Magnums

Counting Sets with Surreal Numbers

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Outline

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Magnums and Subsets of $\mathbb{N}$

The aim of this work is to define a number

$$m(A)$$

for subsets A of $\mathbb{N}$ that corresponds to our intuition about the size or magnitude of A.

We call $m(A)$ the **magnum of A**.

Magnum = Magnitude Number

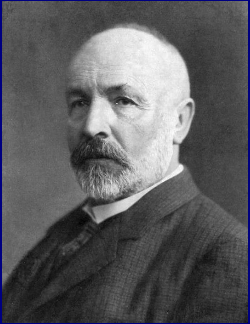
Outline

Galileo Galilei (1564–1642)

Every number *n* can be matched with its square n^2.

In a sense, there are as many squares as whole numbers.

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

Georg Cantor (1845–1918)

- Invented Set Theory.
- One-to-one Correspondence.
- Infinite and Well-ordered Sets.
- Cardinals and Ordinals.
- Proved $\mathrm{card}(\mathbb{Q}) = \mathrm{card}(\mathbb{N})$.
- Proved $\mathrm{card}(\mathbb{R}) > \mathrm{card}(\mathbb{N})$.
- Hierarchy of Infinities.

Set Theory: Controversy

Cantor was strongly criticized by

- Henri Poincaré.
- Leopold Kronecker.
- Ludwig Wittgenstein.

Set Theory is a “grave disease” (HP).
Cantor is a “corrupter of youth” (LK).
“Nonsense; laughable; wrong!” (LW).

Set Theory: A Difficult Birth

Set Theory brought into prominence several paradoxical results.

It was so innovative that many mathematicians could not appreciate its fundamental value and importance.

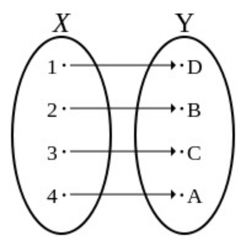
Gösta Mittag-Leffler was reluctant to publish it in his *Acta Mathematica.* He said the work was "100 years ahead of its time".

David Hilbert said:
"We shall not be expelled from the paradise that Cantor has created for us."

Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Infinite Sets

Now we consider sets that are infinite.

We take the natural numbers and the even numbers

$$\mathbb{N} = \{1, 2, 3, ...\}$$

$$\mathbb{E} = \{2, 4, 6, ...\}$$

By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$, we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\text{card}[\mathbb{N}] = \text{card}[\mathbb{E}]$$

Again,

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[\mathbb{E}]$$

But this is paradoxical: The set of natural numbers contains all the even numbers

$$\mathbb{E} \subset \mathbb{N}$$

and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

Outline

Richard Dedekind (1831–1916)

Irrational Numbers

Richard Dedekind defined irrational numbers by means of cuts of the rational numbers $\mathbb{Q}$.

For example, $\sqrt{2}$ is defined as (L, R), where

$$\begin{aligned} L &= \{\text{All rationals less than } \sqrt{2}\} \\ R &= \{\text{All rationals greater than } \sqrt{2}\} \end{aligned}$$

More precisely, and avoiding self-reference,

$$\begin{aligned} L &= \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\} \\ R &= \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\} \end{aligned}$$

Irrational Numbers

For each irrational number there is a corresponding cut (L, R).

We can regard the cut as equivalent to the number.

There are rules to manipulate cuts that are equivalent to the arithmetical rules for numbers.

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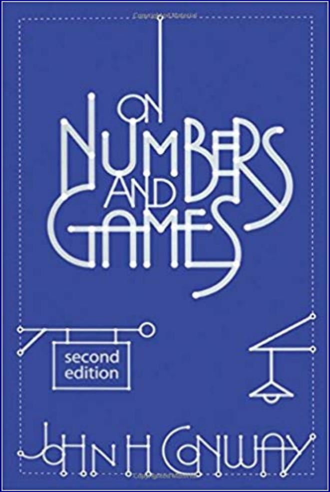
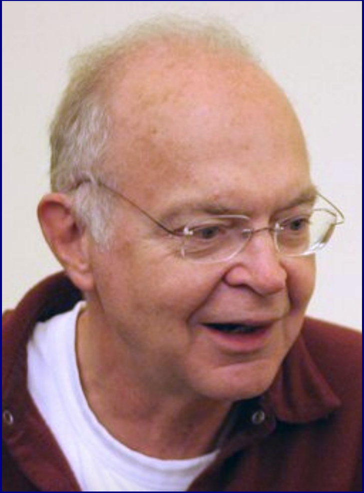
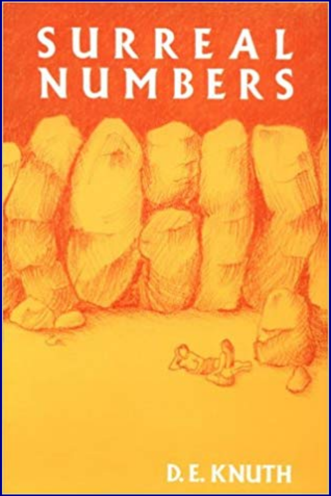
The surreal numbers are based upon a dramatic generalization of Dedekind's cuts.

