between equatorial and polar radii is only about 20 km (about 0.3%). The centrifugal term is combined with the true gravity to produce an 'apparent gravity'. This force is everywhere perpendicular to the spheroid and defines a vertical direction on its surface so that the apparent gravity has no horizontal component on this surface.

We can write the Newtonian gravitational potential as $\Phi_e = -g_0 a^2/r$, where $g_0 = 9.80665\,\mathrm{m\,s^{-2}}$ is the standard value of gravitational acceleration.[3] We also note that the centrifugal acceleration $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$ is the gradient of the potential $\Phi_c = -\frac{1}{2}\Omega^2 R^2$, where R is the distance from the Earth's axis. Thus, the potential of apparent gravity is

$$\Phi = \Phi_e + \Phi_c = -g_0\frac{a^2}{r} - \tfrac{1}{2}\Omega^2 R^2 \, .$$

This is plotted in Fig. 2.1 (top left panel). Near the Earth, the potential surfaces are approximately spherical (actually, spheroidal with minute eccentricity). Further out, they are greatly distorted. The equilibrium point over the equator at about

[3] Variations of surface *g* with latitude are ignored, as we are primarily interested in the *horizontal component* of gravity, not the amplitude of the vertical component.

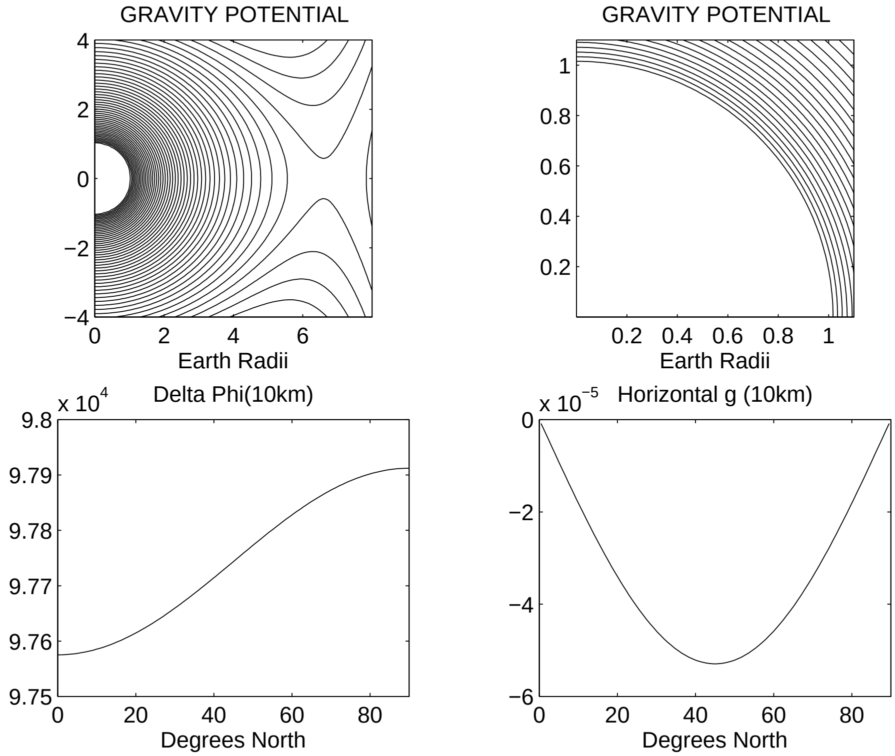

Fig. 2.1. Potential of apparent gravity. Top left: broad field, to eight Earth radii. Top right: close-up view. Bottom left: $\Delta\Phi = \Phi(10\text{km}) - \Phi$(mean sea level). Bottom right: horizontal component $g_H = -\partial\Delta\Phi/\partial y$ of $\mathbf{g}$ at 10 km.

seven Earth radii is valuable real estate, the location for geostationary satellites. Zooming in (top right panel), we see no evidence of assymetry near the Earth. However, examining the difference $\Delta\Phi$ between the potential at 10 km and the value at the surface as a function of latitude (lower left panel), there is a perceptable gradient. This implies a horizontal component of gravity $g_H = -\partial\Delta\Phi/\partial y$ (lower right panel) directed towards the equator. Its maximum amplitude is about $5 \times 10^{-5}\,\mathrm{m\,s^{-2}}$. Richardson was very cautious about making unnecessary assumptions. He remarks (WPNP, page 22) 'As the arithmetical method allows us to take account of the terms which are usually neglected, many of these terms have been included'. In particular, he allows for small variations in the gravitational acceleration. He includes a horizontal component of $\mathbf{g}$ which he denotes by g_N and tabulates as a function of latitude and height. Thus, his northward momentum equation can be written

$$\frac{\partial \rho v}{\partial t} = \rho g_N + \frac{\partial p}{\partial y} + \Big\{\text{Terms involving velocities}\Big\} \tag{2.13}$$

(see his Ch. 4/4#4 on page 31 of WPNP). Since g_N vanishes at the Earth's surface, there will be no acceleration there if the velocity is initially zero and p is independent of y. However, above the surface a meridional pressure gradient is required to balance g_N if a steady state is to be maintained. A mean value of $5 \times 10^{-5}\,\mathrm{m\,s^{-2}}$ would require for balance an equator-to-pole height difference of only about 30 m, a small fraction of the observed mean height difference at 250 hPa (annual mean about 1200 m). Nevertheless, it is a systematic effect. Richardson's tabulated values of g_N include variations in true gravity due to the Earth's eccentricity (as usual, he sought to use the most accurate values available to him). While these values differ somewhat from the values of $g_H = -\partial\Delta\Phi/\partial y$ defined above, the qualitative arguments still apply.

The term ρg_N in (2.13) amounts to about 17% of the total acceleration for the uppermost layer in Richardson's scheme (Form M_{IV}). This effect is usually ignored in current atmospheric models. The 'horizontal' component of gravity can be removed by choosing the gravitational potential Φ as a 'vertical' co-ordinate in place of z. This attractive alternative was briefly considered by Richardson and described by him as 'a rather tempting one', but he resisted the temptation to use it. Geopotential co-ordinates are discussed by Phillips (1973). With an appropriate origin, Φ/g_0 is very close in value to z for the lower atmosphere.

2.3 The vertical velocity equation

> 'If progress is to be possible, it can only be by eliminating the vertical velocity' (WPNP, page 115).

The vertical component of velocity in the atmosphere is typically two or three orders of magnitude smaller than the horizontal components. It is difficult to measure w and in general no observations of this variable are available. In particular, Richardson had no such observations for 0700 UTC on 20 May 1910, the initial time chosen for his forecast. Moreover, even if he had had such observations, he recognized the practical impossibility of computing the tendency $\partial w/\partial t$ which would have to be calculated as a tiny residual term in the vertical dynamical equation.

Richardson acknowledged the influence of Vilhelm Bjerknes' publications *Statics* and *Kinematics* (Bjerknes *et al.*, 1910, 1911) on his work. In his Preface (WPNP, p. *viii*; Dover Edition, p. *xii*) Richardson states that his choice of 'conventional strata', his use of specific momentum rather than velocity, his method of calculating vertical motion at ground level and his adoption of the hydrostatic approximation are all in accordance with Bjerknes' ideas.

The hydrostatic equation results from neglecting the vertical acceleration, and other small terms, in the vertical dynamical equation. But this precludes the possibility of calculating the acceleration $\partial w/\partial t$ directly. It was a stroke of genius for Richardson not only to realize the need to evaluate w diagnostically from the other fields but also to construct a magnificent mathematical equation to achieve this.

2.3.1 The tendency equation

An equation for the pressure tendency can be derived from the hydrostatic equation and continuity equation. The integrated form of the hydrostatic equation is

$$p = \int_z^\infty g\rho\, dz\,. \tag{2.14}$$

This is a mathematical statement of the physical assumption that the pressure at any point is determined by the weight of air above it. The continuity equation (2.9) may be written

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} + \frac{\partial \rho w}{\partial z} = 0$$

where $\mathbf{V}$ is the horizontal velocity and $\nabla\cdot(\quad)$ the horizontal divergence operator. Taking the time-derivative of (2.14), noting that the limits of integration are independent of time, and using the continuity equation, we obtain the following equation:

$$\frac{\partial p}{\partial t} = g\rho w - \int_z^\infty g\nabla\cdot\rho\mathbf{V}\, dz \tag{2.15}$$