John Conway's ONAG



Donald Knuth's *Surreal Numbers*





Constructing the Surreals

The Surreal numbers $\mathbb{S}$ are constructed **inductively.**

- Every number x is defined by a pair of sets, the left set and the right set:

$$x = \{ \, L \mid R \, \}$$

- No element of L is greater than or equal to any element of R.

x is the **simplest** number between L and R.

Constructing the Surreals

We start with 0, defined as

$$0 = \{\varnothing \mid \varnothing\} = \{\ \mid\ \}$$

Then 1, 2, 3 and so on are defined as

$$\{\,0 \mid\,\} = 1 \qquad \{\,1 \mid\,\} = 2 \qquad \{\,2 \mid\,\} = 3 \qquad \ldots$$

Negative numbers are defined inductively as

$$-x = \{-R \mid -L\,\}$$

so that

$$\{\,\mid 0\,\} = -1 \qquad \{\,\mid -1\,\} = -2 \qquad \{\,\mid -2\,\} = -3 \qquad \ldots$$

Constructing the Surreals

Dyadic fractions (of the form $m/2^n$) appear as

$$\{\,0 \mid 1\,\} = \tfrac{1}{2} \quad \{\,1 \mid 2\,\} = \tfrac{3}{2} \quad \{\,0 \mid \tfrac{1}{2}\,\} = \tfrac{1}{4} \quad \{\,\tfrac{1}{2} \mid 1\,\} = \tfrac{3}{4} \quad \ldots$$

After an infinite number of stages,
all the dyadic fractions have emerged.

At the next stage, all other real numbers appear.

Infinite and infinitesimal numbers also appear.

Surreal Numbers

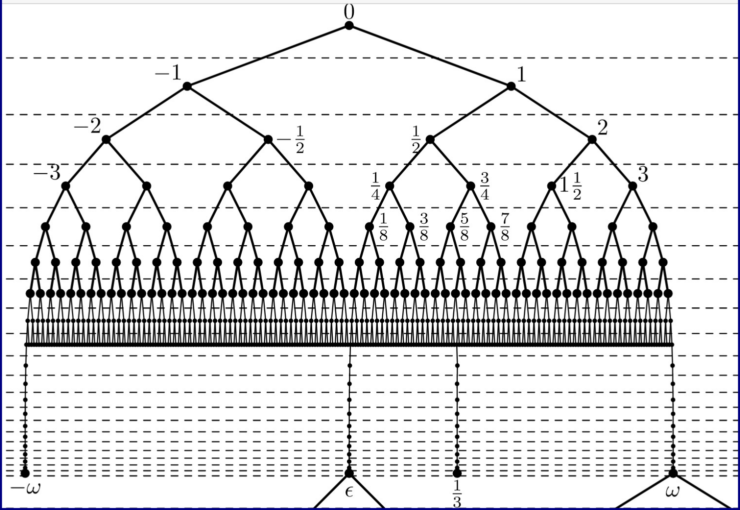


Figure: Surreal network from 0 to the first infinite number ω.

The First Infinite Number

The first infinite number ω is defined as

$$\omega = \{0, 1, 2, 3, \dots \mid \}$$

We can also introduce

$$\omega + 1 = \{0, 1, 2, \dots \ \omega \mid \}, \qquad \omega - 1 = \{0, 1, 2, \dots \mid \omega\}$$

$$2\omega = \{0, 1, 2, \dots \ \omega, \omega{+}1, \dots \mid \} \quad \tfrac{1}{2}\omega = \{0, 1, 2, \dots \mid \omega, \omega{-}1, \dots\}$$

and many other more exotic numbers.