(the boundary condition $\rho w \to 0$ as $z \to \infty$ has been used). This equation may be solved for the pressure tendency if the vertical velocity is known. In particular, if the lower limit is taken at $z = 0$ and the bottom boundary is assumed to be flat so that $w = 0$ there, we get

$$\frac{\partial p_S}{\partial t} = -\int_0^\infty g\nabla\cdot\rho\mathbf{V}\, dz\,, \tag{2.16}$$

sometimes called the *tendency equation*. This equation was discussed by Margules (1904) who recognized the impracticality of using it directly to forecast changes in pressure. He showed that tiny errors in the wind fields can result in spuriously large values for convergence of momentum and correspondingly unrealistic pressure tendency values (see §7.5). This was discovered, to his cost, by Richardson with the result that he obtained an unreasonable value for the pressure change; we will discuss this at length below.

2.3.2 Richardson's equation for vertical velocity

To construct Richardson's w-equation we eliminate the time dependency between the continuity equation and the thermodynamic equation using the hydrostatic equation. Recall that the thermodynamic equation can be written in the form

$$\frac{1}{\gamma p}\left(\frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla p + w\frac{\partial p}{\partial z}\right) - \frac{1}{\rho}\frac{d\rho}{dt} = 0\,, \tag{2.17}$$

and that one of the various forms of the continuity equation is

$$\frac{1}{\rho}\frac{d\rho}{dt} + \left(\nabla\cdot\mathbf{V} + \frac{\partial w}{\partial z}\right) = 0\,. \tag{2.18}$$

We can eliminate the pressure tendency in (2.17) using (2.15) and the density term by means of the continuity equation (2.18) to get

$$\frac{1}{\gamma p}\left(-\int_z^\infty g\nabla\cdot\rho\mathbf{V}\,dz + \mathbf{V}\cdot\nabla p\right) + \left(\nabla\cdot\mathbf{V} + \frac{\partial w}{\partial z}\right) = 0\,.$$

Expanding the integrand, using the hydrostatic equation again and rearranging, we get

$$\frac{\partial w}{\partial z} = -\nabla\cdot\mathbf{V} + \frac{1}{\gamma p}\int_z^\infty \left(g\rho\nabla\cdot\mathbf{V} - \frac{\partial\mathbf{V}}{\partial z}\cdot\nabla p\right)\,dz\,. \tag{2.19}$$

Since the upper limit of the integral is infinite, it is convenient to use pressure as the independent variable in the integral; this is done by using the hydrostatic equation once more, yielding the result:

$$\frac{\partial w}{\partial z} = -\nabla\cdot\mathbf{V} + \frac{1}{\gamma p}\int_0^p \left(\nabla\cdot\mathbf{V} - \frac{\partial\mathbf{V}}{\partial p}\cdot\nabla p\right)\,dp\,. \tag{2.20}$$

This corresponds to (9) on page 124 of WPNP, save that we have omitted the effects of moisture and diabatic forcing which were included by Richardson. Following Eliassen (1949) we call (2.19), or (2.20), *Richardson's Equation*.

The solution of (2.20) for w is straightforward. The gradient $\partial w/\partial z$ is calculated for each layer, working downwards from the stratosphere since the integral vanishes at $p = 0$. Then w may be calculated at the interface of each layer, working upwards, once it is known at the Earth's surface. Richardson followed Bjerknes in taking the surface value

$$w_S = (\mathbf{V}\cdot\nabla h)_S\ ,$$

where h is the surface elevation. This is equivalent to the kinematic condition that the ground is impervious to the wind. However, Richardson does not state how he evaluates $\mathbf{V}_S$, the horizontal wind at the surface; he merely states (WPNP, p. 178) that it has to be estimated from the statistics of its relation to the horizontal wind in the lowest layer. In repeating his forecast we have assumed a simple relationship

$$\mathbf{V}_S = k\mathbf{V}_5 \tag{2.21}$$

where $\mathbf{V}_5$ is the velocity for the lowest layer, and have fixed the constant of proportionality arbitrarily by chosing the value $k = 0.2$.

Richardson described a complicated method of calculating the vertical derivative of w in the stratosphere (his equation (21), WPNP, p. 138). This was required because he used a differentiated form of the vertical velocity equation (his (1) on page 178) and needed a boundary condition to integrate it. He alluded to this on page 119 of WPNP: 'As a matter of fact, the differential form was first derived directly from ... [the hydrostatic equation] and was employed throughout the example of Ch. 9; but a constant of integration kept on appearing inconveniently in places where it could not be determined. This "hysterical manifestation" was eventually traced to the suppression of the limits of integration which are now explicit in equation ... [(2.20)]'. We take the simpler path of using (2.20) directly so that only one boundary condition (w at the ground) is required. The vertical velocities thus obtained will be compared to the values in Richardson's Computing Form $\mathrm{P}_{\mathrm{XVI}}$, and reasonable agreement shown.

The vertical velocity equation was a major contribution by Richardson to dynamic meteorology. In recognizing its essential role in his forecast scheme he observed (WPNP, p. 178) that 'it might be called the keystone of the whole system, as so many other equations remain incomplete until the vertical velocity has been inserted'. In a hand-written note in the Revision File, Richardson draws an analogy between his forecasting algorithm and the workings of a motor-car engine. The 'enormous vertical velocity equation ... corresponds to the connecting rod for transmitting the power from the cylinders to the wheels. It is the sort of connecting rod that Heath Robinson[4] would delight to draw. And yet any connecting rod, even an ungainly one, is better than no connecting rod at all. And I am afraid there are to be found theories which omit this necessary link'.

2.4 Temperature in the stratosphere

Richardson devoted a full chapter of 24 pages to the stratosphere. We will not discuss the bulk of this, but we must consider the means by which the temperature of the uppermost layer is forecast. For, in the scheme adopted by Richardson, the vertical integral of pressure through the stratospheric layer depends on the temperature so that prediction of the latter is essential to ensure a 'lattice-reproducing'

[4] W. Heath Robinson (1872–1944) was a British cartoonist and illustrator who delighted in drawing outlandish and ingeneous mechanical contraptions. His autobiography, *My Line of Life*, was published in 1938.

scheme — that is, an algorithm which, starting with a set of variables at one instant, produces the corresponding set at a later instant.

Richardson calculated the change in stratospheric temperature using two different equations, his elaborate equation (8) on page 147 of WPNP and a much simpler equation corresponding to (14) on page 143. The resulting temperature tendencies, given in his Computing Form $\mathrm{P}_{\mathrm{XIV}}$ on page 201, were $9.1 \times 10^{-4}\,\mathrm{K\,s^{-1}}$ for the elaborate equation and $9.2 \times 10^{-4}\,\mathrm{K\,s^{-1}}$ for the simpler. In view of this close agreement, we will confine attention to the simpler alternative, which will now be derived.

The basic assumption for the stratosphere is that of vertical isothermy,

$$\frac{\partial T}{\partial z} = 0\,. \tag{2.22}$$

Since the effects of radiation are neglected here, the entropy $S = c_p \log\theta$ is conserved following the flow:

$$\frac{\partial S}{\partial t} + \mathbf{V}\cdot\nabla S + w\frac{\partial S}{\partial z} = 0\,. \tag{2.23}$$

Recalling the definition of potential temperature $\theta = T(p/p_0)^{-\kappa}$, we have

$$S = c_p \log T - \Re \log p + \text{constant}$$

so that, using (2.22) and the hydrostatic equation, we get

$$\frac{\partial S}{\partial z} = \frac{g}{T}\,. \tag{2.24}$$

Now take the vertical derivative of (2.23) to obtain

$$\frac{\partial}{\partial t}\left(\frac{g}{T}\right) + \mathbf{V}\cdot\nabla\left(\frac{g}{T}\right) + \frac{\partial \mathbf{V}}{\partial z}\cdot\nabla\left(c_p\log T - \Re\log p\right) + \frac{g}{T}\frac{\partial w}{\partial z} = 0$$

which, by simple rearrangement of terms, may be expressed

$$\frac{\partial T}{\partial t} = \frac{c_p T}{g}\frac{\partial \mathbf{V}}{\partial z}\cdot\nabla T - \mathbf{V}\cdot\nabla T - \frac{T}{g\rho}\frac{\partial \mathbf{V}}{\partial z}\cdot\nabla p + T\frac{\partial w}{\partial z}\,. \tag{2.25}$$