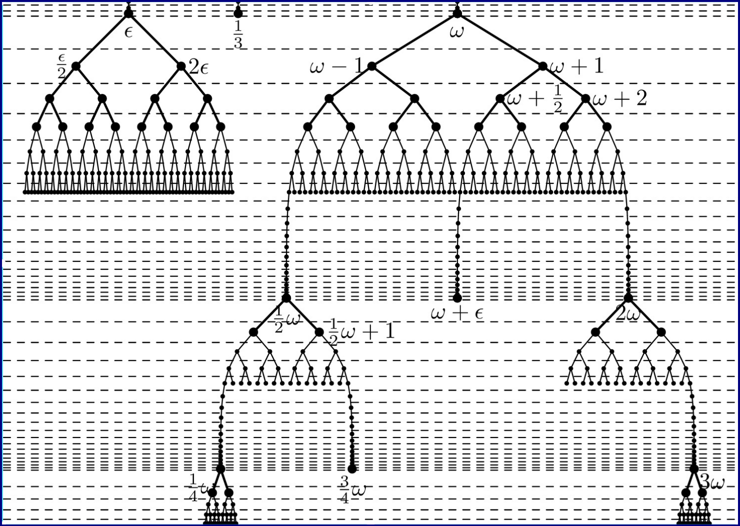


Figure: Network of early infinite and infinitesimal numbers.

Manipulating Infinite Numbers

The surreal numbers behave beautifully:
The class $\mathbb{S}$ is a totally ordered field.

We can define quantities like

$$\omega^2 \qquad \omega^\omega \qquad \sqrt{\omega} \qquad \log \omega$$

and many even stranger numbers.

The First Infinitesimal Number $\epsilon = 1/\omega$

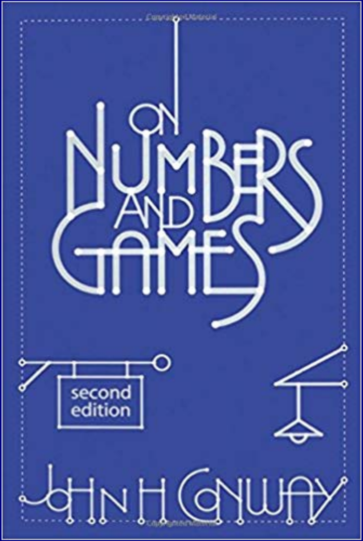
On day ω, the number $\epsilon = 1/\omega$ appears.

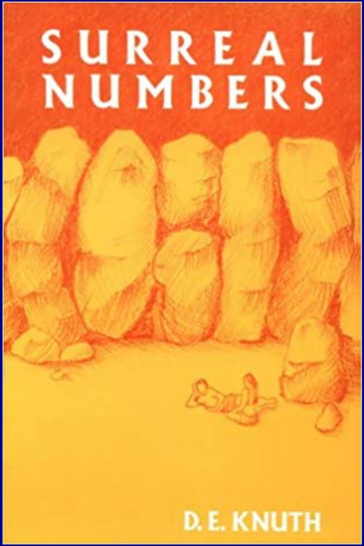
It can be shown that

$$\frac{\omega}{\omega} = \omega \times \epsilon = 1$$

Since we are interested in subsets of $\mathbb{N}$, we will consider surreals less than or equal to ω.

Books about Surreal Numbers





Outline

ABSTRACT

Cardinality is a *blunt instrument*:
The natural numbers, rationals and algebraic numbers all have the same cardinality.

So, $\aleph_0$ fails to discriminate between them.

Our aim is to define a number $m(A)$ for subsets A of $\mathbb{N}$ that corresponds to our intuition about the size or magnitude of A.

We define $m(A)$ as a surreal number.

Desiderata

- **For a finite subset A we have $m(A) = \mathrm{card}(A)$**
- **For a proper subset A of B we have**

$$A \subsetneqq B \implies m(A) < m(B)\,.$$

- **For the odd and even non-negative numbers**

$$\mathbb{N}_O = \{1, 3, 5, \ldots\} \implies m(\mathbb{N}_O) \approx \tfrac{1}{2} m(\mathbb{N})$$
$$\mathbb{N}_E = \{2, 4, 6, \ldots\} \implies m(\mathbb{N}_E) \approx \tfrac{1}{2} m(\mathbb{N})$$

Difficulties with Limits

In ONAG (page 43), Conway states that we cannot assume the limit of the sequence $(1, 2, 3, \dots)$ is ω.

We cannot conclude that $m(\mathbb{N}) = \omega$.
Therefore, we will write $m(\mathbb{N}) = \varpi$.

The precise specification of ϖ as a surreal number in the form $\{ L \mid R \}$ remains to be done.

Euler's Number

The usual definition of Euler's number is

$$e = \lim_{n\to\infty} f(n)\,, \qquad \textbf{where} \qquad f(n) = \left(1 + \frac{1}{n}\right)^n .$$

Evaluating $f(n)$ for $n = \varpi$ we obtain a surreal number

$$e_\varpi = f(\varpi) = \left(1 + \frac{1}{\varpi}\right)^\varpi$$

which is not equal to e.

Extending Functions from $\mathbb{R}$ to $\mathbb{S}$

The extension of many functions from $\mathbb{R}$ to $\mathbb{S}$ can be done without difficulty.

$$f : x \mapsto x^2\,, \;\; x \in \mathbb{R} \qquad \text{to} \qquad f : x \mapsto x^2\,, \;\; x \in \mathbb{S}$$

so we have $f(\varpi) = \varpi^2$ and so on.

This is fine for polynomials, rational functions, the logarithm and trigonometric functions.

Some Examples

$$f(n) = \left(\frac{n-1}{n}\right) = 1 - \frac{1}{n} \qquad \textbf{so} \qquad f(\varpi) = 1 - \frac{1}{\varpi}$$

The value of $f(\varpi)$ may not be defined in all cases:

$$f(n) = (-1)^n \qquad \textbf{extends to} \qquad f(\varpi) = (-1)^\varpi$$

and it is not clear what the value of this should be.

We introduce the notation

$$\Lambda \equiv (-1)^\varpi$$

without (yet) defining the value to be assigned to Λ.

Numerical Examples

For the real numbers, $0.999\cdots = 1$**.**
For the surreals, this is not the case:

$$f(n) = \underbrace{0.999\ldots9}_{n \text{ terms}} = 1-10^{-n}\,, \qquad \textbf{so} \qquad f(\varpi) = 1-10^{-\varpi} < 1\,.$$

Many more examples could be given, such as

$$\begin{aligned} 0.\overline{142857} &= \frac{142{,}857}{1{,}000{,}000}\left[1 + 10^{-6} + 10^{-12} + \ldots\right] \\ &= \frac{1}{7}\left[1 - 10^{-6\varpi}\right]\,. \end{aligned}$$

Outline

Counting Sequence

We define the characteristic function of $A \subset \mathbb{N}$ by

$$\chi_A(n) = \begin{cases} 1, & n \in A \\ 0, & \textbf{otherwise} \end{cases}$$

We assume that $a_1 < a_2 < a_3 < \cdots < a_n < \dots$.

Definition

We define the *counting sequence* κ_A to be the sequence of partial sums of the sequence $\{\chi_A(n)\}$:

$$\kappa_A(n) = \sum_{k=1}^{n} \chi_A(k)$$

Clearly, $\kappa(n) \leq n$ and $\kappa_A(n)$ counts the number of elements of A less than or equal to n.

The Magnum of A

Definition

If $\kappa_A(x)$ is defined for $x = \varpi$, the *magnum* of $A \subset \mathbb{N}$ is

$$m(A) = \kappa_A(\varpi)$$

Note that the magnum is a surreal number.

If A is a finite set, $m(A)$ is just $\mathrm{card}(A)$.

Principal Part of $m(A)$

We denote by $M(A)$ the infinite part of $m(A)$.

We write $m(A)$ in its *normal form*. Then

$$m(A) = \underbrace{M(A)}_{\text{Infinite}} + \underbrace{\big(m(A) - M(A)\big)}_{\text{Finite}}$$

This can be done in a canonical manner.

To compute the magnum, we write

$$\kappa_A(n) = \pi_A(n) + \big(\kappa_A(n) - \pi_A(n)\big)$$

Then $M(A) = \pi_A(\varpi)$ (if this exists).

A Set without a Magnum

Let U be the set of natural numbers with an odd number of decimal digits.

$$\chi_U(n) = \begin{cases} 1 \text{ if } n \text{ has an odd number of decimal digits}, \\ 0 \text{ if } n \text{ has an even number of decimal digits}. \end{cases}$$

If the density of U is $\rho_U(N) = \kappa_U(n)/N$ then

$$\begin{aligned} \rho_U(1) &= 0.0 \\ \rho_U(10) &= 0.9 \\ \rho_U(100) &= 0.09 \\ \rho_U(1000) &= 0.909 \\ \rho_U(10000) &= 0.0909 \end{aligned}$$

Outline

Intuition about Sizes

How do we 'know' that $\mathbb{N}_E$ is half the size of $\mathbb{N}$.

We do not. But we have a 'feeling' about it.

Why?

For any large but finite N, about half the numbers less than N are odd and about half are even.