At this point the geostrophic wind approximation and its vertical derivative, the thermal wind equation, are introduced:

$$\mathbf{V}_{\mathrm{g}} = \frac{1}{f\rho}\mathbf{k}\times\nabla p\,; \qquad \frac{\partial \mathbf{V}_{\mathrm{g}}}{\partial z} = \frac{g}{fT}\mathbf{k}\times\nabla T\,.$$

If these are substituted into (2.25), the first right-hand term vanishes and the second and third terms on that side cancel, leaving the simple relationship

$$\frac{\partial T}{\partial t} = T\frac{\partial w}{\partial z}\,. \tag{2.26}$$

This equation is sufficient for predicting the stratospheric temperature as long as the assumptions of geostrophy and vertical isothermy are acceptable. We will use this simple prognostic equation in the sequel.

2.5 Pressure co-ordinates

We have used geometric height as the vertical co-ordinate. This seems the obvious choice, and was the one made by Richardson. However, he also briefly considered the possibility of interchanging the roles of pressure and height and of using isobaric co-ordinates. If pressure is treated as an independent variable, the heights of the isobaric surfaces become dependent variables, and the rate of change of pressure following the flow replaces w as a measure of vertical velocity. Since ρ and θ are functions only of p and T, isotherms on an isobaric surface are also lines of constant density and potential temperature, facilitating analysis. In the barotropic case, where ρ is a function of pressure only, isobaric (constant p), isothermal (constant T), isopycnal (constant ρ) and isentropic (constant S) surfaces all coincide.

Isobaric co-ordinates were used by Vilhelm Bjerknes in preparing his synoptic charts. He plotted the heights and other variables on ten sheets at the standard levels from 100 hPa to 1000 hPa. Richardson discussed this (WPNP, p. 17) and wrote 'This system readily yields elegant approximations'. But he considered that deformable co-ordinate surfaces that vary in time would be inconvenient. He set down the expressions for the relationships between the derivatives in the two systems:

$$\left(\frac{\partial\psi}{\partial x}\right)_z = \left(\frac{\partial\psi}{\partial x}\right)_p - \frac{\partial\psi}{\partial z}\left(\frac{\partial z}{\partial x}\right)_p \tag{2.27}$$

$$\left(\frac{\partial\psi}{\partial y}\right)_z = \left(\frac{\partial\psi}{\partial y}\right)_p - \frac{\partial\psi}{\partial z}\left(\frac{\partial z}{\partial y}\right)_p . \tag{2.28}$$

He stated that 'The result of these substitutions is to produce a large number of terms. The additional terms are small, but they are not always negligible in comparison with the errors of observations. …On this account I have preferred to use instead … [height co-ordinates]'. In fact, since the Lagrangian time-derivative is invariant with respect to the co-ordinate transformation, we can write

$$\frac{d\psi}{dt} = \left(\frac{\partial\psi}{\partial t}\right)_p + u\left(\frac{\partial\psi}{\partial x}\right)_p + v\left(\frac{\partial\psi}{\partial y}\right)_p + \omega\frac{\partial\psi}{\partial p} ,$$

where $\omega = dp/dt$. This is no more complicated than the corresponding expansion in z-co-ordinates. Thus, Richardson's concern with small additional terms appears to be unwarranted.[5]

[5] Eliassen (1949) showed (in his §§20, 21) that there are additional terms in the continuity equation ((2.29) below), but that they may be disregarded in general.