The Odd Numbers

The characteristic sequence for the *odd numbers* is

$$\chi_O(n) = (1, 0, 1, 0, 1, 0, \dots)$$

and the counting sequence for the odd numbers is

$$\kappa_O(n) = (1, 1, 2, 2, 3, 3, \dots)$$

We can write $\chi_O(n)$ and $\kappa_O(n)$ as

$$\chi_O(n) = \frac{1-(-1)^n}{2} \qquad \text{and} \qquad \kappa_O(n) = \tfrac{1}{2}\left[n + \frac{1-(-1)^n}{2}\right]$$

Evaluating the counting function at ϖ we get

$$m(\mathbb{N}_O) = \kappa_O(\varpi) = \frac{\varpi}{2} + \frac{1}{4}\left[1-(-1)^{\varpi}\right] = \frac{\varpi}{2} + \frac{1}{4} - \frac{\Lambda}{4}.$$

The Even Numbers

We repeat this procedure for the even numbers.

$$\chi_E(n) = (0, 1, 0, 1, 0, 1, \dots)$$

$$\kappa_E(n) = (0, 1, 1, 2, 2, 3, \dots)$$

We can write these sequences as

$$\chi_E(n) = \frac{1 + (-1)^n}{2} \qquad \textbf{and} \qquad \kappa_E(n) = \tfrac{1}{2}\left[n - \frac{1 - (-1)^n}{2}\right]$$

Evaluating the counting function at ϖ we get

$$m(\mathbb{N}_E) = \kappa_E(\varpi) = \frac{\varpi}{2} - \frac{1}{4}\left[1 - (-1)^{\varpi}\right] = \frac{\varpi}{2} - \frac{1}{4} + \frac{\Lambda}{4}\,.$$

All Together

$$m(\mathbb{N}_O) = \frac{\varpi}{2} + \frac{1}{4} - \frac{\Lambda}{4}$$
$$m(\mathbb{N}_E) = \frac{\varpi}{2} - \frac{1}{4} + \frac{\Lambda}{4}$$

Assuming ϖ is an 'even number' $\Lambda = (-1)^\varpi = 1$ so

$$m(\mathbb{N}_O) = \frac{\varpi}{2}$$
$$m(\mathbb{N}_E) = \frac{\varpi}{2}$$

Since $\mathbb{N}_E$ and $\mathbb{N}_O$ are disjoint and $\mathbb{N}_E \cup \mathbb{N}_O = \mathbb{N}$, it is refreshing to observe that

$$m(\mathbb{N}_O) + m(\mathbb{N}_E) = \varpi = m(\mathbb{N}) .$$

Outline

Zeros at the Beginning

Theorem: **Suppose the set A has magnum $m(A)$. Then the shifted sequence B defined by**

$$\chi_B(1) = 0\,, \qquad \chi_B(n) = \chi_A(n-1)\,, \quad n > 1$$

has magnum

$$m(B) = m(A) - \chi_A(\varpi)\,.$$

Corollary: **If the sequence B is shifted from A by k places, we have**

$$m(B) = m(A) - \sum_{j=1}^{k} \chi_A(\varpi + 1 - j)$$

General Arithmetic Sequence

Theorem: The magnum of the arithmetic sequence $A = \{a, a+d, a+2d, a+3d, \dots\}$ is

$$m(s) = \frac{\varpi}{d} + \left(\frac{d+1-2a}{2d}\right)$$

Squares of Natural Numbers

We now consider the set of squares of natural numbers $S = \{1, 4, 9, 16, \dots\}$. The characteristic sequence is

$$\chi_S(n) = (1, \underbrace{0, 0}_{\text{2 zeros}}; 1, \underbrace{0, 0, 0, 0}_{\text{4 zeros}}; 1, \underbrace{0, 0, 0, 0, 0, 0}_{\text{6 zeros}}; 1, \dots)$$

and the sequence of partial sums of this sequence is

$$\kappa(n) = (\underbrace{1, 1, 1}_{\text{3 terms}}, \underbrace{2, 2, 2, 2, 2}_{\text{5 terms}}, \underbrace{3, 3, 3, 3, 3, 3, 3}_{\text{7 terms}}, \dots)$$

Theorem: The magnum of the sequence of squares is

$$m(S) = \sqrt{\varpi} - \frac{1}{2} + \text{HOT}\,.$$

General Geometric Sequence

We now consider the general geometric sequence

$$G = \{\beta r, \beta r^2, \beta r^3, \dots\}$$

Theorem: The magnum of the geometric sequence $G = \{\beta r, \beta r^2, \beta r^3 \dots\}$ is

$$m(G) = \frac{\ln \varpi}{\ln r} - \left(\frac{\ln \beta}{\ln r} + \frac{1}{2}\right) .$$

Thank you