Of greater interest is Richardson's allusion to 'elegant approximations'. What can he have meant? From (2.27) and (2.28) and the hydrostatic relation, it follows that the pressure gradient force in p-co-ordinates is simpler than in z-co-ordinates:

$$\frac{1}{\rho}\nabla_z p = g\nabla_p z\,.$$

The density does not appear on the right, so this term is linear in the dependent variables. The expression for the geostrophic wind is similarly simplified:

$$\mathbf{V}_{\mathrm{g}} = \frac{g}{f}\mathbf{k}\times\nabla_p z$$

and the thermal wind has only a single term

$$\frac{\partial \mathbf{V}_{\mathrm{g}}}{\partial z} = \frac{R}{f}\mathbf{k}\times\nabla_p T$$

which, again, is simpler and more elegant than for height co-ordinates.

The continuity equation takes a particularly elegant form in pressure co-ordinates, becoming a diagnostic equation. We recall equation (2.9), which can be written

$$\frac{\partial\rho}{\partial t} + \mathbf{V}\cdot\nabla_z\rho + \rho\nabla_z\cdot\mathbf{V} + \frac{\partial\rho w}{\partial z} = 0\,.$$

Using the hydrostatic equation to replace ρ by p we can write

$$\frac{\partial}{\partial p}\left\{\frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla_z p + w\frac{\partial p}{\partial z}\right\} + \left[\nabla_z\cdot\mathbf{V} - \frac{\partial\mathbf{V}}{\partial p}\cdot\nabla_z p\right] = 0\,.$$

The term in braces is just $\omega = dp/dt$. By (2.27) and (2.28) the term in square brackets is the horizontal divergence in isobaric co-ordinates. Thus we get the remarkable result:

$$\nabla_p\cdot\mathbf{V} + \frac{\partial\omega}{\partial p} = 0\,. \tag{2.29}$$

The continuity equation contains no explicit time derivative; it is a diagnostic equation. It is formally identical to the continuity equation for an incompressible fluid. The tendency equation in pressure co-ordinates is also greatly simplified. Integrating (2.29) and using the boundary condition $\omega = 0$ at $p = 0$, we get

$$\omega = -\int_0^p \nabla_p\cdot\mathbf{V}\,dp\,, \tag{2.30}$$

which is notably simpler than (2.15). It shows that ω is determined completely once $\mathbf{V}$ is known.

An intriguing question is whether Richardson could possibly have discovered the isobaric form of the continuity equation. Platzman (1967) argues that if he had discovered it he would have presented it in his book and would indeed have used it. It is difficult to dispute this view, particularly when Sydney Chapman's words about Richardson are recalled: 'He once told me that he had put all he knew into the book' (WPNP, Dover Edition, p. *viii*). The equation was derived by Eliassen in his seminal paper of 1949 on the quasi-geostrophic aproximation. It had appeared earlier, in work of Sutcliffe and of a Belgian meteorologist, O. Godard. However, more recently, Eliassen (1999) pointed out what had not been previously realized: the diagnostic relationship in pressure co-ordinates was discovered by Vilhelm Bjerknes, and is to be found, not as a mathematical equation but in textual form, in Bjerknes' *Kinematics* (1911). As Richardson was very familiar with Bjerknes' work, it is indeed conceivable that he knew about the equation and that this was what he meant when he wrote that the pressure system 'yields elegant approximations'. But until we find more conclusive documentary evidence, the matter remains unclear.

Another co-ordinate system introduced by Eliassen (1949) was the log-pressure system. If we define $Z = -H \log(p/p_0)$, where H and p_0 are constants, Z is a re-labelling of the pressure surfaces. With H chosen as the scale-height of the atmosphere and p_0 as the standard surface pressure, Z is approximately equal to the geometric height z. This system was used to great effect by Holton (1975) in his monograph on the statosphere and mesosphere.

The lower boundary condition in pressure co-ordinates is more complicated than in height co-ordinates, because the pressure at the Earth's surface changes with time. This property offsets to some extent the advantages of the pressure system. Phillips (1953) introduced normalized pressure $\sigma = p/p_S$ as the so-called sigma-co-ordinate. This yields the particularly simple lower boundary condition $\dot{\sigma} = 0$ at $\sigma = 1$. It has been used extensively in numerical modelling of the atmosphere. Richardson hinted at the possibility of terrain-following co-ordinates (WPNP, p. 92) but did not follow up on this idea.